O. Aşyrow, H. Soltanow

MATEMATIKI ANALIZ BOÝUNÇA MESELELER WE GÖNÜKMELER

(Bir üýtgeýänli funksiýalaryň differensialy we integraly. Köp üýtgeýänli funksiýalaryň differensialy)

I

Ýokary okuw mekdepleriniň talyplary üçin okuw gollanmasy

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A 79 **Matematiki analiz boýunça meseleler we gönükmeler** (Bir üýtgeýänli funksiýalaryň differensialy we integraly. Köp üýtgeýänli funksiýalaryň differensialy). Ýokary okuw mekdepleriniň talyplary üçin okuw gollanmasy. – A.: Türkmen döwlet neşirýat gullugy, 2019.

Matematiki analiz boýunça meseleleriň we gönükmeleriň bu ýygyndysy köplükler nazaryýetiniň elementleri, yzygiderligiň we bir üýtgeýänli funksiýalaryň predeli, önümi, differensialy we integraly, köp üýtgeýänli funksiýalaryň predeli we differensialy boýunça taýýarlanyldy.

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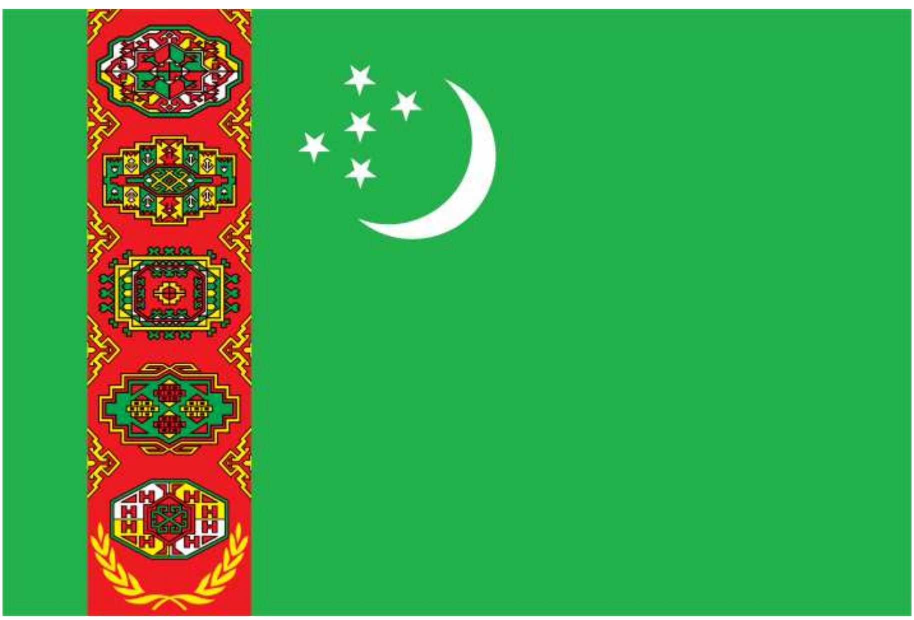
TÜRKMENISTANYŇ PREZIDENTI
GURBANGULY BERDIMUHAMEDOW

TÜRKMENISTANYŇ DÖWLET TUGRASY



TÜRKMENISTANYŇ DÖWLET BAÝDAGY

TÜRKMENISTANYŇ DÖWLET SENASY

Janym gurban saňa, erkana ýurdum,
Mert pederleň ruhy bardyr köňülde.
Bitarap, garaşsyz topragyň nurdur,
Baýdagyň belentdir dünýäň öňünde.

Gaýtalama:

Halkyň guran Baky beýik binasy,
Berkarar döwletim, jigerim-janym.
Başlaryň täji sen, diller senasy,
Dünýä dursun, sen dur, Türkmenistanym!

Gardaşdyr tireler, amandyr iller,
Owal-ahyr birdir biziň ganymyz.
Harasatlar almaz, syndyrmaz siller,
Nesiller döş gerip gorar şanymyz.

Gaýtalama:

Halkyň guran Baky beýik binasy,
Berkarar döwletim, jigerim-janym.
Başlaryň täji sen, diller senasy,
Dünýä dursun, sen dur, Türkmenistanym!

Türkmenistanyň Prezidenti
Gurbanguly Berdimuhamedow:

– Ýurdumyzyň ýokary bilim ulgamynyň ileri tutulýan ugry döwletiň netijeli dolandyryş ulgamyny we ykdysadyýetiň durnukly ösüşini, täze innowasion tehnologiýalaryň önümçilik pudaklaryna ornaşdyrylmagyny üpjün edip biljek we ylmyň ösüşine ýardam berjek, dünýä iş bazarynda bäsleşige ukyply ýokary bilimli hünärmenleri taýýarlamakdan ybaratdyr.

SÖZBAŞY

Türkmenistanyň Prezidenti hormatly Gurbanguly Berdimuhamedow Türkmenistanda ylym-bilim ulgamyny XXI asyryň ösen talaplaryna laýyklykda guramak, ýaşlara berilýän bilimi dünýä ülňülerine laýyk kämilleşdirmek maksady bilen, mugallymlaryň, pedagoglaryň, alymlaryň öňünde uly wezipeler goýdy. Şol wezipeleriň biri hem has kämil okuw maksatnamalaryny işläp düzmek, ylmyň soňky gazananlaryna daýanýan okuw kitaplaryny ýazmakdan ybaratdyr. Bu wajyp meselede ýokary okuw mekdepleriniň işgärleriniň borçlary örän uludyr. Sebäbi olar ýurdumyzyň ykdysadyýetiniň ähli ugurlaryny dünýä derejelerine galdyryp biljek we döwrebap tehnologiýalary özleşdirip biljek hünärmenleri taýýarlamalydyrlar.

Matematiki analiziň birinji bölegi boýunça taýýarlanylan bu gollanma köplükler nazaryýetiniň elementlerinden başlanylyp, onda yzygiderligiň we bir üýtgeýänli funksiýanyň predeli, üznüksizligi, önümi we differensialy, differensirlenýän funksiýalar hakyndaky teoremalar, kesgitsiz we kesgitli integrallar, olaryň ulanylyşlary hem-de köp üýtgeýänli funksiýalaryň predeli, hususy önümleri we differensiallary boýunça meseleler we gönükmeler toplanandyr. Bu okuw gollanmasynda diňe meseleleriň sanawy we olaryň jogaplary getirilmek bilen çäklenilmän, eýsem, her bölümiň başynda gysgaça teoretiki maglumatlar beýan edildi we olaryň dürli görnüşdäki meseleleri çözmekde ulanylyşlaryny görkezýän köpsanly mysallaryň çözülişi getirildi.

Bu okuw gollanmasyndan uniwersitetleriň, mugallymçylyk institutynyň we tehniki ýokary okuw mekdepleriniň talyplary peýdalanyp bilerler.

I. KÖPLÜKLER NAZARYÝETINIŇ ELEMENTLERI

§ 1. Köplükler we olar bilen geçirilýän amallar

1. *Köplük düşünjesi.* Köplükler nazaryýeti matematikanyň dürli şahalarynda ýüze çykýan meseleler çözülende wajyp orny eýeleýär. Matematikada köplük başlangyç düşünjeleriň biri bolup, köplük diýip haýsy-da bolsa bir nyşan (düzgün, häsiýet we ş.m.) boýunça birleşdirilen predmetleriň ýygyndysyna, toplumyna düşünilýär.

Köplügi düzüjilere onuň agzalary ýa-da elementleri diýilýär. Köplükler baş A, B, C, ... harplar bilen, olaryň elementleri bolsa setir a, b, c, ... harplar bilen belgilenýär. Mysal üçin, natural sanlaryň köplügi $\boldsymbol{N}$, bitin sanlaryň köplügi $\boldsymbol{Z}$, rasional sanlaryň köplügi $\boldsymbol{Q}$ we hakyky sanlaryň köplügi $\boldsymbol{R}$ bilen belgilenýär. (Gollanmada mysalyň çözülişiniň, tassyklamanyň subudynyň başyny we soňuny görkezýän ***Ç.B.*** we ***Ç.S.*** belgiler, şeýle hem islendik (her bir) sözüni çalşyrýan $\forall$ belgi we bar bolup (tapylyp) sözüni çalşyrýan $\exists$ belgi ulanylýar). $A \Rightarrow B$ ýazgy A sözlemden B sözlemiň gelip çykýandygyny, $A \Leftrightarrow B$ ýazgy bolsa A sözlemden B sözlemiň we şol bir wagtda B sözlemden A sözlemiň gelip çykýandygyny aňladýar. Eger a element B köplügiň elementi bolsa, onda ol $a \in B$ belgi arkaly aňladylýar we ol «a element B köplüge girýär» diýlip okalýar (ýöne ol «a element B köplüge degişli» diýlip hem okalýar). a elementiň B köplüge girmeýändigi $a \notin B$ ýazgy boýunça aňladylýar we ol «a element B köplüge girmeýär» diýlip okalýar (ýöne ol «a element B köplüge degişli däl» diýlip hem okalýar). A köplügiň $a_1, a_2, ..., a_n$ elementlerden düzülendigi

$$A = \{a_1, a_2, ..., a_n\}$$

görnüşde ýazylýar. Mysal üçin, eger $A = \{1, 2, 3, 4\}$ bolsa, onda $3 \in A$, ýöne $7 \notin A$.

Eger A köplügiň her bir elementi B köplügiň hem elementi bolsa $(x \in A) \rightarrow (x \in B)$, onda A köplüge B köplügiň bölegi (bölek köplügi) diýilýär. Bu halda A köplük B köplükde saklanýar ýa-da B köplük A köplügi özünde saklaýar hem diýilýär we ol $A \subset B$ ýa-da $B \supset A$ görnüşde ýazylýar. Mysal üçin, $N \subset Z \subset Q \subset R$. $\{x : P(x)\}$ we $\{x \in B : P(x)\}$ ýazgylar degişlilikde x elementleriň P häsiýetdäki köplügini we B köplügiň P häsiýetdäki bölek köplügini aňladýar. Mysal üçin, $A = \{x : x^2 - 1 = 0\}$ köplük $x^2 - 1 = 0$ deňlemäniň ähli kökleriniň köplügidir, ýagny $A = \{-1, 1\}$. $B = \{x \in R : x^2 + 1\}$ köplük bolsa $x^2 + 1 = 0$ deňlemäniň ähli hakyky kökleriniň köplügidir we ol köplügiň hiç bir elementi ýokdur. Şeýle köplüge boş köplük diýilýär we ol $\varnothing$ simwol bilen belgilenýär. Eger $A \subset B$ we şol bir wagtda $B \subset A$ bolsa, onda olara deň köplükler diýilýär we ol $A = B$ görnüşde ýazylýar.

K we L sözlemleriň bir wagtda ýerine ýetýändigi sebäpli $K \wedge L$ görnüşde, K ýa-da L sözlemleriň iň bolmanda biriniň ýerine ýetýändigi bolsa $K \vee L$ görnüşde ýazylýar.

Getirilen belgileri ulanyp, A we B köplükleriň deňligini gysgaça

$$(A = B) \Leftrightarrow (A \subset B) \wedge (B \subset A)$$

görnüşde ýazmak bolar.

Hakyky sanlaryň $\boldsymbol{R}$ köplüginiň käbir bölek köplükleriniň ýörite atlary bardyr:

$\{x \in \boldsymbol{R} : a \leq x \leq b\} = [a, b]$ – kesim;
$\{x \in \boldsymbol{R} : a < x < b\} = (a, b)$ – interwal;
$\{x \in \boldsymbol{R} : a \leq x < b\} = [a, b)$ – çep tarapy ýapyk interwal;
$\{x \in \boldsymbol{R} : a < x \leq b\} = (a, b]$ – sag tarapy ýapyk interwal.

Nokady öz içinde saklaýan islendik interwala şol nokadyň golaý töweregi, uzynlygy 2ε bolan $(a-\varepsilon, a+\varepsilon)$ interwal a nokadyň ε golaý töweregi diýlip atlandyrylýar we ol $U(a, \varepsilon)$ bilen belgilenýär. Mysal üçin, (0,98, 1,02) interwal $a=1$ nokadyň $\varepsilon = 0{,}02$ golaý töweregidir. $(a-\varepsilon, a]$ ($[a, a+\varepsilon)$) aralyga a nokadyň çep (sag) ýarym ε golaý töweregi diýilýär.

$$\mathring{U}(a, \varepsilon) = U(a, \varepsilon) \setminus \{a\}$$

köplüge a nokadyň sünjülen golaý töweregi diýilýär. $(a-\varepsilon, a)$ ($(a, a+\varepsilon)$) interwala a nokadyň çep (sag) ýarym sünjülen ε golaý töweregi diýilýär.

Hakyky sanyň moduly şeýle kesgitlenýär:

$$|a| = \begin{cases} a, \text{ eger } a \geq 0 \quad \text{bolsa}, \\ -a, \text{ eger } a < 0 \text{ bolsa}. \end{cases}$$

1-nji mysal. $B = \{1, 2, 3\}$köplügiň bölek köplüklerini görkezmeli.

Ç.B. Bu köplügiň 8 sany bölek köplügi bardyr: $\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1, 2, 3\}$. **Ç.S.**

2-nji mysal. $B = \{x \in \boldsymbol{N} : -4 < x \leq 5\}$ köplügiň elementlerini anyklamaly.

Ç.B. Köplügiň elementleri $-4 < x \leq 5$ deňsizligi kanagatlandyrýan x natural sanlardyr. Şeýle sanlar diňe 1, 2, 3, 4 we 5 bolup biler. Şeýlelikde, $B = \{1, 2, 3, 4, 5\}$. **Ç.S.**

2. ***Köplükler bilen geçirilýän amallar.*** A we B köplükleriň iň bolman-da birine girýän ähli elementleriň köplügine olaryň birleşmesi diýilýär we ol $A \cup B$ görnüşde belgilenýär:

$$A \cup B = \{x : x \in A \vee x \in B\}.$$

A we B köplükleriň bir wagtda ikisine hem girýän ähli elementleriň köplügine olaryň kesişmesi diýilýär we ol $A \cap B$ görnüşde belgilenýär:

$$A \cap B = \{x : x \in A \wedge x \in B\}.$$

A köplüge girip, B köplüge girmeýän ähli elementleriň köplügine şol köplükleriň tapawudy diýilýär we ol $A \backslash B$ görnüşde belgilenýär:

$$A \backslash B = \{x : x \in A \wedge x \notin B\}.$$

A we B köplükleriň islendik birine degişli bolup, beýlekisine degişli bolmadyk ähli elementleriň köplügine A we B köplükleriň simmetrik tapawudy diýilýär we ol $A \Delta B$ görnüşde belgilenýär:

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

Mysal üçin, eger A köplük 11-e çenli jübüt natural sanlaryň köplügi, B bolsa 11-e çenli 3-e bölünýän sanlaryň köplügi bolsa, onda $A \cup B = \{2, 3, 4, 6, 8, 9, 10\}$, $A \cap B = \{6\}$, $A \backslash B = \{2, 4, 8, 10\}$, $B \backslash A = \{3, 9\}$, $A \Delta B = \{2, 3, 4, 8, 9, 10,\}$.

Köplükleriň birleşmesi we kesişmesi düşünjelerini ikiden köp köplükler üçin hem girizmek bolar.

3-nji mysal. Köplükleriň $\{A_n = [1/n, n+1]\}$ yzygiderligi üçin $\bigcup_{n=1}^{\infty} A_n$ we $\bigcap_{n=1}^{\infty} A_n$ köplükleri tapmaly.

Ç.B. Berlen köplükleriň çep uçlaryna seredeliň. $n \to \infty$ bolanda $1/n \to 0$ bolýandygy üçin kesimleriň çep uçlarynyň predel ýagdaýy 0 nokatdyr, ýöne olar ol bahany alyp bilmeýär. Kesimleriň sag uçlary bolsa $n \to \infty$ bolanda $+\infty$ – e ymtylýar. Şonuň üçin hem $\bigcup_{n=1}^{\infty} A_n = (0, +\infty)$ bolar. Berlen kesimleriň iň kiçisi kesimdir we ol beýlekileriň hemmesine girýär. Şonuň üçin $\bigcap_{n=1}^{\infty} A_n = [1, 2]$. **Ç.S.**

Eger $B \subset A$ bolsa, onda $A \backslash B$ tapawuda B köplügiň A köplüge doldurgyjy diýilýär we ol $C_A B$ ýa-da gysgaça CB görnüşde belgilenýär. Bu kesgitlemeden şeýle deňlik alynýar:

$$C\,(CB) = C\,(A \backslash B) = A \backslash (A \backslash B) = B.$$

4-nji mysal. $C\,(A \cup B) = CA \cap CB$, $C(A \cap B) = CA \cup CB$ deňlikleri subut etmeli.

Ç.B. Goý, $x \in C\,(A \cup B)$ bolsun, onda $x \in (A \cup B)$, ýagny $(x \in A) \wedge (x \notin B)$. Onda $(x \in CA) \wedge (x \in CB)$, şonuň üçin hem $x \in (CA \cap CB)$. Diýmek,

$$C\,(A \cup B) \subset (CA \cap CB). \tag{1}$$

Eger $x \in (CA \cap CB)$ bolsa, onda $(x \in CA) \wedge (x \in CB)$, ýagny $(x \in A) \wedge (x \in B)$. Bu ýerden $x \in (A \cup B)$ alynýar. Şonuň üçin hem onuň esasynda $x \in C\,(A \cup B)$ bolar. Diýmek,

$$(CA \cap CB) \subset C(A \cup B). \tag{2}$$

(1) we (2) – deňliklerden bolsa $C(A \cup B) = CA \cap CB$ deňlik alynýar. Ikinji $C(A \cap B) = CA \cup CB$ deňlik bolsa şonuň ýaly görkezilýär. **Ç.S.**

Eger a we b sanlaryň haýsysynyň birinji, haýsysynyň ikinjidigi görkezilen bolsa, onda olara tertipleşdirilen jübüt sanlar diýilýär we (a, b) görnüşde belgilenýär. Şunlukda,

$$((a_1, b_1) = (a_2, b_2)) \Leftrightarrow ((a_1 = a_2) \wedge (b_1 = b_2)).$$

Goý, A we B erkin köplükler bolsun. Onda $a \in A,\ b \in B$ bolandaky mümkin bolan hemme tertipleşdirilen (a, b) jübütleriň köplügine A we B köplükleriň dekart köpeltmek hasyly diýilýär we $A \times B$ bilen belgilenýär:

$$A \times B := \{(a, b) : (a \in A) \wedge (b \in B)\}.$$

Mysal üçin, eger $A = \{2, 3\}$ we $B = \{4, 5\}$ bolsa, onda

$$A \times B = \{(2, 4), (2,5), (3, 4), (3, 5)\},$$
$$B \times A = \{(4, 2), (4, 3), (5, 2), (5, 3)\}.$$

Umuman aýdylanda, $A \times B \neq B \times A$. Eger $A = B$ bolsa, onda $A \times A$ ýazgynyň ýerine A^2 ýazylýar we oňa A köplügiň dekart kwadraty diýilýär. Mysal üçin, $\boldsymbol{R}$ köplügiň $\boldsymbol{R}^2 = \boldsymbol{R} \times \boldsymbol{R}$ dekart kwadraty tekizligiň nokatlarynyň ähli (x, y) dekart koordinatalarynyň köplügidir.

3. *Hasaply we hasapsyz köplükler.* Eger A köplügiň her bir elementine B köplügiň diňe bir elementi we tersine, B köplügiň her bir elementine A köplügiň diňe bir elementi degişli edilen bolsa, onda ol köplükleriň elementleriniň arasyndaky *özara birbahaly degişlilik* gurnalan diýilýär. Bu halda A we B köplükleriň özlerine *deňgüýçli* (*ekwiwalent*) *köplükler* diýilýär we $A \sim B$ görnüşde belgilenýär. Bu halda olara deň kuwwatly köplükler hem diýilýär. Mysal üçin, natural sanlaryň köplügi jübüt natural sanlaryň köplügine deňgüýçlüdir. Olaryň arasyndaky özara birbahaly degişliligi gysgaça şeýle ýazmak bolar: $n \leftrightarrow 2n$, $n = 1, 2, ...$.

Eger şeýle $n \in \boldsymbol{N}$ san tapylyp, $A \sim \{1, 2, 3, ..., n\}$ bolsa, onda A köplüge tükenikli köplük diýilýär. Bu köplügiň kuwwaty n-e deň ýa-da onuň n elementi bar diýilýär. Boş köplük hem tükenikli köplük hasap edilýär we onuň kuwwaty nola deňdir. Tükenikli bolmadyk köplüge tükeniksiz köplük diýilýär.

Eger $A \sim \boldsymbol{N}$ bolsa, onda A köplüge hasaply köplük diýilýär. Hasaply köplüge kuwwaty hasaply köplük hem diýilýär. Kuwwaty $\boldsymbol{N}$ köplügiňkiden uly bolan köplüge hasapsyz köplük diýilýär.

Ähli rasional sanlaryň köplügi hasaply köplükdir.

Ähli hakyky sanlaryň köplügi hasapsyz köplükdir.

4. *Çäkli we çäksiz köplükler.* Eger X köplük üçin şeýle c san tapylyp, köplügiň ähli x elementleri üçin $x \leq c$ $(x \geq c)$ bolsa, onda X köplüge *ýokardan* (*aşakdan*) *çäkli köplük* diýilýär. Hem aşakdan, hem ýokardan *çäkli köplüge* çäkli köplük diýilýär. Çäkli bolmadyk köplüge çäksiz köplük diýilýär. Mysal üçin, natural sanlaryň köplügi aşakdan 1 san bilen çäkli bolup, ol ýokardan çäkli däldir, diýmek, ol çäksiz köplükdir. Kesgitlemedäki c sana *X köplügiň ýokarky* (*aşaky*) *çägi ýa-da X köplügi ýokardan* (*aşakdan*) *çäklendiriji san* diýilýär. Eger c san X köplüge girýän bolsa, onda bu halda oňa X köplügiň iň uly (iň kiçi) elementi diýilýär. Mysal üçin, eger X iki belgili jübüt položitel sanlaryň köplügi bolsa, onda ol aşakdan 10 san bilen, ýokardan 98 san bilen çäklidir. Şonuň üçin ol çäkli köplükdir we 10 onuň iň kiçi elementi, 98 bolsa onuň iň uly elementidir.

X köplügiň çäkli bolmagy üçin şeýle $K > 0$ san tapylyp, islendik $x \in X$ üçin $|x| \leq K$ deňsizligiň ýerine ýetmegi zerur we ýeterlikdir.

X köplügiň iň uly (iň kiçi) elementi ýeke-täk kesgitlenip, onuň ýokarky (aşaky) çägi ýeke-täk kesgitlenýän däldir.

Eger X köplügiň ýokarky M çägi üçin:

1. $\forall x \in X$ üçin $x \leq M$;
2. $\forall \varepsilon > 0$ üçin $X \ni x_\varepsilon$ tapylyp, $x_\varepsilon > M - \varepsilon$

şertler ýerine ýetse, onda M sana X köplügiň takyk ýokarky çägi ýa-da X köplügiň ýokarky çäkleriniň iň kiçisi diýilýär.

Eger-de X köplügiň aşaky m çägi üçin:

1. $\forall x \in X$ üçin $x \geq m$;
2. $\forall \varepsilon > 0$ üçin $X \ni x_\varepsilon$ tapylyp, $x_\varepsilon < m + \varepsilon$

şertler ýerine ýetse, onda m sana X köplügiň *takyk aşaky* çägi ýa-da X köplügiň aşaky çäkleriniň iň ulusy diýilýär.

X köplügiň takyk ýokarky we takyk aşaky çäkleri degişlilikde

$$\sup X, \quad \sup_{x \in X} \{x\} \quad \text{we} \quad \inf X, \quad \inf_{x \in X} \{x\}$$

bilen belgilenýär. Kesgitlemelerden görnüşi ýaly, köplügiň iň uly we iň kiçi elementleri degişlilikde onuň takyk ýokarky we takyk aşaky çäkleridir. Beýle ýagdaý hemme tükenikli köplükler üçin dogrudyr. Eger $X = [a, b]$, $X_1 = (a, b)$ bolsa, onda olaryň ikisiniň hem takyk aşaky çägi a sana, takyk ýokarky çägi bolsa b sana deňdir. Bu mysallar köplügiň takyk çäkleriniň şol köplüge degişli bolup hem, degişli bolman hem bilýändigini görkezýär.

Ýokardan (aşakdan) çäkli köplügiň takyk ýokarky (takyk aşaky) çägi bardyr.

Islendik položitel hakyky a san üçin şeýle natural n san bar bolup, $a \leq n$ deňsizlik ýerine ýetýär (*Arhimediň prinsipi*).

Hakyky sanlardan düzülip, biri-biriniň içinde saklanýan kesimleriň $\{[a_n, b_n]\}$ ulgamy, ýagny islendik n üçin $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ şerti kanagatlandyrýan kesimleriň islendik ulgamy üçin olaryň hemmesine degişli bolan c san bardyr (*saklanýan kesimler prinsipi*).

5. *Matematiki induksiýa usuly.* Natural n sana bagly bolan dürli $A(n)$ sözlemleri subut etmeklikde matematiki induksiýa usuly ulanylýar. Onuň manysy şeýledir: eger $A(m)(m \geq 1)$ sözlem ýerine ýetýän bolsa we $k > m$ üçin $A(k)$ sözlemiň ýerine ýetýändiginden $A(k+1)$ sözlemiň hem ýerine ýetýändigi gelip çykýan bolsa, onda $A(n)$ sözlem islendik natural $n \geq m$ san üçin dogrudyr.

Bu usulyň ulanylyşyny görkezýän mysallara garalyň.

5-nji mysal. Nýutonyň binomynyň

$$(a+b)^n = \sum_{m=0}^{n} C_n^m a^{n-m} b^m, \qquad n \in N$$

formulasyny subut etmeli, bu ýerde $C_n^m = \dfrac{n!}{m!(n-m)!}, \qquad 0! = 1.$

Ç.B. Bu formula $n = 1$ bolanda dogrudyr, çünki

$$a + b = \sum_{m=0}^{1} C_1^m a^{1-m} b^m = C_1^0 a + C_1^1 b = a + b.$$

Goý, ol formula käbir $n = k$ üçin dogry bolsun, onda

$$(a+b)^{k+1} = (a+b)(a+b)^k = (a+b)\sum_{m=0}^{k} C_k^m a^{k-m} b^m =$$

$$= \sum_{m=0}^{k} C_k^m a^{k+1-m} b^m + \sum_{m=0}^{k} C_k^m a^{k-m} b^{m+1} = \sum_{m=0}^{k} C_k^m a^{k+1-m} b^m + \sum_{m=1}^{k+1} C_k^{m-1} a^{k+1-m} b^m =$$

$$= C_k^0 a^{k+1} + \sum_{m=1}^{k} C_k^m a^{k+1-m} b^m + \sum_{m=1}^{k} C_k^{m-1} a^{k+1-m} b^m + C_k^k b^{k+1}.$$

Bu ýerden

$$C_k^0 = C_{k+1}^0 = C_k^k = C_{k+1}^{k+1} = 1, \qquad C_k^m + C_k^{m-1} = C_{k+1}^m$$

deňliklerden peýdalanyp,

$$(a+b)^{k+1} = C_{k+1}^0 a^{k+1} + \sum_{m=1}^{k} C_{k+1}^m a^{k+1-m} b^m + C_{k+1}^{k+1} b^{k+1} = \sum_{m=0}^{k+1} C_{k+1}^m a^{k+1-m} b^m$$

deňligi alarys. Şoňa görä-de, matematiki induksiýa usulynyň esasynda Nýutonyň binomynyň formulasynyň ýerine ýetýändigi gelip çykýar. **Ç.S.**

6-njy mysal. Eger $\alpha > 0$ bolsa, onda $\forall n \in N$ üçin

$$(1+\alpha)^n \geq 1 + n\alpha, \qquad (1+\alpha)^n > \frac{n(n-1)}{2}\alpha^2$$

deňsizlikler dogrudyr.

Ç.B. Nýutonyň binomynyň formulasy esasynda

$$(1+\alpha)^n = 1 + n\alpha + \frac{n(n-1)}{2}\alpha^2 + \dots + \alpha^n.$$

Bu deňligiň sag böleginiň hemme goşulyjylary položiteldir, şonuň üçin hem ol deňlikden subut edilmeli deňsizlikleriň alynýandygy aýdyňdyr. **Ç.S.**

Deňsizlikleriň birinjisine Bernulliniň deňsizligi diýilýär, ol diňe $n = 1$ bolanda deňlige öwrülýär.

7-nji mysal. Eger islendik položitel $x_1, x_2, ..., x_n$ sanlar üçin deňlik $x_1 \cdot x_2 \cdot ... \cdot x_n = 1$ ýerine ýetse, onda $x_1 + x_2 +...+ x_n \geq n$ deňsizligiň dogrudygyny subut etmeli.

Ç.B. $n = 1$ bolanda $x_1 = 1$ şertiň esasynda deňsizlik dogrudyr. Goý, deňsizlik $n = k$ bolanda dogry bolsun we islendik položitel $x_1, x_2, ..., x_n, x_{k+1}$ sanlar üçin

$$x_1 x_2 ... x_k x_{k+1} = 1$$

deňlik ýerine ýetsin. Bu ýerde iki ýagdaý bolup biler: 1) sanlaryň ählisi bire deň, onda olaryň jemi $k + 1$-e deňdir we deňsizlik subut edildi; 2) ol sanlaryň içinde iň bolmanda biri bire deň däl, onda ol sanlaryň içinde iň bolmanda ýene biri bire deň däldir. Şunlukda, eger olaryň biri birden kiçi bolsa, onda beýlekisi birden uludyr. Umumylygy kemeltmezden $x_1 > 1, x_2 < 1$ hasap edeliň. Kabul etmämize görä $(x_1 x_2)$, $x_3 ... x_k, x_{k+1}$ položitel k sanlar üçin

$$x_1 x_2 + x_3 + ... + x_{k+1} \geq k$$

deňsizlik dogrudyr. Bu deňsizligiň iki bölegine-de $x_1 + x_2$ jemi goşup, $x_1 x_2$ aňlatmany deňsizligiň sag tarapyna geçirip, onuň sag bölegini özgerdeliň:

$$x_1 + x_2 + x_3 + ... + x_{k+1} \geq k - x_1 x_2 + x_1 + x_2 =$$

$$= k + 1 + x_1(1 - x_2) + x_2 - 1 = k + 1 + x_1(1 - x_2) - (1 - x_2) =$$

$$= k + 1 + (x_1 - 1)(1 - x_2) \geq k + 1.$$

Bu deňsizlik subut edilmeli deňsizligiň $n = k + 1$ bolanda hem dogry bolýandygyny görkezýär. Şonuň üçin hem matematiki induksiýa usulynyň esasynda deňsizlik islendik n üçin dogrudyr. Subut edilen deňsizlikde deňlik diňe

$$x_1 = x_2 =...= x_n = 1$$

bolanda ýerine ýetýändir. **Ç.S.**

Gönükmeler

1. Eger fakultetiň talyp toparlarynyň biriniň talyplarynyň köplügi A, fakultetiň başlykçyleriniň köplügi B bolsa, onda $A \cap B$, $A \backslash B$, $B \backslash A$ köplükler talyplaryň nähili köplüklerini düzer?

2. Berlen $A = \{x : 0 < x < 2\}$, $B = \{x : 1 \leq x \leq 3\}$ köplükler boýunça $A \cup B$, $A \cap B$, $A \backslash B$, $B \backslash A$ köplükleri tapmaly.

2.1. Berlen A we B köplükler boýunça $A \cup B$, $A \cap B$, $A \backslash B$, $B \backslash A$ köplükleri aňlatmaly:

a) $A = \{x \in \boldsymbol{R} : x^2 + 6x + 8 < 0\}$, $B = \{x \in \boldsymbol{R} : x^2 + 3x < 0\}$;
b) $A = \{x \in \boldsymbol{R} : 1 < |x - 3| \leq 2\}$, $B = \{x \in \boldsymbol{R} : 2|x| < 3\}$;
ç) $A = \{(x, y) \in \boldsymbol{R}^2 : x^2 + y^2 \leq 1\}$, $B = \{(x, y) \in \boldsymbol{R}^2 : xy \geq 0\}$.

3. Deňlikleri subut etmeli:

a) $(A \backslash B) \cap (C \backslash D) = (A \cap C) \backslash (B \cap D)$; b) $A \Delta B = (A \cup B) \backslash (A \cap B)$.

4. $0 < m < n$ deňsizligi kanagatlandyrýan natural m we n sanlar üçin ähli dogry m/n rasional droblaryň köplüginiň iň kiçi we iň uly elementiniň ýokdugyny subut etmeli. Ol köplügiň takyk aşaky we takyk ýokarky çäklerini tapmaly.

5. $r^2 < 2$ deňsizligi kanagatlandyrýan rasional r san köplüginiň takyk aşaky we takyk ýokarky çäklerini tapmaly.

6. Goý, $\{-x\}$ köplük $x \in \{x\}$ sanlara garşylykly bolan sanlaryň köplügi bolsun. Subut etmeli:

a) $\inf\{-x\} = -\sup\{x\}$; b) $\sup\{-x\} = -\inf\{x\}$.

7. Goý, $\{x + y\}$ köplük otrisatel däl $x \in \{x\}$ we $y \in \{y\}$ sanlaryň ähli $x + y$ jemleriniň köplügi bolsun. Deňlikleri subut etmeli:

a) $\inf\{x + y\} = \inf\{x\} + \inf\{y\}$; b) $\sup\{x + y\} = \sup\{x\} + \sup\{y\}$.

8. Goý, $\{xy\}$ köplük otrisatel däl $x \in \{x\}$ we $y \in \{y\}$ sanlaryň ähli xy köpeltmek hasyllarynyň köplügi bolsun. Deňlikleri subut etmeli:

a) $\inf\{xy\} = \inf\{x\} \cdot \inf\{y\}$; b) $\sup\{xy\} = \sup\{x\} \cdot \sup\{y\}$.

9. Deňsizlikleri subut etmeli:

a) $|x - y| \geq ||x| - |y||$; b) $|x + x_1 + \ldots + x_n| \geq |x| - (|x_1| + \ldots + |x_n|)$.

Deňsizlikleri çözmeli:

10. $|x + 1| < 0{,}01$.

11. $|x - 2| \geq 10$.

12. $|x| > |x + 1|$.

13. $|2x - 1| < |x - 1|$.

14. $|x + 2| + |x - 2| \leq 12$.

15. $|x + 2| - |x| > 1$.

16. $|x + 1| - |x - 1| < 1$.

17. $|x(1 - x)| < 0{,}05$.

18. Toždestwony subut etmeli:

$$\left(\frac{x + |x|}{2}\right)^2 + \left(\frac{x - |x|}{2}\right)^2 = x^2.$$

19. 10 *sm* uzynlyk ölçelende absolýut ýalňyşlyk 0,5 *mm*; 500 *km* uzaklyk ölçelende absolýut ýalňyşlyk 200 *m* bolupdyr. Bu ölçegleriň haýsysy has takyk?

20. Gönüburçlugyň taraplary

$$x = 2{,}50\ sm \pm 0{,}01\ sm; \quad y = 4{,}00\ sm \pm 0{,}02\ sm.$$

Onuň S meýdany haýsy çäklerde bolar? Gönüburçlugyň taraplary hökmünde onuň orta bahalary alnanda, onuň meýdanynyň absolýut Δ we otnositel δ ýalňyşlyklary nähili bolar?

21. Jisimiň agramy $p = 12{,}59\ g \pm 0{,}01\ g$, onuň göwrümi $V = 3{,}2\,sm^3 \pm 0{,}2\,sm^3$. Jisimiň udel agramyny kesgitlemeli, jisimiň agramy we göwrümi hökmünde orta bahalary alnanda udel agramyň absolýut we otnositel ýalňyşlyklaryny bahalandyrmaly.

22. Tegelegiň radiusy $r = 7{,}2\ m \pm 0{,}1\ m$. Eger $\pi = 3{,}14$ alynsa, onda tegelegiň meýdanynyň iň kiçi otnositel ýalňyşlygy nähili bolar?

23. Göni burçly parallelepipediň ölçegleri:

$$x = 24{,}7\ m \pm 0{,}2\ m;$$
$$y = 6{,}5\ m \pm 0{,}1\ m;$$
$$z = 1{,}2\ m \pm 0{,}1\ m.$$

Bu parallelepipediň υ göwrümi haýsy çäklerde bolar? Ölçegleri hökmünde onuň orta bahalary alnanda, parallelepipediň göwrüminiň absolýut we otnositel ýalňyşlyklary nähili bolar?

24. Kwadratyň x taraplary ($2\ m < x < 3\ m$) haýsy absolýut ýalňyşlyk bilen ölçelende, onuň meýdanyny $0{,}001\ m^2$ takyklykda kesgitlemek bolar?

25. Taraplarynyň her biri 10 m-den uly bolmadyk halynda gönüburçlugyň x we y taraplary haýsy absolýut ýalňyşlyk bilen ölçelende onuň meýdanyny $0{,}01\ m^2$ takyklykda kesgitlemek bolar?

26. Eger $\delta(x)$, $\delta(y)$ we $\delta(xy)$ degişlilikde x, y we xy sanlaryň otnositel ýalňyşlyklary bolsa, onda

$$\delta(xy) \le \delta(x) + \delta(y) + \delta(x)\delta(y)$$

deňsizligi subut etmeli.

Matematiki induksiýa usulyny ulanyp, islendik natural n san üçin deňlikleri subut etmeli:

27. $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$.

28. $1 + 3 + 5 + \ldots + (2n - 1) = n^2$.

29. $1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

30. $1^2 + 3^2 + \ldots + (2n-1)^2 = \dfrac{n(4n^2-1)}{3}$.

31. $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$.

32. $1^3 + 3^3 + 5^3 + \ldots + (2n - 1)^3 = n^2(2n^2 - 1)$.

33. $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \ldots + \dfrac{1}{(2n-2)(2n+1)} = \dfrac{n}{2n+1}$.

34. $\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+...+\frac{1}{(4n-3)(4n+1)}=\frac{n}{4n+1}.$

35. Şol bir alamatly –1-den uly bolan $x_1, x_2, ..., x_n$ sanlar üçin

$$(1+x_1)(1+x_2)...(1+x_n)\geq 1+x_1+x_2+...+x_n$$

Bernulliniň deňsizligini subut etmeli.

Deňsizlikleri subut etmeli:

36. $n!<\left(\frac{n+1}{2}\right)^n$ $(n>1)$. (*Görkezme*: $\left(\frac{n+2}{n+1}\right)^{n+1}=\left(1+\frac{1}{n+1}\right)^{n+1}>2$ *deňsizlikden peýdalanmaly*).

37. $2!\cdot 4!...(2n)!>[(n+1)!]^n$ $(n>1)$.

38. $\frac{1}{2}\cdot\frac{3}{4}...\frac{2n-1}{2n}<\frac{1}{\sqrt{2n+1}}.$

39. $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{n}}>\sqrt{n}$ $(n\geq 2)$.

40. $n^{n+1}>(n+1)^n$ $(n\geq 3)$.

41. $2^n n!<n^n$.

42. $\left|\sin\left(\sum_{k=1}^{n}x_k\right)\right|\leq\sum_{k=1}^{n}\sin x_k$ $(0\leq x_k\leq\pi;\ k=\overline{1,n})$.

43. $(2n)!<2^{2n}(n!)^2$.

44. Goý, položitel x_i $(i=\overline{1,n})$ sanlar üçin

$$a_n=\frac{x_1+x_2+...+x_n}{n}\text{ – orta arifmetik,}$$

$$b_n=\sqrt[n]{x_1x_2...x_n}\text{ – orta geometrik,}$$

$$c_n=\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_n}}\text{ – orta garmonik}$$

sanlar bolsun. $c_n\leq b_n\leq a_n$ deňsizligi subut etmeli (bu ýerde deňlik diňe $x_1=x_2=...=x_n$ bolanda dogrudyr). (*Görkezme*: *7-nji mysaldan peýdalanmaly*).

45. Natural m san üçin deňligi subut etmeli:

$$\sum_{k=1}^{n}k(k+1)...(k+m-1)=\frac{1}{m+1}n(n+1)...(n+m).$$

Bu deňlikden peýdalanyp, aşakdaky jemleri hasaplamaly:

a) $1 \cdot 2 + 2 \cdot 3 + ... + n \cdot (n+1)$;
b) $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + ... + n \cdot (n+1) \cdot (n+2)$;
ç) $1 \cdot 2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 \cdot 5 + ... + n \cdot (n+1) \cdot (n+2) \cdot (n+3)$.

46. Natural m san üçin deňligi subut etmeli:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)...(k+m)} = \frac{1}{m}\left(\frac{1}{m!} - \frac{1}{(n+1)(n+2)...(n+m-1)}\right).$$

Bu deňlikden peýdalanyp, aşakdaky jemleri hasaplamaly:

a) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + ... + \frac{1}{n \cdot (n+1)}$;

b) $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + ... + \frac{1}{n \cdot (n+1) \cdot (n+2)}$;

ç) $\frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + ... + \frac{1}{n \cdot (n+1) \cdot (n+2) \cdot (n+3)}$.

47. Deňlikleri subut etmeli:
a) $1 \cdot 1! + 2 \cdot 2! + ... + n \cdot n! = (n+1)! - 1$;

b) $1^4 + 2^4 + ... + n^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1)$;

ç) $1^5 + 2^5 + ... + n^5 = \frac{1}{12}n^2(n+1)^2(2n^2 + 2n - 1)$.

II. YZYGIDERLIGIŇ PREDELI

§ 1. San yzygiderlikleri we olaryň häsiýetleri

1. *Yzygiderlik düşünjesi.* Eger her bir n natural sana käbir x_n hakyky san degişli edilse, onda

$$x_1, x_2, \ldots, x_n, \ldots$$

sanlaryň toplumyna hakyky sanlaryň yzygiderligi diýilýär we ol gysgaça $\{x_n\}$ bilen belgilenýär.

Eger $\forall\varepsilon > 0$ san üçin $n_0 = n_0(\varepsilon)$ nomer tapylyp, $\forall n > n_0$ üçin

$$|x_n - a| < \varepsilon \tag{1}$$

deňsizlik ýerine ýetse, onda a sana $\{x_n\}$ yzygiderligiň predeli (ýa-da limiti) diýilýär.

a sanyň $\{x_n\}$ yzygiderligiň predeli bolýandygy

$$\lim x_n = \lim_{n\to\infty} x_n = a$$

ýazgyda aňladylýar.

Şeýlelikde,

$$\left(\lim_{n\to\infty} x_n = a\right) \Leftrightarrow (\forall\varepsilon > 0)(\exists\, n_0 \in N)(\forall n > n_0) : |x_n - a| < \varepsilon.$$

a sanyň $\{x_n\}$ yzygiderligiň predeli bolmagynyň geometrik manysy şeýledir: $\forall\varepsilon > 0$ üçin şeýle n_0 nomer tapylyp, yzygiderligiň n_0-dan soňky nomerli ähli agzalary a sanyň ε golaý töwereginde ýerleşýär, ýagny yzygiderligiň tükenikli sany agzalary $U(a, \varepsilon)$ golaý töweregiň daşynda ýerleşýär.

Eger yzygiderligiň tükenikli predeli bar bolsa, onda oňa ýygnanýan yzygiderlik, predeli ýok ýa-da tükeniksizlige deň bolsa, onda oňa dargaýan yzygiderlik diýilýär.

1-nji mysal. $\left\{\dfrac{n+c}{n}\right\}$ yzygiderligiň predeliniň bire deňdigini subut etmeli, bu ýerde c erkin hakyky san.

Ç.B. $\forall\varepsilon>0$ üçin Arhimediň prinsipi esasynda şeýle n_0 natural san tapylyp, $\dfrac{1}{n_0} < \varepsilon$ bolar. Şonuň üçin hem $\forall n>n_0$ üçin $\left|\dfrac{n+c}{n} - 1\right| = \dfrac{|c|}{n} < \dfrac{|c|}{n_0} < |c|\varepsilon = \bar{\varepsilon}$ deňsizlik ýerine ýetýär we kesgitleme boýunça $\lim\limits_{n\to\infty}\dfrac{n+c}{n} = 1$. **Ç.S.**

Bu mysaldan hemişelik c üçin $\lim\limits_{n\to\infty}\dfrac{c}{n} = 0$ deňlik gelip çykýar.

2-nji mysal. $\{(-1)^n\}$ yzygiderligiň predeliniň ýokdugyny subut etmeli.

Ç.B. Käbir a sany şol yzygiderligiň predeli hökmünde kabul edip, onuň $U(a,1/3)$ golaý töweregine garalyň. Bu golaý töweregiň, ýagny $(a-1/3,\ a+1/3)$ interwalyň uzynlygy 2/3-ä deň. Şonuň üçin bu golaý töwerek bir wagtda –1 we +1 nokatlary özünde saklap bilmez, sebäbi ol nokatlaryň arasyndaky uzaklyk 2-ä deňdir. Eger 1 bu golaý töwerege degişli däl diýip güman etsek, onda $n = 2, 4, 6, ...$ bolanda $x_n = 1$ bolýandygy üçin, bu golaý töweregiň daşynda yzygiderligiň tükeniksiz köp agzalary ýerleşer. Diýmek, a yzygiderligiň predeli bolup bilmez, onuň erkinliginden bolsa yzygiderligiň predeliniň ýokdugy gelip çykýar. **Ç.S.**

3-nji mysal. $x_n = \dfrac{5n-3}{2n}$ yzygiderligiň predeliniň 2,5-e deňdigini subut etmeli we $\varepsilon = 0{,}01$ üçin n_0 nomeri kesgitlemeli.

Ç.B. Kesgitleme boýunça $\forall\varepsilon\, 0$ üçin $\forall n > n_0$ bolanda, $|x_n - 2{,}5| < \varepsilon$ bolar ýaly n_0 nomeri tapalyň. Onuň üçin deňsizligiň çep bölegini özgerdeliň:

$$|x_n - 2{,}5| = \left|\frac{5n-3}{2n} - 2{,}5\right| = \left|\frac{-3}{2n}\right| = \frac{3}{2n} < \varepsilon.$$

Bu ýerden görnüşi ýaly, $n > \dfrac{3}{2\varepsilon}$ bolanda $|x_n - 2{,}5| < \varepsilon$ bolar. Şonuň üçin n_0 nomer hökmünde $\dfrac{3}{2\varepsilon}$ sany ýa-da onuň bitin bölegini almak bolar. Şonda $\forall n > n_0$ üçin $|x_n - 2{,}5| < \varepsilon$ bolar, ýagny $\lim\limits_{n\to\infty} x_n = 2{,}5$. Eger $\varepsilon = 0{,}01$ bolsa, onda $n_0 = 3/(2\cdot 0{,}01) = 150$ bolar. **Ç.S.**

Eger şeýle c san tapylyp, $\forall n \in N$ üçin $x_n \le c$ ($x_n \ge c$) bolsa, onda $\{x_n\}$ yzygiderlige ýokardan (aşakdan) çäkli yzygiderlik diýilýär. Hem ýokardan, hem aşakdan çäkli yzygiderlige çäkli yzygiderlik diýilýär. $\{x_n\}$ yzygiderligiň çäkli bolmagy üçin şeýle $K > 0$ san tapylyp, $\forall n \in N$ üçin $|x_n| \le K$ deňsizligiň ýerine ýetmegi zerur we ýeterlikdir. Çäkli bolmadyk yzygiderlige çäksiz yzygiderlik diýilýär.

2. ***Yzygiderligiň esasy häsiýetleri.*** **1)** eger yzygiderligiň predeli bar bolsa, onda yzygiderlik çäklidir we onuň predeli ýeke-täkdir.

2) eger $\{x_n\}$ yzygiderligiň $a \ne 0$ predeli bar bolsa, onda şeýle n_0 nomer tapylyp, $\forall n > n_0$ üçin $|x_n| > |a|/2$ deňsizlik ýerine ýetýär. Has anykragy, görkezilen n üçin $a > 0$ bolanda $x_n > a/2$ we $a < 0$ bolanda $x_n < a/2$. Diýmek, käbir nomerden başlap $\{x_n\}$ yzygiderligiň alamaty a sanyň alamaty bilen gabat gelýär.

3) eger $\lim\limits_{n\to\infty} x_n = a$ we $\lim\limits_{n\to\infty} y_n = b$ predeller bar bolup, $x_n > y_n$ ýa-da $x_n \ge y_n$ bolsa, onda $a \ge b$.

4) eger $\{x_n\}$ we $\{y_n\}$ yzygiderlikleriň ikisiniň hem predeli şol bir a sana deň bolsa we $\forall n \in N$ üçin $x_n \le z_n \le y_n$ deňsizlikler ýerine ýetse, onda a san $\{z_n\}$ yzygiderligiň hem predelidir.

5) eger $\{x_n\}$ we $\{y_n\}$ yzygiderlikleriň predelleri bar bolsa, onda $\{x_n \pm y_n\}$, $\{x_n \cdot y_n\}$ we $\{x_n / y_n\}$ (paý kesgitlenende $y_n \neq 0$) yzygiderlikleriň hem predelleri bardyr:

$$\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n;$$

$$\lim_{n\to\infty}(x_n \cdot y_n) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n;$$

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n} \quad \left(\lim_{n\to\infty} y_n \neq 0\right).$$

3. ***Tükeniksiz kiçi we tükeniksiz uly yzygiderlikler.*** Eger yzygiderligiň predeli nola deň bolsa, onda oňa tükeniksiz kiçi yzygiderlik diýilýär.

a sanyň $\{x_n\}$ yzygiderligiň predeli bolmagy üçin

$$x_n = a + \alpha_n, \quad \lim_{n\to\infty} \alpha_n = 0 \tag{2}$$

deňlikleriň ýerine ýetmegi zerur we ýeterlikdir.

Tükenikli sany tükeniksiz kiçi yzygiderlikleriň algebraik jemi we köpeltmek hasyly hem-de tükeniksiz kiçi yzygiderligiň çäkli yzygiderlige köpeltmek hasyly tükeniksiz kiçi yzygiderlikdir.

Eger $\forall K > 0$ üçin $n_0 = n_0(K)$ nomer tapylyp, $\forall n > n_0$ üçin $|x_n| > K$ deňsizlik ýerine ýetse, onda $\{x_n\}$ yzygiderlige tükeniksiz uly yzygiderlik diýilýär. Ol şeýle ýazylýar:

$$\lim_{n\to\infty} x_n = \infty \quad \text{ýa-da} \quad x_n \to \infty.$$

4-nji mysal. $\{q^n\}$ yzygiderligiň $|q| > 1$ bolanda tükeniksiz uly, $|q| < 1$ bolanda tükeniksiz kiçi yzygiderlikdigini subut etmeli.

Ç.B. Eger $|q| > 1$ bolsa, onda $|q| = 1 + \alpha$, $\alpha > 0$ we Bernulliniň deňsizligi esasynda

$$(1+\alpha)^{n_0} \geq 1 + n_0\alpha > n_0\alpha.$$

Eger $\forall K > 0$ üçin n_0 nomeri $n_0\alpha > K$ bolar ýaly saýlap alsak, onda $\forall n > n_0$ üçin

$$|q|^n > |q|^{n_0} = (1+\alpha)^{n_0} > n_0\alpha > K,$$

ýagny $\{q^n\}$ – tükeniksiz uly yzygiderlikdir.

Eger $|q| < 1$ bolsa, onda $|1/q| > 1$ bolýandygy üçin bu halda yzygiderligiň tükeniksiz kiçi bolýandygy subut edilenden gelip çykýar. **Ç.B.**

5-nji mysal. Degişlilikde k we l derejeli $P(n)$ we $Q(n)$ köpagzalar üçin $P(n)/Q(n)$ paýyň predelini tapmaly.

Ç.B. Bu predeli tapmak üçin ony

$$\frac{P(n)}{Q(n)} = n^{k-l}\frac{a_0 + \frac{a_1}{n} + \ldots + \frac{a_k}{n^k}}{b_0 + \frac{b_1}{n} + \ldots + \frac{b_l}{n^l}}$$

görnüşde ýazýarlar. Bu deňlige görä alarys:

$$\lim_{n\to\infty}\frac{P(n)}{Q(n)} = \begin{cases} 0, & \text{eger } k < l \text{ bolsa,} \\ a_0/b_0, & \text{eger } k = l \text{ bolsa,} \\ \infty, & \text{eger } k > l \text{ bolsa.} \end{cases}$$ **Ç.S.**

6-njy mysal. $\lim \sqrt[n]{n} = 1$ deňligi subut etmeli.

Ç.B. Eger $n \geq 2$ bolsa, onda $\lim \sqrt[n]{n} > 1$. Şonuň esasynda $\forall n \geq 2$ üçin şeýle $\alpha_n > 0$ san tapylyp, $\sqrt[n]{n} = 1 + \alpha_n$ deňlik ýerine ýeter. Ondan bolsa $n = (1 + \alpha_n)^n$ deňlik alynýar. Bu deňlik esasynda

$$n = (1 + \alpha_n)^n > \frac{n(n-1)}{2}\alpha_n^2$$

deňsizligi alarys. Bu ýerden $\forall n \geq 2$ üçin $0 < \alpha_n < \sqrt{\frac{2}{n-1}}$ deňsizlik alynýar. Ondan bolsa $\lim_{n\to\infty}\sqrt{\frac{2}{n-1}} = 0$ bolýandygyndan peýdalanyp, $\lim_{n\to\infty}\alpha_n = 0$ deňligi alarys. Diýmek,

$$\lim_{n\to\infty}\sqrt[n]{n} = \lim_{n\to\infty}(1 + \alpha_n) = 1.$$ **Ç.S.**

7-nji mysal. $\forall a > 0$ üçin $\lim_{n\to\infty} a^{1/n} = 1$, $\lim_{n\to\infty} a^{-1/n} = 1$ deňlikleri subut etmeli.

Ç.B. Eger $1 < a < n$ bolsa, onda $1 < \sqrt[n]{a} < \sqrt[n]{n}$ bolar. Bu ýerden 5-nji mysal we 4-nji häsiýet esasynda $\lim_{n\to\infty}\sqrt[n]{a} = 1$ alynýar. Eger-de $0 < a < 1$ bolsa, onda $1/a > 1$ we $\lim_{n\to\infty}(1/a)^{1/n} = 1$ bolýandygy üçin

$$\lim_{n\to\infty}\sqrt[n]{a} = \lim_{n\to\infty}\frac{1}{\sqrt[n]{1/a}} = 1.$$

Subut edilen deňlikden bolsa $\lim_{n\to\infty} a^{-\frac{1}{n}} = 1$ gelip çykýar. **Ç.S.**

8-nji mysal. $\lim\limits_{n\to\infty}\frac{n}{b^n} = 0$ $(b > 1)$ deňligi subut etmeli.

Ç.B. $$0 < \frac{n}{b^n} = \frac{n}{(1+(b-1))^n} =$$
$$= \frac{n}{1+n(b-1)+\frac{n(n-1)}{2}(b-1)^2+...+(b-1)^n} <$$
$$< \frac{2n}{n(n-1)(b-1)^2} = \frac{2}{(n-1)(b-1)^2} \to 0.$$

Bu ýerden yzygiderligiň 4-nji häsiýeti esasynda subut edilmeli deňlik gelip çykýar. **Ç.S.**

Bu mysaldan we yzygiderligiň 5-nji häsiýetinden bitin položitel m san üçin $\lim\limits_{n\to\infty}\left(\frac{n}{b^n}\right)^m = 0$ $(b > 1)$ deňlik alynýar.

4. *Monoton yzygiderlikler we olaryň ýygnanma nyşany.* Eger $\forall n \in N$ üçin $x_n \leq x_{n+1}$ $(x_n \geq x_{n+1})$ deňsizlik ýerine ýetse, onda $\{x_n\}$ yzygiderlige kemelmeýän (artmaýan) yzygiderlik diýilýär. Eger-de $\forall n \in N$ üçin $x_n < x_{n+1}$ $(x_n > x_{n+1})$ deňsizlik ýerine ýetse, onda $\{x_n\}$ yzygiderlige artýan (kemelýän) yzygiderlik diýilýär.

Weýerştrasyň teoremasy. Eger kemelmeýän (artmaýan) yzygiderlik ýokardan (aşakdan) çäkli bolsa, onda onuň predeli bardyr.

9-njy mysal. $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = e$ deňlikleri subut etmeli.

Ç.B. Goý, $\{x_n\} = \left\{\left(1+\frac{1}{n}\right)^n\right\}$ bolsun. Nýutonyň binomy esasynda

$$x_n = \left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} +$$
$$+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + ... + \frac{n(n-1)...[n-(n-1)]}{n!}\cdot\frac{1}{n^n} =$$
$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + ... + \frac{1}{n!}\left(1-\frac{1}{n}\right)...\left(1-\frac{n-1}{n}\right). \tag{3}$$

Eger bu deňligi x_{n+1} üçin hem ýazyp, ony (3) deňlik bilen deňeşdirsek, onda $\forall k=\overline{1, n-1}$ üçin $1 - k/n < 1 - k/(n + 1)$ bolýandygy we x_{n+1} üçin ýazylan deňligiň sag bölegine položitel goşulyjynyň goşulýandygy sebäpli, $x_n < x_{n+1}$ deňsizlik alnar, ol bolsa $\{x_n\}$ yzygiderligiň artýandygyny aňladýar. (3) deňlikden $1 - k/n < 1$ $(k=\overline{1, n-1})$ deňsizligiň esasynda

$$x_n < 1 + 1 + \frac{1}{2!} + ... + \frac{1}{n!} \tag{4}$$

deňsizlik alynýar. $1/k! \le 1/2^{k-1}$ $(k \ge 2)$ deňsizligi ulanyp, (4) deňsizligi

$$x_n < 1 + 1 + \frac{1}{2!} + ... + \frac{1}{n!}$$

görnüşde ýazyp bolar. Sag bölekdäki jem ikinjiden başlap maýdalawjysy 1/2-e deň geometrik progressiýanyň jemidir. Şonuň üçin

$$x_n < 1 + \frac{1 - 1/2^n}{1 - 1/2} = 3 - \frac{1}{2^{n+1}} < 3.$$

Şeýlelikde, $\{x_n\}$ yzygiderligiň artýandygyny we ýokardan çäklidigini görkezdik (ol yzygiderlik (3) deňlikden görnüşi ýaly, aşakdan hem çäklidir: $x_n \ge 2$). Şeýlelikde, monoton we çäkli bolan $\{(1+1/n)^n\}$ yzygiderligiň predeli bardyr, ony Eýleriň teklibi boýunça e san bilen belgilemeklik kabul edilendir: $\lim\limits_{n\to\infty}(1 + 1/n)^n = e$.

Goý, $y_n = \left(1 + \frac{1}{n}\right)^{n+1}$ bolsun. Bu yzygiderligiň kemelýändigini we onuň hem predeliniň e sana deňdigini görkezeliň. Bernulliniň deňsizligi esasynda

$$\frac{y_n}{y_{n+1}} = \frac{(1 + 1/n)^{n+1}}{(1 + 1/(n+1))^{n+2}} = \left[\frac{(n+1)^2}{n(n+2)}\right]^{n+1} \frac{n+1}{n+2} =$$

$$= \left(1 + \frac{1}{n^2 + 2n}\right)^{n+1} \frac{n+1}{n+2} \ge \left(1 + \frac{n+1}{n^2 + 2n}\right)\frac{n+1}{n+2} =$$

$$= \frac{n^3 + 4n^2 + 4n + 1}{n^3 + 4n^2 + 4n} > 1,$$

ýagny $\forall n \in N$ üçin $y_n > y_{n+1}$. Bu bolsa $\{y_n\}$ yzygiderligiň kemelýändigini görkezýär.

$$y_n = \left(1 + \frac{1}{n}\right)^{n+1} = \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)$$

deňlikde predele geçip, $\lim\limits_{n\to\infty}(1 + 1/n)^n = e$ deňlikden peýdalanyp, $\lim\limits_{n\to\infty} y_n = e$ deňligi alarys. **Ç.S.**

Bu mysaldaky $\{x_n\}$ yzygiderligiň artýandygy we $\{y_n\}$ yzygiderligiň kemelýändigi sebäpli, $\forall n \in N$ üçin

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1} \Rightarrow \frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}. \tag{5}$$

10-njy mysal. $x_n = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{4}\right)...\left(1 + \frac{1}{2^n}\right)$, $n \in N$ yzygiderligiň ýygnanýandygyny subut etmeli.

Ç.B. $\frac{x_{n+1}}{x_n} = 1 + \frac{1}{2^{n+1}} > 1$, ýagny $\{x_n\}$ yzygiderlik artýar. (5) deňsizligiň esasynda

$$\ln x_n = \ln\left(1 + \frac{1}{2}\right) + \ln\left(1 + \frac{1}{4}\right) + ... + \ln\left(1 + \frac{1}{2^n}\right) <$$
$$< \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^n} < \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^n} + ... = \frac{1}{2}\frac{1}{1-1/2} = 1, \quad x_n < e,$$

ýagny $\{x_n\}$ yzygiderlik çäklidir. Şonuň üçin hem Weýerştrasyň teoremasy boýunça $\{x_n\}$ yzygiderlik ýygnanýar. **Ç.S.**

11-nji mysal. $x_n = \frac{n!}{(2n+1)!!}$, $n \in N$ yzygiderligiň predelini tapmaly.

Ç.B. Ilki bilen aşakdaky gatnaşygy düzeliň:

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)!\cdot(2n+1)!!}{n!\cdot(2n+3)!!} = \frac{n+1}{2n+3}.$$

Bu deňlik esasynda islendik $n \geq 1$ üçin $x_{n+1} < \frac{1}{2}x_n < x_n$ bolar, ýagny yzygiderlik kemelýär we $0 < x_n \leq x_1 = 1/3$ bolany üçin ol çäklidir. Şonuň üçin hem onuň predeli bardyr: $\lim\limits_{n\to\infty} x_n = c$. Onda $x_{n+1} = x_n \cdot \frac{n+1}{2n+3}$ deňlikde predele geçip, $c = \frac{1}{2}c$, ýagny $c = 0$ deňligi alarys. Şeýlelikde, $\lim\limits_{n\to\infty} x_n = 0$. **Ç.S.**

12-nji mysal. $x_n = \sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}$, $n \in N$ (n kök) üçin yzygiderligiň ýygnanýandygyny subut etmeli we predelini tapmaly.

Ç.B. n-iň artmagy bilen x_n-iň artýandygy aýdyňdyr. Iň soňky kökde 2-ni 4 bilen çalşyryp, islendik n üçin $x_n < 2$ deňsizligi alarys. Şeýlelikde, yzygiderlik artýar we ýokardan çäkli, şoňa görä onuň predeli bar: $\lim\limits_{n\to\infty} x_n = a$. Ony tapmak üçin

$$x_n = \sqrt{2 + x_{n-1}}$$

deňlikde predele geçip, $a = \sqrt{2 + a}$ deňligi alarys, ýagny $a = 2$. Şeýlelikde, $\lim\limits_{n\to\infty} x_n = 2$. **Ç.B.**

5. *Bölek yzygiderligiň predeli.* Goý, $n_1, n_2, ..., n_k, ...$ natural sanlaryň artýan yzygiderligi bolsun. $\{x_n\}$ yzygiderlikden nomerleri $n_1, n_2, ..., n_k, ...$ deň bolan agzalary alyp, olary şol natural sanlaryň ýerleşiş tertibinde ýazalyň:

$$x_{n_1}, x_{n_2}, ..., x_{n_k}, ...$$

Berlen yzygiderlikden şu görnüşde alnan yzygiderlige $\{x_n\}$ yzygiderligiň bölek yzygiderligi diýilýär we ol $\{x_{n_k}\}$ ýazgyda belgilenýär.

Mysal üçin, täk natural sanlaryň 1, 3, 5, 7,… yzygiderligi natural sanlaryň yzygiderliginiň bölek yzygiderligidir, ýöne 3, 5, 1, 9, 7,… yzygiderlik onuň bölek yzygiderligi däldir.

Eger $\{x_{n_k}\}$ bölek yzygiderlik ýygnanýan bolsa, onda onuň predeline $\{x_n\}$ yzygiderligiň bölekleýin predeli diýilýär.

Mysal üçin, predeli ýok $\{(-1)^n\}$ yzygiderligiň iki sany –1 we 1 bolan bölekleýin predelleri bardyr, çünki $\{-1\} = -1, -1, ..., -1, ...$ we $\{1\} = 1, 1, ..., 1, ...$ yzygiderlikler berlen yzygiderligiň bölek yzygiderlikleridir we olaryň predelleri –1 we 1 bolýandyr.

Eger yzygiderligiň predeli bar bolsa, onda onuň islendik bölek yzygiderliginiň hem predeli bardyr we ol predel yzygiderligiň predeline deňdir.

Bolsano-Weýerştrasyň teoremasy. Islendik çäkli yzygiderlikden ýygnanýan bölek yzygiderligi almak bolar.

Yzygiderligiň bölekleýin predelleriniň iň ulusyna (iň kiçisine) yzygiderligiň ýokarky (aşaky) predeli diýilýär we ol degişlilikde

$$\overline{\lim_{n\to\infty}} x_n, \quad \left(\underline{\lim_{n\to\infty}} x_n\right)$$

görnüşde belgilenilýär.

Mysal üçin, 1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6, ... yzygiderlik üçin $\overline{\lim_{n\to\infty}} x_n = 1, \quad \underline{\lim_{n\to\infty}} x_n = 0.$

13-nji mysal. $x_n = (-1)^{n-1}(2 + 3/n)$, $n \in N$ üçin $\inf\{x_n\}$, $\sup\{x_n\}$, $\overline{\lim_{n\to\infty}} x_n$, $\underline{\lim_{n\to\infty}} x_n$ aňlatmalary tapmaly.

Ç.B. Ilki bilen $\{x_n\}$ yzygiderligiň agzalaryndan ýygnanýan

$$x_{2n} = -2 - \frac{3}{2n}, \quad x_{2n-1} = 2 + \frac{3}{2n-1}, \quad n \in N$$

iki bölekden ybarat yzygiderlikleri düzeliň. Şunlukda, $x_{2n} < x_{2n-1}$ we $\{x_{2n-1}\}$ kemelýändir, $\{x_{2n}\}$ bolsa artýandyr. Şonuň esasynda $\{x_n\}$ yzygiderligiň iň kiçi agzasy x_2 we iň uly agzasy x_1 bolar, ýagny

$$\inf\{x_n\} = x_2 = -7/2, \quad \sup\{x_n\} = x_1 = 5.$$

$x_{2n} < x_{2n-1}$ deňsizligiň esasynda

$$\overline{\lim_{n\to\infty}} x_n = \lim_{n\to\infty} x_{2n-1} = 2, \quad \underline{\lim_{n\to\infty}} x_n = \lim_{n\to\infty} x_{2n} = -2. \quad \textbf{Ç.S.}$$

14-nji mysal. $x_n = \dfrac{(3\cos(n\pi/2) - 1)n + 1}{n}$, $n \in N$ üçin bölekleýin yzygiderlikleri, $\overline{\lim_{n\to\infty}} x_n$, $\underline{\lim_{n\to\infty}} x_n$ we $\inf\{x_n\}$, $\sup\{x_n\}$ tapmaly.

Ç.B. $n = 4k$ bolanda

$$x_n = \frac{2n+1}{n} = 2 + \frac{1}{n}$$

bolýandygy üçin $\lim_{k\to\infty} x_{4k} = 2,\ 2 < x_{4k} \le 2 + \frac{1}{4}$ we $x_4 = \frac{9}{4}$.

$n = 4k + 1$ ýa-da $n = 4k + 3$ bolanda

$$x_n = \frac{-n+1}{n} = -1 + \frac{1}{n}$$

bolýandygy üçin $\lim_{k\to\infty} x_{4k+1} = \lim_{k\to\infty} x_{4k+3} = -1$, $-1 < x_n < 0$ bolar.

$n = 4k + 2$ bolanda

$$x_n = \frac{-4n+1}{n} = -4 + \frac{1}{n}$$

bolýandygy üçin $\lim_{k\to\infty} x_{4k+2} = -4$, $-4 < x_n < 0$ bolar. Diýmek, berlen yzygiderligiň bölekleýin predelleri 2, –1, –4 bolýandyr. Seredilen $\{x_{4k}\}$, $\{x_{4k+1}\}$, $\{x_{4k+2}\}$, $\{x_{4k+3}\}$ bölek yzygiderlikleriň hemmesi bilelikde berlen ähli yzygiderligi düzýär. Şonuň üçin hem yzygiderligiň başga bölek yzygiderlikleri ýokdur. Şunlukda,

$$\overline{\lim_{k\to\infty}} x_n = \lim_{k\to\infty} x_{4k} = 2, \quad \underline{\lim_{n\to\infty}} x_n = \lim_{k\to\infty} x_{4k+2} = -4.$$

Şonuň üçin ýokardakylar esasynda

$$\inf\{x_n\} = -4, \quad \sup\{x_n\} = \frac{9}{4}. \quad \textbf{Ç.S.}$$

6. *Yzygiderligiň ýygnanma şertleri.* Eger $\forall \varepsilon > 0$ üçin n_0 nomer tapylyp, $\forall n > n_0$ we $\forall m > n_0$ üçin $|x_n - x_m| < \varepsilon$ deňsizlik ýerine ýetse, onda $\{x_n\}$ yzygiderlige fundamental yzygiderlik diýilýär.

Koşiniň ölçegleri. Yzygiderligiň ýygnanmagy üçin onuň esaslaýyn bolmagy zerur we ýeterlikdir.

Ştolsuň teoremasy. Eger:

1. $\{y_n\}$ artýan tükeniksiz uly yzygiderlik bolsa;
2. $\left\{\frac{x_n - x_{n-1}}{y_n - y_{n-1}}\right\}$ yzygiderligiň a sana deň predeli bar bolsa, onda $\{x_n/y_n\}$ yzygiderligiň hem predeli a sana deňdir.

15-nji mysal. Eger $\lim_{n\to\infty} a_n = a$ predel bar bolsa, onda

$$d_n = \frac{a_1 + a_2 + \ldots + a_n}{n}, \qquad n \in N$$

orta arifmetik $\{d_n\}$ yzygiderligiň hem predeliniň a sana deňdigini subut etmeli, ýagny $\lim_{n\to\infty} d_n = \lim_{n\to\infty} a_n = a$.

Ç.B. Goý, $x_n = a_1 + a_2 + ... + a_n$, $y_n = n$ bolsun, onda

$$\lim_{n\to\infty}\frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty} a_n = a$$

deňligiň esasynda Ştolsuň teoremasyny ulanyp,

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{a_1 + a_2 + ... + a_n}{n} = \lim_{n\to\infty} d_n = \lim_{n\to\infty} a_n = a \text{ alarys.}$$

16-njy mysal. Eger $\lim\limits_{n\to\infty} a_n = a$ predel bar we $a_n > 0$, $n \in N$ bolsa, onda

$$c_n = \frac{n}{1/a_1 + 1/a_2 + ... + 1/a_n}, \quad n \in N$$

orta garmonik $\{c_n\}$ yzygiderligiň hem predeliniň a sana deňdigini subut etmeli, ýagny $\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} a_n = a.$

Ç.B. $\lim\limits_{n\to\infty}\dfrac{1}{a_n} = \dfrac{1}{a}$ deňligiň esasynda 15-nji mysal boýunça

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty}\frac{n}{1/a_1 + 1/a_2 + ... + 1/a_n} =$$
$$= \lim_{n\to\infty}\frac{1}{\dfrac{1/a_1 + 1/a_2 + ... + 1/a_n}{n}} =$$
$$= \frac{1}{\lim\limits_{n\to\infty}\dfrac{1/a_1 + 1/a_2 + ... + 1/a_n}{n}} = \frac{1}{\dfrac{1}{a}} = a. \quad \textbf{Ç.S.}$$

17-nji mysal. Eger $\lim\limits_{n\to\infty} a_n = a$ predel bar we $a_n > 0$, $n \in N$ bolsa, onda

$$b_n = \sqrt[n]{a_1 a_2 ... a_n}, \quad n \in N$$

orta geometrik $\{b_n\}$ yzygiderligiň hem predeliniň a sana deňdigini subut etmeli, ýagny $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} a_n = a.$

Ç.B. Subudy 15-nji we 16-njy mysallaryň esasynda yzygiderligiň 4-nji häsiýeti boýunça 1-nji bölümiň 44-nji mysalynda subut edilen $c_n \le b_n \le d_n$ deňsizliklerden gelip çykýar. **Ç.S.**

18-nji mysal. $\lim\limits_{n\to\infty}\dfrac{\ln n}{n} = 0$ deňligi subut etmeli.

Ç.B. Eger $x_n = \ln n$ we $y_n = n$ bolsa, onda Ştolsuň teoremasynyň hemme şertleri ýerine ýetýär, çünki

$$\left\{\frac{x_{n+1} - x_n}{y_{n+1} - y_n}\right\} = \left\{\frac{\ln(n+1) - \ln n}{n + 1 - n}\right\} = \left\{\ln\left(1 + \frac{1}{n}\right)\right\}$$

yzygiderligiň predeli nola deňdir. Onuň şeýledigi (5) deňsizlikden gelip çykýar. Şonuň üçin hem Ştolsuň teoremasy boýunça

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{\ln n}{n} = 0. \quad \textbf{Ç.S.}$$

Gönükmeler

1. Berlen x_n ($n = 1, 2, ...$) we a san hem-de her bir $\varepsilon > 0$ üçin $n_0 = n_0(\varepsilon)$ nomeri tapyp, $\forall n > n_0$ üçin $|x_n - a| < \varepsilon$ deňsizligi subut edip, $\{x_n\}$ yzygiderligiň predeliniň a sana deňdigini subut etmeli we $\varepsilon = 0{,}1$; $\varepsilon = 0{,}01$ üçin $n_0 = n_0(\varepsilon)$ nomerlerini tapmaly:

a) $x_n = \dfrac{n}{n+1}$, $a = 1$; ç) $x_n = \dfrac{1}{n!}$, $a = 0$; e) $x_n = (-1)^n 0{,}999^n$, $a = 0$.

b) $x_n = \dfrac{(-1)^{n+1}}{n}$, $a = 0$; d) $x_n = \dfrac{2n}{n^3+1}$, $a = 0$.

2. Berlen yzygiderlikler we her bir $K > 0$ üçin $n_0 = n_0(K)$ nomeri tapyp, $\forall n > n_0$ üçin $|x_n| > K$ deňsizligi görkezip, $\{x_n\}$ yzygiderlikleriň predelleriniň tükeniksizlige deňdigini subut etmeli we $K = 10$; $K = 100$ üçin $n_0 = n_0(K)$ nomerleri tapmaly:

a) $x_n = (-1)^n n$; b) $x_n = 2^{\sqrt{n}}$; ç) $x_n = \lg(\lg n)$ ($n \geq 2$).

3. $x_n = n^{(-1)^n}$ ($n = 1, 2, ...$) yzygiderligiň çäksiz, ýöne $n \to \infty$ bolanda tükeniksiz uly däldigini subut etmeli.

4. Aşakdaky tassyklamalary deňsizlikleriň kömegi bilen ýazmaly:

a) $\lim\limits_{n\to\infty} x_n = \infty$; b) $\lim\limits_{n\to\infty} x_n = -\infty$; ç) $\lim\limits_{n\to\infty} x_n = +\infty$.

5. a sanyň $\{x_n\}$ yzygiderligiň predeli bolmaýandygyny $\left(\lim\limits_{n\to\infty} x_n \neq a\right)$ deňsizligiň kömegi bilen ýazmaly.

Natural bahalary alýan n üçin aňlatmalaryň bahalaryny kesgitlemeli:

6. $\lim\limits_{n\to\infty}\dfrac{10000n}{n^2+1}$. **7.** $\lim\limits_{n\to\infty}(\sqrt{n+1} - \sqrt{n})$.

8. $\lim\limits_{n\to\infty}\dfrac{\sqrt[3]{n^2}\sin n!}{n+1}$. **9.** $\lim\limits_{n\to\infty}\dfrac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}}$.

10. $\lim\limits_{n\to\infty}\dfrac{1 + a + a^2 + ... + a^n}{1 + b + b^2 + ... + b^n}$ ($|a| < 1$, $|b| < 1$).

11. $\lim\limits_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n-1}{n^2}\right).$

12. $\lim\limits_{n\to\infty}\left|\frac{1}{n}-\frac{2}{n}+\frac{3}{n}-\ldots+\frac{(-1)^{n-1}n}{n}\right|.$

13. $\lim\limits_{n\to\infty}\left[\frac{1^2}{n^3}+\frac{2^2}{n^3}+\ldots+\frac{(n-1)^2}{n^3}\right].$

14. $\lim\limits_{n\to\infty}\left[\frac{1^2}{n^3}+\frac{3^2}{n^3}+\ldots+\frac{(2n-1)^2}{n^3}\right].$

15. $\lim\limits_{n\to\infty}\left(\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\ldots+\frac{2n-1}{2^n}\right).$

16. $\lim\limits_{n\to\infty}\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}\right].$

17. $\lim\limits_{n\to\infty}\left(\sqrt{2}\,\sqrt[4]{2}\,\sqrt[8]{2}\ldots\sqrt[2^n]{2}\right).$

Deňlikleri subut etmeli:

18. $\lim\limits_{n\to\infty}\frac{n}{2^n}=0.$

19. $\lim\limits_{n\to\infty}\frac{2^n}{n!}=0.$

20. $\lim\limits_{n\to\infty}\frac{n^k}{a^n}=0\ (a>1).$

21. $\lim\limits_{n\to\infty}\frac{a^n}{n!}=0.$

22. $\lim\limits_{n\to\infty}nq^n=0,\quad |q|<1.$

23. $\lim\limits_{n\to\infty}\sqrt[n]{a}=1,\quad (a>0).$

24. $\lim\limits_{n\to\infty}\frac{\log_a n}{n}=0,\quad (a>1).$

25. $\lim\limits_{n\to\infty}\sqrt[n]{n}=1.$

26. $\lim\limits_{n\to\infty}\frac{1}{\sqrt[n]{n!}}=0.$

27. Ýeterlik uly n üçin aňlatmalaryň haýsysy uly:
a) $100n+200$ ýa-da $0{,}01n^2$; b) 2^n ýa-da n^{1000}; ç) 1000^n ýa-da $n!$?

28. $\lim\limits_{n\to\infty}\left(\frac{1}{2}\cdot\frac{3}{4}\ldots\frac{2n-1}{2n}\right)=0$ deňligi subut etmeli. (*Görkezme*: *1.1-nji bölümiň 38-nji mysalyna seret*).

29. $x_n=\left(1+\frac{1}{n}\right)^n$ $(n=1, 2, \ldots)$ yzygiderligiň artýan we ýokardan çäklidigini, $y_n=\left(1+\frac{1}{n}\right)^{n+1}$ $(n=1, 2, \ldots)$ yzygiderligiň kemelýän we aşakdan çäklidigini subut etmeli. Bu ýerden ol yzygiderlikleriň umumy

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=e$$

predeliniň bardygyny subut etmeli. (*Görkezme*: $\frac{x_{n+1}}{x_n}$, $\frac{y_n}{y_{n-1}}$ *gatnaşyklary düzmeli we 9-njy mysaldan peýdalanmaly*).

30. $0 < e - \left(1 + \frac{1}{n}\right)^n < \frac{3}{n}$ $(n=1, 2, ...)$ deňsizligi subut etmeli. n görkezijiniň haýsy bahalarynda $\left(1 + \frac{1}{n}\right)^n$ aňlatmanyň e sandan tapawudy 0,001-den kiçi bolar?

31. $+\infty$-ge ymtylýan p_n $(n=1, 2, ...)$ we $-\infty$-ge ymtylýan q_n $(n=1,2, ...)$ sanlaryň erkin yzygiderligi üçin $(p_n, q_n \notin [-1, 0])$ $\lim\limits_{n\to\infty}\left(1 + \frac{1}{p_n}\right)^{p_n} = \lim\limits_{n\to\infty}\left(1 + \frac{1}{q_n}\right)^{q_n} = e$ deňligi subut etmeli.

32. $\lim\limits_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$ deňligi ulanyp,

$$\lim_{n\to\infty}\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}\right) = e$$

deňligi subut etmeli. Ondan

$$e = 2 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!} + \frac{\theta_n}{n!n}, \quad 0 < \theta_n < 1$$

formulany getirip çykarmaly we e sany 10^{-5} takyklykda hasaplamaly.

33. e sanyň irrasionaldygyny subut etmeli.

34. $\left(\frac{n}{e}\right)^n < n! < e\left(\frac{n}{2}\right)^n$ deňsizligi subut etmeli.

35. Deňsizlikleri subut etmeli:

a) $\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$, n – islendik natural san;

b) $1 + \alpha < e^{\alpha}$, α – noldan tapawutly hakyky san.

36. $\lim\limits_{n\to\infty} n(a^{1/n} - 1) = \ln a$ $(a > 0)$ deňligi subut etmeli, bu ýerde $\ln a$ berlen a sanyň $e = 2{,}718...$ esasly logarifmi.

Monoton we çäkli yzygiderligiň predeli hakyndaky Weýerştrasyň teoremasyndan peýdalanyp, aşakdaky yzygiderlikleriň ýygnanýandygyny subut etmeli:

37. $x_n = p_0 + \frac{p_1}{10} + ... + \frac{p_n}{10^n}$ $(n = 1, 2, ...)$, bu ýerde p_i $(i=1,2,...)$ p_1-den başlap, 9-dan uly bolmadyk otrisatel däl bitin sanlar.

38. $x_n = \frac{10}{1} \cdot \frac{11}{3} ... \frac{n+9}{2n-1}$.

39. $x_n = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)...\left(1 - \frac{1}{2^n}\right)$. **40.** $x_n = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{4}\right)...\left(1 + \frac{1}{2^n}\right)$.

41. $x_1 = \sqrt{2}, x_2 = \sqrt{2+\sqrt{2}}, ..., x_n = \underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{n \text{ kökler}}, ...$

Koşiniň ölçeglerinden peýdalanyp, aşakdaky yzygiderlikleriň ýygnanýandygyny subut etmeli:

42. $x_n = a_0 + a_1 q + ... + a_n q^n$, bu ýerde $|a_k| < M$ $(k = 0, 1, 2, ...)$, $|q| < 1$.

43. $x_n = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + ... + \frac{\sin n}{2^n}$.

44. $x_n = \frac{\cos 1!}{1 \cdot 2} + \frac{\cos 2!}{2 \cdot 3} + ... + \frac{\cos n!}{n(n+1)}$.

45. $x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2}$. (*Görkezme*: $\frac{1}{n^2} < \frac{1}{n-1} - \frac{1}{n}$ $(n = 2, 3, ...)$ *deňsizlikden peýdalanmaly*).

46. Eger şeýle C san tapylyp, $|x_2 - x_1| + |x_3 - x_2| + ... + |x_n - x_{n-1}| < C$ $(n = 2, 3, ...)$ deňsizlik ýerine ýetse, onda x_n $(n = 1, 2, ...)$ yzygiderligiň çäkli üýtgemesi bar diýilýär.

Çäkli üýtgemesi bar bolan yzygiderligiň ýygnanýandygyny subut etmeli.

Ýygnanýan, ýöne çäkli üýtgemesi bolmadyk yzygiderlige mysal getirmeli.

47. Yzygiderlik üçin Koşiniň ölçegleriniň ýerine ýetmeýändigini nämäni aňladýandygyny ýazmaly.

48. Koşiniň ölçeglerinden peýdalanyp, $x_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ yzygiderligiň dargaýandygyny subut etmeli.

49. Eger x_n $(n = 1, 2, ...)$ yzygiderlik ýygnanýan bolsa, onda onuň islendik x_{p_n} bölek yzygiderliginiň hem ýygnanýandygyny we $\lim\limits_{k \to \infty} x_{p_n} = \lim\limits_{n \to \infty} x_n$ deňligi subut etmeli.

50. Käbir bölek yzygiderligi ýygnanýan monoton yzygiderligiň ýygnanýandygyny subut etmeli.

51. Eger $\lim\limits_{n \to \infty} x_n = a$ predel bar bolsa, onda $\lim\limits_{n \to \infty} |x_n| = |a|$ deňligi subut etmeli.

52. Eger $x_n \to a$ bolsa, onda $\lim\limits_{n \to \infty} \frac{x_{n+1}}{x_n}$ predel barada näme diýmek bolar?

53. Ýygnanýan yzygiderligiň çäklidigini subut etmeli.

54. Ýygnanýan yzygiderligiň ýa takyk ýokarky çägini, ýa-da takyk aşaky çägini, ýa-da bolmasa, olaryň ikisini hem alýandygyny subut etmeli. Şeýle yzygiderlikleriň üç görnüşine hem mysal getiriň.

55. $+\infty$-ge ýygnanýan x_n $(n = 1, 2, \ldots)$ san yzygiderligiň hökman özüniň takyk aşaky çägini alýandygyny subut etmeli.

x_n $(n = 1, 2, \ldots)$ yzygiderligiň iň uly agzasyny tapmaly:

56. $x_n = \dfrac{n^2}{2^n}$. **57.** $x_n = \dfrac{\sqrt{n}}{100 + n}$. **58.** $x_n = \dfrac{1000^n}{n!}$.

x_n $(n = 1, 2, \ldots)$ yzygiderligiň iň kiçi agzasyny tapmaly:

59. $x_n = n^2 - 9n - 100$. **60.** $x_n = n + \dfrac{100}{n}$.

x_n $(n = 1, 2, \ldots)$ yzygiderlik üçin $\inf x_n$, $\sup x_n$, $\varliminf\limits_{n\to\infty} x_n$, $\varlimsup\limits_{n\to\infty} x_n$ tapmaly:

61. $x_n = 1 - \dfrac{1}{n}$. **62.** $x_n = (-1)^{n-1}\left(2 + \dfrac{3}{n}\right)$.

63. $x_n = \dfrac{(-1)^n}{n} + \dfrac{1 + (-1)^n}{2}$. **64.** $x_n = 1 + \dfrac{n}{n+1}\cos\dfrac{n\pi}{2}$.

65. $x_n = 1 + 2(-1)^{n+1} + 3\cdot(-1)^{\frac{n(n-1)}{2}}$.

66. $x_n = \dfrac{n-1}{n+1}\cos\dfrac{2n\pi}{3}$. **67.** $x_n = (-1)^n n$.

68. $x_n = -n[2 + (-1)^n]$. **69.** $x_n = n^{(-1)^n}$.

70. $x_n = 1 + n\sin\dfrac{n\pi}{2}$. **71.** $x_n = \dfrac{1}{n - 10{,}2}$.

Aşakdaky yzygiderlikler üçin $\varliminf\limits_{n\to\infty} x_n$, $\varlimsup\limits_{n\to\infty} x_n$ predelleri tapmaly.

72. $x_n = \dfrac{n^2}{1+n}\cos\dfrac{2n\pi}{3}$. **73.** $x_n = \left(1 + \dfrac{1}{n}\right)^n \cdot (-1)^n + \sin\dfrac{n\pi}{4}$.

74. $x_n = \dfrac{n}{n+1}\sin^2\dfrac{n\pi}{4}$. **75.** $x_n = \sqrt[n]{1 + 2^{n\cdot(-1)^n}}$.

76. $x_n = \cos^n\dfrac{2n\pi}{3}$.

Aşakdaky yzygiderlikleriň bölekleýin predellerini tapmaly:

77. $\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{3}{4}, \dfrac{1}{8}, \dfrac{7}{8}, \ldots, \dfrac{1}{2^n}, \dfrac{2^n - 1}{2^n}, \ldots$.

78. $1, \frac{1}{2}, 1+\frac{1}{2}, \frac{1}{3}, 1+\frac{1}{3}, \frac{1}{2}+\frac{1}{3}, \frac{1}{4}, 1+\frac{1}{4}, \frac{1}{2}+\frac{1}{4}, \frac{1}{3}+\frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, 1+\frac{1}{n}, \frac{1}{2}+\frac{1}{n}, \ldots, \frac{1}{n-1}+\frac{1}{n}, \frac{1}{n+1}, \ldots .$

79. $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots .$

80. $x_n = 3\cdot\left(1-\frac{1}{n}\right)+2(-1)^n.$

81. $x_n = \frac{1}{2}[(a+b)+(-1)^n(a-b)].$

82. Bölekleýin predelleri a_1, a_2, ... a_p sanlara deň bolan san yzygiderliginiň mysalyny düzmeli.

83. Berlen a_1, a_2, ..., a_n, ... yzygiderligiň ähli agzalary onuň bölekleýin predelleri bolýan san yzygiderligini düzmeli. Ol yzygiderligiň ýene nähili bölekleýin predelleri bolup biler?

84. Yzygiderlikleri düzmeli:

a) tükenikli bölekleýin predelleri ýok bolan;

b) bir tükenikli predeli bar bolan, ýöne ýygnanmaýan yzygiderligi.

85. x_n we $y_n = x_n\sqrt[n]{n}$ $(n = 1, 2, \ldots)$ yzygiderlikleriň şol bir bölekleýin predelleriniň bardygyny subut etmeli.

86. Çäkli x_n $(n = 1, 2, \ldots)$ yzygiderlikden ýygnanýan bölek x_{n_k} $(k=1,2,\ldots)$ yzygiderligi alyp bolýandygyny subut etmeli.

87. Eger x_n $(n = 1, 2, \ldots)$ yzygiderlik çäkli bolmasa, onda $\lim\limits_{k\to\infty} x_{n_k} = \infty$ bolýan x_{n_k} bölek yzygiderligiň bardygyny subut etmeli.

88. Goý, x_n $(n = 1, 2, \ldots)$ ýygnanýan, y_n $(n = 1, 2, \ldots)$ bolsa dargaýan yzygiderlik bolsun. Onda a) $x_n + y_n$; b) $x_n y_n$ yzygiderlikleriň ýygnanýandygy barada näme aýtmak bolar?

89. Goý, x_n we y_n $(n = 1, 2, \ldots)$ yzygiderlikler dargaýan bolsun. a) $x_n + y_n$; b) $x_n y_n$ yzygiderliklere dargaýan yzygiderlik diýip bolarmy?

90. Goý, $\lim\limits_{n\to\infty} x_n = 0$ we y_n $(n = 1, 2, \ldots)$ erkin yzygiderlik bolsun. Onda $\lim\limits_{n\to\infty} x_n y_n = 0$ diýip tassyklamak bolarmy? Degişli mysallary getirmeli.

91. Eger $\lim\limits_{n\to\infty} x_n y_n = 0$ bolsa, onda bu ýerden ýa $\lim\limits_{n\to\infty} x_n = 0$, ýa-da $\lim\limits_{n\to\infty} y_n = 0$ gelip çykýar diýip bolarmy? $x_n = \frac{1+(-1)^n}{2}$, $y_n = \frac{1-(-1)^n}{2}$ $(n = 1, 2, \ldots)$ mysallara serediň.

92. Subut etmeli:

a) $\varliminf\limits_{n\to\infty} x_n + \varliminf\limits_{n\to\infty} y_n \le \varliminf\limits_{n\to\infty}(x_n + y_n) \le \varliminf\limits_{n\to\infty} x_n + \varlimsup\limits_{n\to\infty} y_n$;

b) $\varliminf\limits_{n\to\infty} x_n + \varlimsup\limits_{n\to\infty} y_n \le \varlimsup\limits_{n\to\infty}(x_n + y_n) \le \varlimsup\limits_{n\to\infty} x_n + \varlimsup\limits_{n\to\infty} y_n$.

Bu ýerde berk deňsizlikleriň ýerine ýetýän mysallaryny getiriň.

93. Goý, $x_n \ge 0$ we $y_n \ge 0$ $(n = 1, 2, ...)$ bolsun. Subut etmeli:

a) $\varliminf\limits_{n\to\infty} x_n \cdot \varliminf\limits_{n\to\infty} y_n \le \varliminf\limits_{n\to\infty}(x_n y_n) \le \varliminf\limits_{n\to\infty} x_n \cdot \varlimsup\limits_{n\to\infty} y_n$;

b) $\varliminf\limits_{n\to\infty} x_n \cdot \varlimsup\limits_{n\to\infty} y_n \le \varlimsup\limits_{n\to\infty}(x_n y_n) \le \varlimsup\limits_{n\to\infty} x_n \cdot \varlimsup\limits_{n\to\infty} y_n$.

Bu ýerde berk deňsizlikleriň ýerine ýetýän mysallaryny getiriň.

94. Eger $\lim\limits_{n\to\infty} x_n$ predel bar bolsa, onda islendik y_n $(n = 1, 2, ...)$ yzygiderlik üçin deňlikleri subut etmeli:

a) $\varlimsup\limits_{n\to\infty}(x_n + y_n) = \lim\limits_{n\to\infty} x_n + \varlimsup\limits_{n\to\infty} y_n$; b) $\varlimsup\limits_{n\to\infty}(x_n y_n) = \lim\limits_{n\to\infty} x_n \cdot \varlimsup\limits_{n\to\infty} y_n$ $(x_n \ge 0)$.

95. Eger käbir x_n $(n = 1, 2, ...)$ yzygiderlik üçin y_n $(n = 1, 2, ...)$ yzygiderligiň nähili boljakdygyna garamazdan:

a) $\varlimsup\limits_{n\to\infty}(x_n + y_n) = \varlimsup\limits_{n\to\infty} x_n + \varlimsup\limits_{n\to\infty} y_n$ ýa-da

b) $\varlimsup\limits_{n\to\infty}(x_n y_n) = \varlimsup\limits_{n\to\infty} x_n \cdot \varlimsup\limits_{n\to\infty} y_n$ $(x_n \ge 0)$

deňlikleriň haýsy-da bolsa biri ýerine ýetýän bolsa, onda x_n $(n=1,2,...)$ yzygiderligiň ýygnanýandygyny subut etmeli.

96. Eger $x_n > 0$ $(n=1,2,...)$ we $\varlimsup\limits_{n\to\infty} x_n \cdot \varlimsup\limits_{n\to\infty} \frac{1}{x_n} = 1$ bolsa, onda x_n $(n=1,2,...)$ yzygiderligiň ýygnanýandygyny subut etmeli.

97. Eger x_n $(n=1,2,...)$ yzygiderlik çäkli we $\lim\limits_{n\to\infty}(x_{n+1} - x_n) = 0$ bolsa, onda ol yzygiderligiň bölekleýin predelleri onuň aşaky $l = \varliminf\limits_{n\to\infty} x_n$ we ýokarky $L = \varlimsup\limits_{n\to\infty} x_n$ predelleriniň arasynda dykyz ýerleşýändigini, ýagny $[l, L]$ kesimiň islendik sanynyň berlen yzygiderligiň bölekleýin predeli bolýandygyny subut etmeli.

98. $x_1, x_2, ..., x_n, ...$ yzygiderlik üçin

$$0 \le x_{m+n} \le x_m + x_n \quad (m, n = 1, 2, ...)$$

şert ýerine ýetende $\lim\limits_{n\to\infty} \frac{x_n}{n}$ predeliň bardygyny subut etmeli.

99. Eger x_n $(n=1,2,...)$ yzygiderlik ýygnanýan bolsa, onda $\xi_n = \frac{1}{n}(x_1 + x_2 + ... + x_n)$ $(n=1,2,...)$ orta arifmetik yzygiderliginiň hem ýygnanýandygyny we

$$\lim_{n\to\infty}\frac{x_1 + x_2 + ... + x_n}{n} = \lim_{n\to\infty} x_n$$

deňligi subut etmeli. Ters tassyklama dogry däldir, oňa degişli mysal getiriň.

100. Eger $\lim_{n\to\infty} x_n = +\infty$ bolsa, onda

$$\lim_{n\to\infty}\frac{x_1 + x_2 + ... + x_n}{n} = +\infty$$

bolýandygyny subut etmeli.

101. Eger x_n $(n = 1, 2, ...)$ yzygiderlik ýygnanýan we $x_n > 0$ bolsa, onda

$$\lim_{n\to\infty}\sqrt[n]{x_1 x_2 ... x_n} = \lim_{n\to\infty} x_n$$

bolýandygyny subut etmeli.

102. Eger $x_n > 0$ $(n = 1, 2, ...)$ we $\lim_{n\to\infty}\frac{x_{n+1}}{x_n}$ predel bar bolsa, onda

$$\lim_{n\to\infty}\sqrt[n]{x_n} = \lim_{n\to\infty}\frac{x_{n+1}}{x_n}$$

bolýandygyny subut etmeli.

103. $\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} = e$ deňligi subut etmeli.

104. Ştolsuň teoremasyny subut etmeli. Eger:

1. $\{y_n\}$ artýan tükeniksiz uly yzygiderlik bolsa;
2. $\left\{\frac{x_n - x_{n-1}}{y_n - y_{n-1}}\right\}$ yzygiderligiň a sana deň predeli bar bolsa.

Onda $\{x_n / y_n\}$ yzygiderligiň hem predeli a sana deňdir.

105. Predelleri tapmaly:

a) $\lim_{n\to+\infty}\frac{n^2}{a^n}$ $(a > 1)$; b) $\lim_{n\to+\infty}\frac{\lg n}{n}$.

106. Natural p san üçin deňlikleri subut etmeli:

a) $\lim_{n\to\infty}\frac{1^p + 2^p + ... + n^p}{n^{p+1}} = \frac{1}{p+1}$;

b) $\lim_{n\to\infty}\left(\frac{1^p + 2^p + ... + n^p}{n^p} - \frac{n}{p+1}\right) = \frac{1}{2}$;

ç) $\lim\limits_{n\to\infty}\dfrac{1^p+3^p+...+(2n-1)^p}{n^{p+1}}=\dfrac{2^p}{p+1}$.

107. $x_n=1+\dfrac{1}{2}+\dfrac{1}{3}+...+\dfrac{1}{n}-\ln n$ $(n = 1, 2, ...)$ yzygiderligiň ýygnanýandygyny subut etmeli.

Şeýlelikde,

$$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}=C+\ln n+\varepsilon_n,$$

bu ýerde $C = 0{,}577216...$ – bu san Eýleriň sanydyr we $n\to\infty$ bolanda $\varepsilon_n\to 0$.

108. $\lim\limits_{n\to\infty}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+...+\dfrac{1}{2n}\right)$ predeli tapmaly.

109. x_n $(n = 1, 2, ...)$ yzygiderlik şeýle kesgitlenýär:

$$x_1=a,\quad x_2=b,\quad x_n=\frac{x_{n-1}+x_{n-2}}{2}\quad (n=3, 4, ...)$$

$\lim\limits_{n\to\infty}x_n$ predeli tapmaly.

110. x_n $(n = 1, 2, ...)$ yzygiderlik şeýle kesgitlenýär:

$$x_0>0,\quad x_{n+1}=\frac{1}{2}\left(x_n+\frac{1}{x_n}\right)\quad (n=0, 1, 2, ...)$$

$\lim\limits_{n\to\infty}x_n=1$ deňligi subut etmeli.

111. x_n we y_n $(n = 0, 1, 2, ...)$ yzygiderlikler şeýle kesgitlenýär:

$$x_1=a,\quad y_1=b,\quad x_{n+1}=\sqrt{x_ny_n},\quad y_{n+1}=\frac{x_n+y_n}{2}.$$

Olaryň umumy $\mu(a,b)=\lim\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}y_n$ predeliniň bardygyny subut etmeli. ($\mu(a, b)$ sana a we b sanlaryň orta arifmetik-geometrik sany diýilýär).

112. $\lim\limits_{n\to\infty}\dfrac{1\cdot 1!+2\cdot 2!+...+n\cdot n!}{(n+1)!}=1$ deňligi subut etmeli.

Natural m san üçin deňlikleri subut etmeli:

113. $\lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=1}^{n}k(k+1)...(k+m-1)}{n(n+1)...(n+m)}=\dfrac{1}{m+1}$.

114. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{1}{k(k+1)...(k+m+1)}=\dfrac{1}{m\cdot m!}$.

III. FUNKSIÝANYŇ PREDELI

§1. Funksiýa we onuň grafigi

1. ***Funksiýanyň kesgitlenişi.*** X köplügiň her bir x elementine Y köplügiň kesgitli y elementini degişli edýän f düzgüne X köplükde kesgitlenen, bahasy Y köplükde bolan funksiýa (ýa-da öwürme) diýilýär.

Funksiýany (öwürmäni) aşakdaky ýaly belgilemek kabul edilen:

$$f : X \to Y, \quad X \xrightarrow{f} Y.$$

Funksiýa ululygy üçin $x \to f(x)$ ýa-da $y = f(x)$ ýazgy hem ulanylýar.

$y = f(x)$ ýazgydaky her bir x ululyga f funksiýanyň argumenti ýa-da üýtgeýäni, $f(x)$ ululyga bolsa f funksiýanyň x ululyga degişli bahasy diýilýär. Şonda onuň kesgitlenen X köplügine kesgitleniş ýaýlasy, bahalar köplügine bolsa bahalar ýaýlasy diýilýär.

Eger f funksiýa X köplügiň her bir elementine Y köplügiň diňe bir elementini degişli edýän bolsa, onda oňa birbahaly funksiýa, eger-de birden köp agzasyny degişli edýän bolsa – köpbahaly funksiýa diýilýär.

Eger f funksiýa X köplügiň Y köplüge öwürmesi, F bolsa Y köplügiň Z köplüge öwürmesi bolsa, onda $z = F[f(x)]$ funksiýa x-a görä çylşyrymly funksiýa (funksiýanyň funksiýasy) diýilýär we $F \circ f$ ýazgyda belgilenýär. Ol X köplügiň Z köplüge öwürmesidir.

2. ***Funksiýanyň grafigi.*** Tertipleşdirilen ähli (x, y) jübütleriň $x \in X$ we $y = f(x) \in Y$ şertleri kanagatlandyrýan köplügine $y = f(x)$ funksiýanyň grafigi diýilýär. Başgaça aýdylanda, ol tekizlikdäki koordinatalary $y = f(x)$ baglanyşykda bolýan ähli (x, y) nokatlaryň köplügidir.

$y = f(x)$ funksiýanyň grafiginden peýdalanyp, funksiýanyň ýa-da argumentiň özgerdilmegi esasynda alynýan

$$y = g(x) \quad (g(x) = mf(ax + b) + k)$$

görnüşdäki funksiýanyň grafigini gurmaklygy aşakdaky 1-nji tablisany ulanyp, şeýle tertipde ýerine ýetirmeli:

$$f(x) \to f(ax) \to f\left[a\left(x + \frac{b}{a}\right)\right] = f(ax + b) \to$$
$$\to mf(ax + b) \to mf(ax + b) + k = g(x).$$

Funksiýanyň grafiginiň geometrik özgertmeleri

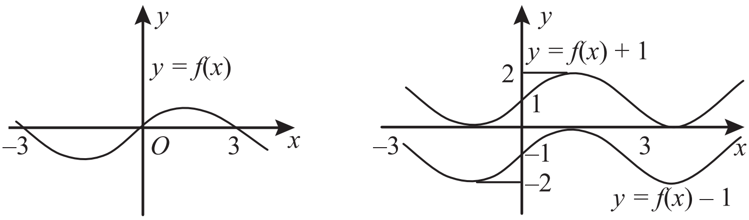
$y = g(x)$ funksiýanyň grafigi	$y = f(x)$ funksiýanyň grafigi bilen geçirilmeli özgertmeler
$g(x) = f(x) + k$	*Oy* oky boýunça $k > 0$ bolanda grafigi k birlik ýokaryk, $k < 0$ bolanda grafigi $\|k\|$ birlik aşak süýşürmeli (*1-nji surat*).
$g(x) = f(x - a)$ $(a \neq 0)$	*Ox* oky boýunça $a > 0$ bolanda grafigi a birlik saga, $a < 0$ bolanda grafigi $\|a\|$ birlik çepe süýşürmeli (*2-nji surat*).
$g(x) = mf(x)$ $(m > 0, m \neq 1)$	*Oy* okunyň ugruna *Ox* okuna görä $m > 1$ bolanda grafigi m esse süýndürmeli, $0 < m < 1$ bolanda grafigi m esse gysmaly (*3-nji surat*).
$g(x) = f(ax)$ $(a > 0, a \neq 1)$	*Ox* okunyň ugruna $a > 1$ bolanda *Oy* okuna görä grafigi a esse gysmaly, $0 < a < 1$ bolanda grafigi $1/a$ esse süýndürmeli (*4-nji surat*).
$g(x) = -f(x)$	*Ox* okuna görä grafigi simmetrik öwürmeli (*5-nji surat*).
$g(x) = f(-x)$	*Oy* okuna görä grafigi simmetrik öwürmeli (*6-njy surat*).
$g(x) = \|f(x)\|$	Grafigiň *Ox* okundan ýokarda ýerleşen bölegi öňküligine galdyrylyp, aşakda ýerleşen bölegini şol oka görä simmetrik öwürmeli (*7-nji surat*).
$g(x) = -f(\|x\|)$	Grafigiň *Oy* okundan sagda ýerleşen bölegi öňküligine galdyrylyp, şol bölegi *Oy* okuna görä simmetrik öwürmeli (*8-nji surat*).

Bellik. Funksiýanyň grafigi gurlanda ýalňyşlyk goýbermezlik üçin *Ox* oky boýunça süýşürilmeli ululyk ax argumente goşulýan san bilen kesgitlenmän, x argumente goşulýan san bilen kesgitlenýändigini ýatladýarys. Mysal üçin, $y = \log_3(1-2x)$ funksiýanyň grafigini gurmak $y = \log_3 x$ funksiýanyň grafiginden peýdalanyp, şeýle shema boýunça ýerine ýetirilýär:

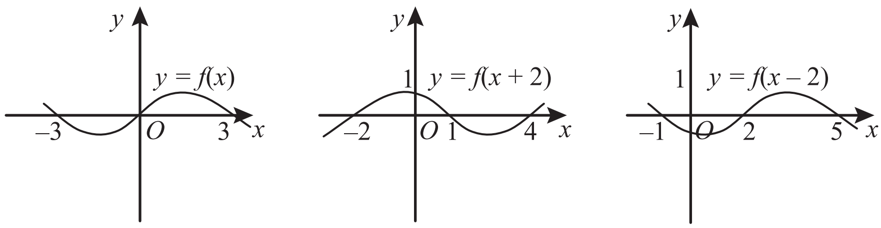
$$\log_3 x \to \log_3(2x) \to \log_3(-2x) \to \log_3\left[-2\left(x - \frac{1}{2}\right)\right] \equiv \log_3(1 - 2x),$$

ýagny berlen funksiýanyň grafigini gurmaklygy $y = \log_3 x$ funksiýanyň grafigini gurmakdan başlamaly, soňra grafigi *Ox* okunyň ugruna *Oy* okuna görä iki esse gysmaly, soňra ol grafigi *Oy* okuna görä simmetrik öwürmeli we iň soňunda alnan grafigi *Ox* oky boýunça 1/2 birlik saga süýşürmeli.

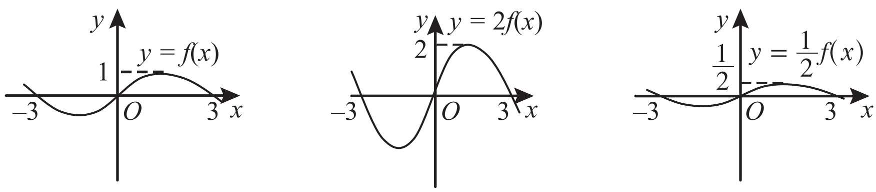
Mysal. $y = \dfrac{ax + b}{cx + d}$ drob çyzykly funksiýanyň grafigini gurmaly.



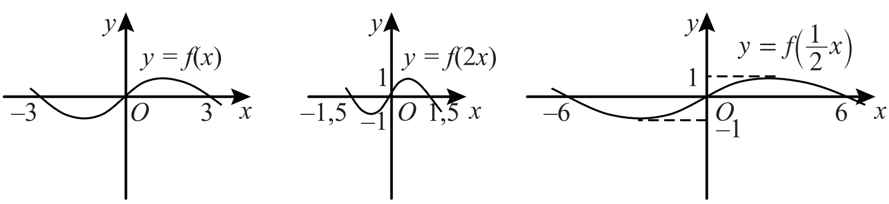
1-nji surat



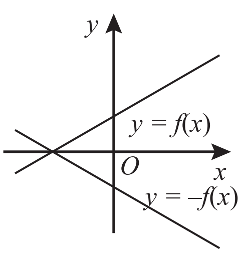
2-nji surat



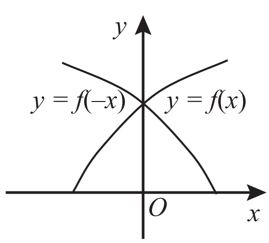
3-nji surat



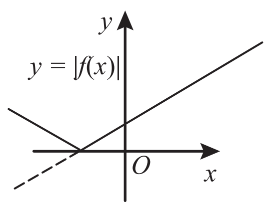
4-nji surat



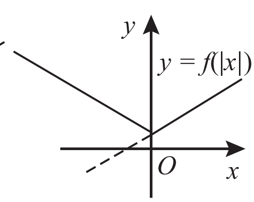
5-nji surat



6-njy surat



7-nji surat



8-nji surat

Ç.B. Drob gysgalmaýan halynda (ýagny $bc \neq ad$ bolanda) özgertme geçirmek bilen ony

$$\frac{ax+b}{cx+d} = \frac{a\left(x+\frac{b}{a}\right)}{c\left(x+\frac{d}{c}\right)} = \frac{a\left(\left(x+\frac{d}{c}\right)+\left(\frac{b}{a}-\frac{d}{c}\right)\right)}{c\left(x+\frac{d}{c}\right)} =$$

$$= \frac{a}{c} + \frac{\frac{bc-ad}{c^2}}{x+\frac{d}{c}} = \frac{a}{c} + \frac{k}{x+\frac{d}{c}}, \quad k = \frac{bc-ad}{c^2}$$

görnüşde ýazmak bolar. Şonuň esasynda berlen funksiýanyň grafigi $y = \frac{k}{x}$ funksiýanyň (giperbolanyň) grafiginden $\frac{d}{c}$ sanyň alamatyna baglylykda Ox oky boýunça $|d/c|$ birlik saga ýa-da çepe süýşürilip, a/c sanyň alamatyna baglylykda Oy oky boýunça birlik ýokaryk, ýa-da aşak süýşürilip alynýar. Şonuň üçin berlen drob çyzykly funksiýanyň grafigini gurmak üçin onuň asimptotalaryny bilmek we şolara görä giperbolanyň şahalarynyň biriniň ýerleşişini bilmek ýeterlikdir, çünki onuň ikinji şahasy asimptotalaryň kesişme nokadyna görä birinji şahasyna simmetrikdir. Grafigiň asimptotalary $y = k/x$ funksiýanyň grafiginiň degişli asimptotalaryndan süýşürilip alynýan $x = -d/c$ we $y = a/c$ göni çyzyklardyr, şahalarynyň biriniň ýerleşişi bolsa giperbolanyň Ox ýa-da Oy oky bilen kesişme nokady boýunça kesgitlenýär.
Ç.S.

3. *Ters we monoton funksiýalar*. Goý, $f: X \to Y$ we her bir $Y_1 = f(X) \ni y$ ululyga $y = f(x)$ bolýan $X \ni x$ ululyk degişli bolsun. Onda Y_1 köplükde, umuman aýdylanda, köpbahaly $x = \varphi(y)$ funksiýa kesgitlenendir. Oňa $y = f(x)$ funksiýanyň ters funksiýasy diýilýär.

Eger $\forall x_1, x_2 \in X$ üçin $x_1 < x_2$ bolanda

$$f(x_1) < f(x_2) \quad (f(x_1) > f(x_2))$$

deňsizlik ýerine ýetse, onda f funksiýa X köplükde artýan (kemelýän) funksiýa diýilýär, eger-de

$$f(x_1) \leq f(x_2) \quad (f(x_1) \geq f(x_2))$$

deňsizlik ýerine ýetse, onda f funksiýa X köplükde kemelmeýän (artmaýan) funksiýa diýilýär.

Eger f funksiýa X köplükde artýan (kemelýän) bolsa, onda onuň $Y = f(X)$ köplükde kesgitlenen birbahaly artýan (kemelýän) ters funksiýasy bardyr. Ol, köplenç, f^{-1} bilen belgilenýär.

Mysal üçin, $[0, 2]$ kesimde $y = x^2$ funksiýa artýar we onuň bahalar köplügi $[0, 4]$ kesim bolýar. Şonuň üçin şol kesimde berlen funksiýa ters bolan ýeke-täk $x = f^{-1}(y) = \sqrt{y}$ funksiýa kesgitlenendir.

4. *Giperbolik we beýleki funksiýalar.* Aşakdaky deňlikler arkaly kesgitlenýän

$$\operatorname{sh}x = \frac{e^{x} - e^{-x}}{2}, \quad \operatorname{ch}x = \frac{e^{x} + e^{-x}}{2}$$

funksiýalara degişlilikde giperbolik sinus we giperbolik kosinus funksiýalar diýilýär. Olar arkaly kesgitlenýän

$$\operatorname{th}x = \frac{\operatorname{sh}x}{\operatorname{ch}x}, \quad \operatorname{cth}x = \frac{\operatorname{ch}x}{\operatorname{sh}x}$$

funksiýalara degişlilikde giperbolik tangens we giperbolik kotangens funksiýalar diýilýär. Bu funksiýalar üçin şeýle formulalar dogrudyr:

$$\operatorname{ch}^2x - \operatorname{sh}^2x = 1;$$

$$\operatorname{ch}(x + y) = \operatorname{ch}x \operatorname{ch}y + \operatorname{sh}x \operatorname{sh}y;$$

$$\operatorname{sh}(x + y) = \operatorname{sh}x \operatorname{ch}y + \operatorname{ch}x \operatorname{sh}y;$$

$$\operatorname{ch}2x = \operatorname{ch}^2x + \operatorname{sh}^2x, \quad \operatorname{sh}2x = 2\operatorname{sh}x \operatorname{ch}y.$$

Eger x we y üýtgeýän ululyklaryň arasyndaky baglylyk

$$F(x, y) = 0$$

deňlik arkaly berlen bolsa, onda funksiýa anyk däl görnüşde berlen funksiýalar diýilýär. Käbir ýagdaýlarda üýtgeýän x we y ululyklaryň arasyndaky baglylyk goşmaça t parametr arkaly, ýagny

$$x = \varphi(t), \quad y = \psi(t) \quad (t_1 \le t \le t_2)$$

deňlikleriň kömegi bilen berilýär. Bu halda funksiýa parametrik görnüşde berlen funksiýa diýilýär. Häzirki wagtda kompýuterlerde dürli hasaplamalary geçirmek üçin üýtgeýän ululyklaryň arasyndaky funksional baglylyklar programmalar arkaly hem berilýär.

Gönükmeler

Funksiýalaryň kesgitleniş ýaýlalaryny tapmaly:

1. $y = \dfrac{x^2}{1 + x}$. **2.** $y = \sqrt{3x - x^3}$. **3.** $y = (x - 2)\sqrt{\dfrac{1 + x}{1 - x}}$.

4. a) $y = \log(x^2 - 4)$; b) $y = \log(x + 2) + \log(x - 2)$.

5. $y = \sqrt{\sin(\sqrt{x})}$. **6.** $y = \sqrt{\cos x^2}$.

7. $y = \lg\left(\sin\dfrac{\pi}{x}\right)$. **8.** $y = \dfrac{\sqrt{x}}{\sin \pi x}$.

9. $y = \arcsin\dfrac{2x}{1 + x}$. **10.** $y = \arccos(2\sin x)$.

11. $y = \lg[\cos(\lg x)]$.

12. $y = (x + |x|)\sqrt{x\sin^2 \pi x}$.

13. $y = \mathrm{ctg}\pi x + \arccos(2^x)$.

14. $y = \arcsin(1 - x) + \lg(\lg x)$.

15. $y = (2x)!$

16. $y = \log_2\log_3\log_4 x$.

17. $y = \sqrt[4]{\lg \mathrm{tg} x}$.

18. $y = \sqrt{\sin 2x} + \sqrt{\sin 3x}$ $(0 \le x \le 2\pi)$.

Funksiýalaryň kesgitleniş we bahalar ýaýlalaryny tapmaly:

19. $y = \sqrt{2 + x - x^2}$.

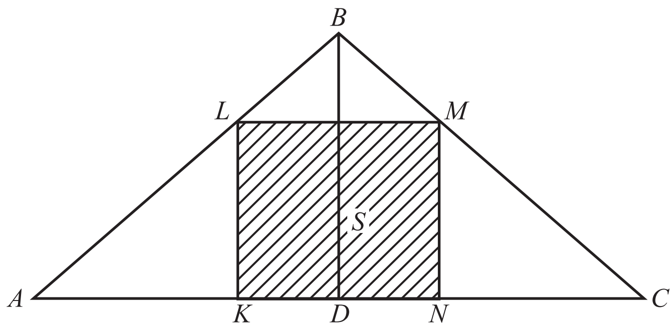
20. $y = \lg(1 - 2\cos x)$.

21. $y = \arccos\dfrac{2x}{1 + x^2}$.

22. $y = \arcsin\left(\lg\dfrac{x}{10}\right)$.

23. $y = (-1)^x$.

24. Esasy $AC = b$ we beýikligi $BD = h$ bolan ABC üçburçlugyň içinden beýikligi $NM = x$ bolan $KLMN$ gönüburçluk çyzylan (*9-njy surat*). $KLMN$ gönüburçlugyň P perimetrini we onuň S meýdanyny x-iň funksiýasy hökmünde aňlatmaly.

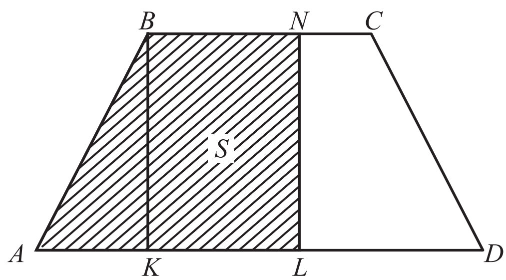


9-njy surat

$P = P(x)$ we $S = S(x)$ funksiýalaryň grafiklerini gurmaly.

25. ABC üçburçlugyň $AB = 6$ *sm*, $AC = 8$ *sm* taraplary we $BAC = x$ burçy berlen. $BC = a$ tarapy we ABC üçburçlugyň S meýdanyny x-iň funksiýasy hökmünde aňlatmaly. $a = a(x)$ we $S = S(x)$ funksiýalaryň grafiklerini gurmaly.

26. Esaslary $AD = a$ we $BC = b$ $(a > b)$, beýikligi $BK = h$ bolan deňýanly $ABCD$ trapesiýada A depeden $AL = x$ uzaklykda $LN \| KB$ göni çyzyk geçirilen (*10-njy surat*). $ABNLA$ figuranyň S meýdanyny x-iň funksiýasy hökmünde aňlatmaly. $S = S(x)$ funksiýanyň grafigini gurmaly.



10-njy surat

27. Ox okunyň $0 \le x \le 1$ kesiminde 2 *g* massa deňölçegli paýlanan, ol okuň $x = 2$ we $x = 3$ nokatlarynda bolsa her biri 1 *g* bolan massalar jemlenen. San bahasy $(-\infty, x)$ interwaldaky massa deň bolan $m = m(x)$ $(-\infty < x < +\infty)$ funksiýanyň analitiki aňlatmasyny düzmeli we ol funksiýanyň grafigini gurmaly.

28. $y = \operatorname{sgn} x$ funksiýa şeýle kesgitlenýär:

$$\operatorname{sgn} x = \begin{cases} -1, \text{ eger } x < 0; \\ 0, \quad \text{eger } x = 0; \\ 1, \quad \text{eger } x > 0 \quad \text{bolsa.} \end{cases}$$

Şu funksiýanyň grafigini gurmaly.

$$|x| = x\operatorname{sgn}x$$

deňligi subut etmeli.

29. $y = [x]$ (x-iň bitin bölegi) funksiýa şeýle kesgitlenýär: eger bitin n san üçin $x = n + r$ we $0 \le r < 1$ bolsa, onda $[x] = n$.

Şu funksiýanyň grafigini gurmaly.

30. Goý, $y = \pi(x)$ $(x \ge 0)$ funksiýa x-dan uly bolmadyk ýönekeý sanlaryň mukdary bolsun. Argumentiň $0 \le x \le 20$ bahalary üçin ol funksiýanyň grafigini gurmaly.

$y = f(x)$ funksiýanyň E_x köplügi haýsy E_y köplüge öwürýändigini anyklamaly.

31. $y = x^2$, $E_x = \{-1 \le x \le 2\}$.

32. $y = \lg x$, $E_x = \{10 < x < 1000\}$.

33. $y = \dfrac{1}{\pi}\operatorname{arctg}x$, $E_x = \{-\infty < x < +\infty\}$.

34. $y = \operatorname{ctg}\dfrac{\pi x}{4}$, $E_x = \{0 < |x| \le 1\}$.

35. $y = |x|$, $E_x = \{1 \le |x| \le 2\}$.

Üýtgeýän x ululyk $0 < x < 1$ interwalda üýtgände berlen üýtgeýän y ululyk haýsy köplükde üýtgeýändigini anyklamaly.

36. $y = a + (b - a)x$.

37. $y = \dfrac{1}{1 - x}$.

38. $y = \dfrac{x}{2x - 1}$.

39. $y = \sqrt{x - x^2}$.

40. $y = \operatorname{ctg}\pi x$.

41. $y = x + [2x]$.

42. Berlen $f(x) = x^4 - 6x^3 + 11x^2 - 6x$ funksiýanyň $f(0), f(1), f(2), f(3), f(4)$ bahalaryny tapmaly.

43. Berlen $f(x) = \lg(x^2)$ funksiýanyň $f(-1), f(-0{,}001), f(100)$ bahalaryny tapmaly.

44. Berlen $f(x) = 1 + [x]$ funksiýanyň $f(0{,}9), f(0{,}99), f(0{,}999), f(1)$ bahalaryny tapmaly.

45. Berlen $f(x) = \begin{cases} 1 + x, & \text{eger } -\infty < x \le 0, \\ 2^x, & \text{eger } 0 < x < +\infty \end{cases}$ funksiýanyň $f(-2), f(-1), f(0), f(1), f(2)$ bahalaryny tapmaly.

46. Berlen $f(x)=\dfrac{1-x}{1+x}$ funksiýa boýunça aşakdaky funksiýanyň $f(0), f(-x)$, $f(x+1), f(x)+1,\ f\left(\dfrac{1}{x}\right),\ \dfrac{1}{f(x)}$ bahalaryny tapmaly.

47. Berlen funksiýalar boýunça x-iň 1) $f(x)=0$; 2) $f(x)>0$; 3) $f(x)<0$ bolýan bahalaryny tapmaly:

a) $f(x)=x-x^3$; b) $f(x)=\sin\dfrac{\pi}{x}$; ç) $f(x)=(x+|x|)(1-x)$.

48. Berlen a) $f(x)=ax+b$; b) $f(x)=x^2$; ç) $f(x)=a^x$ funksiýalar boýunça $\varphi(x)=\dfrac{f(x+h)-f(x)}{h}$ funksiýany tapmaly.

49. $f(x)=ax^2+bx+c$ funksiýa üçin

$$f(x+3)-3f(x+2)+3f(x+1)-f(x)\equiv 0$$

deňligi subut etmeli.

50. Berlen $f(0)=-2$ we $f(3)=5$ bahalar boýunça bitin çyzykly $f(x)=ax+b$ funksiýany tapmaly. $f(1)$ we $f(2)$ bahalar (çyzykly interpolýasiýa) näçä deň?

51. Berlen $f(-2)=0$, $f(0)=1$, $f(1)=5$ bahalar boýunça ikinji derejeli bitin rasional $f(x)=ax^2+bx+c$ funksiýany tapmaly.

$f(-1)$ we $f(0{,}5)$ bahalar (kwadrat interpolýasiýa) näçä deň?

52. Berlen $f(-1)=0$, $f(0)=2$, $f(1)=-3$, $f(2)=5$ bahalar boýunça üçünji derejeli bitin rasional $f(x)=ax^3+bx^2+cx+d$ funksiýany tapmaly.

53. Berlen $f(0)=15$, $f(2)=30$, $f(4)=90$ bahalar boýunça $f(x)=a+bc^x$ funksiýany tapmaly.

54. Eger çyzykly $f(x)=ax+b$ funksiýanyň $x=x_n$ $(n=1, 2, ...)$ argumentiniň bahalary arifmetik progressiýany emele getirýän bolsa, onda funksiýanyň degişli $y_n=f(x_n)$ $(n=1, 2, ...)$ bahalarynyň hem arifmetik progressiýany emele getirýändigini subut etmeli.

55. Eger görkezijili $f(x)=a^x$ $(a>0)$ funksiýanyň $x=x_n$ $(n=1, 2, ...)$ argumentiniň bahalary arifmetik progressiýany emele getirýän bolsa, onda funksiýanyň degişli $y_n=f(x_n)$ $(n=1, 2, ...)$ bahalarynyň geometrik progressiýany emele getirýändigini subut etmeli.

56. Goý, $f(u)$ funksiýa $0<u<1$ üçin kesgitlenen bolsun. Funksiýalaryň kesgitleniş ýaýlalaryny tapmaly:

a) $f(\sin x)$; b) $f(\ln x)$; ç) $f([x]/x)$.

57. $f(x) = \frac{1}{2}(a^x + a^{-x})$ $(a > 0)$ funksiýa üçin

$$f(x+y) + f(x-y) = 2f(x)f(y)$$

deňligi subut etmeli.

58. Goý, $f(x) + f(y) = f(z)$ bolsun. Berlen funksiýalar üçin z-i tapmaly:

a) $f(x) = ax$;

b) $f(x) = \frac{1}{x}$;

ç) $f(x) = \operatorname{arctg} x$ $(|x| < 1)$;

d) $f(x) = \log\frac{1+x}{1-x}$.

Berlen funksiýalar boýunça çylşyrymly

$$\varphi[\varphi(x)], \quad \psi[\psi(x)], \quad \varphi[\psi(x)] \quad \text{we} \quad \psi[\varphi(x)]$$

funksiýalary tapmaly:

59. $\varphi(x) = x^2$ we $\psi(x) = 2^x$.

60. $\varphi(x) = \operatorname{sgn} x$ we $\psi(x) = \frac{1}{x}$.

61. $\varphi(x) = \begin{cases} 0, & \text{eger } x \le 0, \\ x, & \text{eger } x > 0 \end{cases}$ we $\psi(x) = \begin{cases} 0, & \text{eger } x \le 0, \\ -x^2, & \text{eger } x > 0. \end{cases}$

62. Berlen $f(x) = \frac{1}{1-x}$ funksiýa boýunça

$$f[f(x)], \quad f\{f[f(x)]\}$$

funksiýalary tapmaly.

63. Goý, $f_n(x) = \underbrace{f(f(...f(x)))}_{n \text{ gezek}}$ bolsun. Berlen $f(x) = \frac{x}{\sqrt{1+x^2}}$ üçin $f_n(x)$ funksiýany tapmaly.

64. Berlen $f(x+1) = x^2 - 3x + 2$ boýunça $f(x)$-i tapmaly.

65. Berlen $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$ $(|x| \ge 2)$ boýunça $f(x)$-i tapmaly.

66. Berlen $f\left(\frac{1}{x}\right) = x + \sqrt{1+x^2}$ $(x > 0)$ boýunça $f(x)$-i tapmaly.

67. Berlen $f\left(\frac{x}{x+1}\right) = x^2$ boýunça $f(x)$-i tapmaly.

Funksiýalaryň görkezilen aralyklarda artýandygyny subut etmeli:

68. $f(x) = x^2$ $(0 \le x < +\infty)$.

69. $f(x) = \sin x$ $\left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right)$.

70. $f(x) = \operatorname{tg} x$ $\left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$.

71. $f(x) = 2x + \sin x$ $(-\infty < x < +\infty)$.

Funksiýalaryň görkezilen aralyklarda kemelýändigini subut etmeli:

72. $f(x) = x^2 \quad (-\infty < x \leq 0)$. **73.** $f(x) = \cos x \quad (0 \leq x \leq \pi)$.

74. $f(x) = \mathrm{ctg}\, x \quad (0 < x < \pi)$.

75. Funksiýalaryň monotonlygyny derňemeli:

a) $f(x) = ax + b$; ç) $f(x) = x^3$; e) $f(x) = a^x \quad (a > 0)$.

b) $f(x) = ax^2 + bx + c$; d) $f(x) = \frac{ax + b}{cx + d}$;

76. Deňsizligi agzama-agza logarifmläp bolarmy?

77. Goý, $\varphi(x)$, $\psi(x)$ we $f(x)$ monoton artýan funksiýalar bolsun. Eger $\varphi(x) \leq f(x) \leq \psi(x)$ bolsa, onda

$$\varphi[\varphi(x)] \leq f[f(x)] \leq \psi[\psi(x)]$$

deňsizligiň dogrudygyny subut etmeli.

Berlen funksiýalaryň $x = \varphi(y)$ ters funksiýalaryny we olaryň kesgitleniş ýaýlalaryny tapmaly:

78. $y = 2x + 3 \quad (-\infty < x < +\infty)$.

79. $y = x^2$; a) $-\infty < x \leq 0$; b) $0 \leq x < +\infty$.

80. $y = \frac{1 - x}{1 + x} \quad (x \neq -1)$.

81. $y = \sqrt{1 - x^2}$; a) $-1 \leq x \leq 0$; b) $0 \leq x \leq 1$.

82. $y = \mathrm{sh}\, x$; $\mathrm{sh}\, x = \frac{1}{2}(e^x - e^{-x}) \quad (-\infty < x < +\infty)$.

83. $y = \mathrm{th}\, x$; $\mathrm{th}\, x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad (-\infty < x < +\infty)$.

84. $y = \begin{cases} x, & \text{eger } -\infty < x < 1; \\ x^2, & \text{eger } 1 \leq x \leq 4; \\ 2^x, & \text{eger } 4 < x < +\infty. \end{cases}$

85. Eger simmetrik $(-l, l)$ interwalda kesgitlenen $f(x)$ funksiýa üçin $f(-x) \equiv f(x)$ bolsa, onda oňa jübüt funksiýa, eger-de $f(-x) \equiv -f(x)$ bolsa, onda oňa täk funksiýa diýilýär.

Berlen funksiýalaryň haýsylarynyň jübüt, haýsylarynyň täk funksiýadygyny anyklamaly:

a) $f(x) = 3x - x^3$; d) $f(x) = \ln\frac{1 - x}{1 + x}$;

b) $f(x) = \sqrt[3]{(1 - x)^2} + \sqrt[3]{(1 + x)^2}$; e) $f(x) = \ln\left(x + \sqrt{1 + x^2}\right)$;

ç) $f(x) = a^x + a^{-x} \; (a > 0)$.

86. Simmetrik $(-l, l)$ interwalda kesgitlenen islendik $f(x)$ funksiýany şol kesimde jübüt we täk funksiýalaryň jemi görnüşinde aňladyp bolýandygyny subut etmeli.

87. Eger X köplükde kesgitlenen $f(x)$ funksiýa üçin $T > 0$ san tapylyp, $x \in X$ üçin $f(x \pm T) \equiv f(x)$ deňlik ýerine ýetse, onda oňa T periodly periodik funksiýa (gysgaça T – periodik funksiýa) diýilýär.

Aşakdakylaryň haýsylarynyň periodik funksiýalardygyny anyklamaly we olaryň iň kiçi periodyny kesgitlemeli:

a) $f(x) = A\cos\lambda x + B\sin\lambda x$;

b) $f(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x$;

ç) $f(x) = 2\operatorname{tg}\frac{x}{2} - 3\operatorname{tg}\frac{x}{3}$;

d) $f(x) = \sin^2 x$;

e) $f(x) = \sin x^2$;

ä) $f(x) = \sqrt{\operatorname{tg}x}$;

f) $f(x) = \operatorname{tg}\sqrt{x}$;

g) $f(x) = \sin x + \sin\left(x\sqrt{2}\right)$.

88. Islendik rasional sanyň Dirihläniň

$$D(x) = \begin{cases} 1, & \text{eger } x \text{ rasional bolsa;} \\ 0, & \text{eger } x \text{ irrasional bolsa} \end{cases}$$

funksiýasynyň periody bolýandygyny subut etmeli.

89. Kesgitleniş ýaýlasy umumy, periodlary ölçegdeş bolan iki funksiýanyň jeminiň we köpeltmek hasylynyň periodik funksiýa bolýandygyny subut etmeli.

90. Eger $f(x)$ funksiýa üçin $f(x + T) \equiv -f(x)$ $(T > 0)$ deňlik ýerine ýetse, onda oňa antiperiodik funksiýa diýilýär. Ol funksiýanyň $2T$-periodik funksiýa bolýandygyny subut etmeli.

91. Eger $f(x)$ $(-\infty < x < +\infty)$ funksiýa we položitel k we T sanlar üçin $f(x + T) \equiv kf(x)$ deňlik ýerine ýetse, onda $f(x) = a^x\varphi(x)$ deňlik dogrudyr, bu ýerde: a – hemişelik san, $\varphi(x)$ bolsa T – periodik funksiýa.

92. Çyzykly birjynsly $y = ax$ funksiýanyň $a = 0$; $1/2$; 1; 2; -1 bolandaky grafigini gurmaly.

93. Çyzykly $y = x + b$ funksiýanyň $b = 0$; 1; 2; -1 bolandaky grafigini gurmaly.

94. Çyzykly funksiýalaryň grafiklerini gurmaly:

a) $y = 2x + 3$; b) $y = 2 - 0{,}1x$; ç) $y = -\frac{x}{2} - 1$.

95. Demriň çyzyklaýyn giňelme koeffisiýenti $a = 1{,}2 \cdot 10^{-6}$. Amatly bolan masştabda $l = f(T)$ $(-40^\circ \le T \le 100^\circ)$ funksiýanyň grafigini gurmaly, bu ýerde T graduslarda aňladylan temperatura we l demir sterženiň T temperaturadaky uzynlygy ($l = 100\, sm$, $T = 0^\circ$ almaly).

96. San oky boýunça iki material nokat hereket edýär. Birinjisi $t = 0$ başlangyç wagtda koordinatalar başlangyjyndan 20 m çepde bolup, tizligi $\vartheta_1 = 10\ m/s$. Ikinjisi bolsa $t = 0$ wagtda O nokatdan 30 m sagda bolup, onuň tizligi $\vartheta_2 = -20\ m/s$. Ol nokatlaryň hereketleriniň deňlemeleriniň grafiklerini gurmaly we olaryň duşuşýan wagtlaryny we ýerlerini tapmaly.

97. Ikinji derejeli bitin rasional funksiýalaryň (parabolalaryň) grafiklerini gurmaly:
a) $y = ax^2$; $a = 1, 1/2, 2, -1$;
b) $y = (x - x_0)^2$; $x_0 = 0, 1, 2, -1$;
ç) $y = x^2 + c$; $c = 0, 1, 2, -1$.

98. $y = ax^2 + bx + c$ kwadrat üçagzany $y = y_0 + a(x - x_0)^2$ görnüşe getirip, grafigini gurmaly. Aşakdaky mysallary hem şu görnüşe getirip, grafigini guruň:
a) $y = 8x - 2x^2$;
ç) $y = -x^2 + 2x - 1$;
b) $y = x^2 - 3x + 2$;
d) $y = \frac{1}{2}x^2 + x + 1$.

99. Material nokat başlangyç $\vartheta_0 = 600\ m/s$ tizlik bilen gorizontyň tekizligine $\alpha = 45^\circ$ burç bilen zyňylan. Onuň hereketiniň traýektoriýasynyň grafigini gurmaly we galan iň ýokary beýikligini we uçuşyň daşlygyny tapmaly (takmynan $g \approx 10\ m/s^2$ diýip hasap etmeli, howanyň garşylygyny hasaba almaly däl).

Ikiden ýokary derejeli bitin rasional funksiýalaryň grafiklerini gurmaly:

100. $y = x^3 + 1$.
101. $y = (1 - x^2)(2 + x)$.
102. $y = x^2 - x^4$.
103. $y = x(a - x)^2(a + x)^3$ $(a > 0)$.

Drob çyzykly funksiýalaryň (giperbolalaryň) grafiklerini gurmaly:

104. $y = \frac{1}{x}$.
105. $y = \frac{1 - x}{1 + x}$.

106. Drob çyzykly $y = \frac{ax + b}{cx + d}$ $(ad - bc \neq 0,\ c \neq 0)$ funksiýanyň grafigini $y = y_0 + \frac{m}{x - x_0}$ görnüşe getirip gurmaly. Aşakdaky mysaly hem şol görnüşe getirip, grafigini gurmaly: $y = \frac{3x + 2}{2x - 3}$.

107. Gaz $p_0 = 1\ kg/sm^2$ basyşda $v_0 = 12\ m^3$ göwrümi alýar. Gazyň temperaturasy hemişelik bolanda v göwrüminiň p basyşa baglylykda üýtgeýşiniň grafigini gurmaly (Boýl-Mariottyň kanuny).

Drob rasional funksiýalaryň grafiklerini gurmaly:

108. $y = x + \frac{1}{x}$ (giperbola).

109. $y = x^2 + \frac{1}{x}$ (Nýutonyň trezubesi).

110. $y = x + \frac{1}{x^2}$.

111. $y = \frac{1}{1 + x^2}$ (Anýeziniň çyzygy).

112. $y = \frac{2x}{1 + x^2}$ (Nýutonyň serpantini).

113. $y = \frac{1}{1 - x^2}$.

114. $y = \frac{x}{1 - x^2}$.

115. $y = \frac{1}{(1 + x)} - \frac{2}{x} + \frac{1}{1 - x}$.

116. $y = \frac{1}{1 + x} - \frac{2}{x^2} + \frac{1}{1 - x}$.

117. $y = \frac{(x + 1)(x - 2)}{(x - 1)(x + 2)}$.

118. $y = \frac{ax^2 + bx + c}{a_1 x + b_1}$ $(a_1 \neq 0)$ funksiýany $y = kx + m + \frac{n}{x - x_0}$ görnüşe getirip, grafigini gurmaly. Aşakdaky mysaly hem şu görnüşe getirip grafigini gurmaly: $y = \frac{x^2 - 4x + 3}{x + 1}$.

119. Çekiji merkezinden x uzaklykda bolan F çekiji güýjüň absolýut ululygynyň grafigini gurmaly: $F = 10$ *kg*, $x = 1$ *m* (Nýutonyň kanuny).

120. Wander-Waalsyň kanuny boýunça real gazyň v göwrümi we onuň p basyşy $\left(p + \frac{a}{v^2}\right)(v - b) = c$ formula arkaly baglanyşýar.

$p = p(v)$ funksiýanyň $a = 2$, $b = 0{,}1$ we $c = 10$ bolandaky grafigini gurmaly.

Irrasional funksiýalaryň grafiklerini gurmaly:

121. $y = \pm\sqrt{-x - 2}$ (parabola).

122. $y = \pm x\sqrt{x}$ (Neýliň parabolasy).

123. $y = \pm\frac{1}{2}\sqrt{100 - x^2}$ (ellips).

124. $y = \pm\sqrt{x^2 - 1}$ (giperbola).

125. $u = \pm\sqrt{\frac{1 - x}{1 + x}}$.

126. $y = \pm x\sqrt{100 - x^2}$.

127. $y = \pm x\sqrt{\frac{x}{10 - x}}$ (sissoida).

128. $y = \pm\sqrt{(x^2 - 1)(9 - x^2)}$.

129. $y = x^n$ derejeli funksiýanyň grafigini gurmaly:

a) $n = 1, 3, 5$;

b) $n = 2, 4, 6$.

130. $y = x^n$ derejeli funksiýanyň grafigini gurmaly:

a) $n = -1, -3$;

b) $n = -2, -4$.

131. $y = \sqrt[m]{x}$ radikalyň grafigini gurmaly:

a) $m = 2, 4$; b) $m = 3, 5$.

132. $y = \sqrt[m]{x^k}$ radikalyň grafigini gurmaly:

a) $m = 2, k = 1$; ç) $m = 3, k = 1$; e) $m = 3, k = 4$; f) $m = 4, k = 3$.
b) $m = 2, k = 3$; d) $m = 3, k = 2$; ä) $m = 4, k = 2$;

133. $y = a^x$ görkezijili funksiýanyň $a = 1/2, 1, 2, e, 10$ bolandaky grafigini gurmaly.

134. $y = e^{y_1}$ çylşyrymly görkezijili funksiýanyň grafigini gurmaly:

a) $y_1 = x^2$; ç) $y_1 = \frac{1}{x}$; e) $y_1 = -\frac{1}{x^2}$;

b) $y_1 = -x^2$; d) $y_1 = \frac{1}{x^2}$; ä) $y_1 = \frac{2x}{1-x^2}$.

135. $y = \log_a x$ logarifmik funksiýanyň grafigini gurmaly:

$$a = 1/2, 2, e, 10.$$

136. Funksiýalaryň grafiklerini gurmaly:
a) $y = \ln(-x)$; b) $y = -\ln x$.

137. $y = \ln y_1$ çylşyrymly logarifmik funksiýanyň grafigini gurmaly:

a) $y_1 = 1 + x^2$; ç) $y_1 = \frac{1-x}{1+x}$; e) $y_1 = 1 + e^x$.

b) $y_1 = (x-1)(x-2)^2(x-3)^3$; d) $y_1 = \frac{1}{x^2}$;

138. $y = \log_x 2$ funksiýanyň grafigini gurmaly.

139. $y = A\sin x$ funksiýanyň $A = 1, 10, -2$ üçin grafigini gurmaly.

140. $y = \sin(x - x_0)$ funksiýanyň $x_0 = 0, \pi/4, \pi/2, 3\pi/4, \pi$ üçin grafigini gurmaly.

141. $y = \sin nx$ funksiýanyň $n = 1, 2, 3, 1/2, 1/3$ üçin grafigini gurmaly.

142. $y = a\cos x + b\sin x$ funksiýany $y = A\sin(x - x_0)$ görnüşe getirip, grafigini gurmaly hem-de aşakdaky mysaly hem şu görnüşe getirip, grafigini gurmaly: $y = 6\cos x + 8\sin x$.

Trigonometrik funksiýalaryň grafiklerini gurmaly:

143. $y = \cos x$. **144.** $y = \mathrm{tg}\, x$.

145. $y = \mathrm{ctg}\, x$. **146.** $y = \sec x$.

147. $y = \csc x$.

148. $y = \sin^2 x$.

149. $y = \sin^3 x$.

150. $y = \mathrm{ctg}^2 x$.

151. $y = \sin x \cdot \sin 3x$.

152. $y = \pm\sqrt{\cos x}$.

Funksiýalaryň grafiklerini gurmaly:

153. $y = \sin x^2$.

154. $y = \sin\frac{1}{x}$.

155. $y = \cos\frac{\pi}{x}$.

156. $y = \sin x \cdot \sin\frac{1}{x}$.

157. $y = \mathrm{tg}\frac{\pi}{x}$.

158. $y = \sec\frac{1}{x}$.

159. $y = x\left(2 + \sin\frac{1}{x}\right)$.

160. $y = \pm\sqrt{1 - x^2}\sin\frac{\pi}{x}$.

161. $y = \frac{\sin x}{x}$.

162. $y = e^x \cos x$.

163. $y = \pm 2^{-x}\sqrt{\sin \pi x}$.

164. $y = \frac{\cos x}{1 + x^2}$.

165. $y = \ln(\cos x)$.

166. $y = \cos(\ln x)$.

167. $y = e^{1/\sin x}$.

Ters trigonometrik funksiýalaryň grafiklerini gurmaly:

168. $y = \arcsin x$.

169. $y = \arccos x$.

170. $y = \mathrm{arctg} x$.

171. $y = \mathrm{arcctg} x$.

172. $y = \arcsin\frac{1}{x}$.

173. $y = \arccos\frac{1}{x}$.

174. $y = \mathrm{arcctg}\frac{1}{x}$.

175. $y = \arcsin(\sin x)$.

176. $y = \arcsin(\cos x)$.

177. $y = \arccos(\cos x)$.

178. $y = \mathrm{arctg}(\mathrm{tg} x)$.

179. $y = \arcsin(2\sin x)$.

180. $y = \arcsin y_1$ funksiýanyň grafigini gurmaly:

a) $y_1 = 1 - \frac{x}{2}$; b) $y_1 = \frac{2x}{1 + x^2}$; ç) $y_1 = \frac{1 - x}{1 + x}$; d) $y_1 = e^x$.

181. $y = \mathrm{arctg} y_1$ funksiýanyň grafigini gurmaly:

a) $y_1 = x^2$; b) $y_1 = \frac{1}{x^2}$; ç) $y_1 = \ln x$; d) $y_1 = \frac{1}{\sin x}$.

182. Funksiýalaryň grafiklerini gurmaly:

a) $y = x^3 - 3x + 2$;

b) $y = \dfrac{x^3}{(1-x)(1+x)^2}$;

ç) $y = \dfrac{x^2}{|x| - 1}$;

d) $y = \sqrt{x(1-x^2)}$;

e) $y = 3\sin\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)$;

ä) $y = \operatorname{ctg}\dfrac{\pi x}{1+x^2}$;

f) $y = \dfrac{1}{1 - 2^{\frac{x}{1-x}}}$;

g) $y = \lg(x^2 - 3x + 2)$;

h) $y = \arcsin\left(\dfrac{3}{2} - \sin x\right)$;

i) $y = \operatorname{arctg}\left(\dfrac{1}{x-1} + \dfrac{1}{x-2} + \dfrac{1}{x-3}\right)$;

j) $y = \log_{\cos x}\sin x$;

ž) $y = (\sin x)^{\operatorname{ctg} x}$.

183. $y = f(x)$ funksiýanyň grafigini ulanyp, aşakdaky funksiýalaryň grafiklerini gurmaly:

a) $y = -f(x)$; b) $y = f(-x)$; ç) $y = -f(-x)$.

184. $y = f(x)$ funksiýanyň grafigini ulanyp, aşakdaky funksiýalaryň grafiklerini gurmaly:

a) $y = f(x - x_0)$; ç) $y = f(2x)$;

b) $y = y_0 + f(x - x_0)$; d) $y = f(kx + b)$ $(k \neq 0)$.

185. $f(x) = \begin{cases} 1 - |x|, & \text{eger } |x| \le 1; \\ 0, & \text{eger } |x| > 1 \end{cases}$ funksiýa üçin $y = \dfrac{1}{2}[f(x-t) + f(x+t)]$ funksiýanyň $t = 0$, $t = 1$, $t = 2$ bolandaky grafiklerini gurmaly.

186. Funksiýalaryň grafiklerini gurmaly:

a) $y = 2 + \sqrt{1-x}$; ç) $y = \ln(1 + x)$; e) $y = 3 + 2\cos 3x$.

b) $y = 1 - e^{-x}$; d) $y = -\arcsin(1 + x)$;

187. $y = f(x)$ funksiýanyň grafigini ulanyp, aşakdaky funksiýalaryň grafiklerini gurmaly:

a) $y = |f(x)|$; b) $y = \dfrac{1}{2}(|f(x)| + f(x))$; ç) $y = \dfrac{1}{2}(|f(x)| - f(x))$.

188. $y = f(x)$ funksiýanyň grafigini ulanyp, aşakdaky funksiýalaryň grafiklerini gurmaly:

a) $y = f^2(x)$; ç) $y = \ln f(x)$; e) $y = \operatorname{sgn} f(x)$;

b) $y = \sqrt{f(x)}$; d) $y = f(f(x))$; ä) $y = [f(x)]$.

189. $f(x) = (x - a)(b - x)$ $(a < b)$ üçin funksiýalaryň grafiklerini gurmaly:

a) $y = f(x)$; ç) $y = \frac{1}{f(x)}$; e) $y = e^{f(x)}$; f) $y = \text{arcctg} f(x)$.

b) $y = f^2(x)$; d) $y = \sqrt{f(x)}$; ä) $y = \lg f(x)$;

190. 1) $f(x) = x^2$; 2) $f(x) = x^3$ üçin funksiýalaryň grafiklerini gurmaly:
a) $y = \arcsin[\sin f(x)]$; ç) $y = \arccos[\sin f(x)]$; e) $y = \text{arctg}[\text{tg} f(x)]$.
b) $y = \arcsin[\cos f(x)]$; d) $y = \arccos[\cos f(x)]$;

191. $y = f(x)$ we $y = g(x)$ funksiýalaryň grafiklerini ulanyp, aşakdaky funksiýalaryň grafiklerini gurmaly:
a) $y = f(x) + g(x)$; b) $y = f(x)g(x)$; ç) $y = f(g(x))$.

Grafikleri goşmak düzgüninden peýdalanyp, funksiýalaryň grafiklerini gurmaly:

192. $y = 1 + x + e^x$;

193. $y = (x + 1)^{-2} + (x - 1)^{-2}$.

194. $y = x + \sin x$.

195. $y = x + \text{arctg} x$.

196. $y = \cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x$.

197. $y = \sin x - \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$.

198. $y = \sin^4 x + \cos^4 x$.

199. $y = |1 - x| + |1 + x|$.

200. $y = |1 - x| - |1 + x|$.

201. Giperbolik funksiýalaryň grafiklerini gurmaly:

a) $y = \text{ch} x$; $\text{ch} x = \frac{1}{2}(e^x + e^{-x})$;

b) $y = \text{sh} x$; $\text{sh} x = \frac{1}{2}(e^x - e^{-x})$;

ç) $y = \text{th} x$; $\text{th} x = \frac{\text{sh} x}{\text{ch} x}$.

Köpeltmek düzgüninden peýdalanyp, funksiýalaryň grafiklerini gurmaly:

202. $y = x\sin x$.

203. $y = x\cos x$.

204. $y = x^2\sin^2 x$.

205. $y = \frac{\sin x}{1 + x^2}$.

206. $y = e^{-x^2}\cos 2x$.

207. $y = x\,\text{sgn}(\sin x)$.

208. $y = [x]|\sin \pi x|$.

209. $y = \cos x \cdot \text{sgn}(\sin x)$.

210. $f(x)=\begin{cases}1-|x|, & \text{eger } |x|\le 1;\\ 0, & \text{eger } |x|>1\end{cases}$ funksiýa üçin $y=f(x)f(a-x)$ funksiýanyň grafigini gurmaly:

a) $a=0$; b) $a=1$; ç) $a=2$.

211. $y=x+\sqrt{x}\,\mathrm{sgn}(\sin\pi x)$ funksiýanyň grafigini gurmaly.

Berlen $f(x)$ funksiýa üçin $y=\dfrac{1}{f(x)}$ funksiýanyň grafigini gurmaly:

212. $f(x)=x^2(1-x^2)$. **213.** $f(x)=x(1-x)^2$.

214. $f(x)=\sin^2 x$. **215.** $f(x)=\ln x$.

216. $f(x)=e^x\sin x$.

217. $f(u)=\begin{cases}-1, & \text{eger } -\infty<u<-1;\\ u, & \text{eger } -1\le u\le 1;\\ 1, & \text{eger } 1<u<+\infty\end{cases}$ bolsa,

funksiýa üçin çylşyrymly $y=f(u)$, $u=2\sin x$ funksiýanyň grafigini gurmaly.

218. $\varphi(x)=\dfrac{1}{2}(x+|x|)$ we $\psi(x)=\begin{cases}x, & \text{eger } x<0;\\ x^2, & \text{eger } x\ge 0\end{cases}$ bolsa,

funksiýalar üçin olaryň grafiklerini gurmaly:

a) $y=\varphi[\varphi(x)]$; b) $y=\varphi[\psi(x)]$; ç) $y=\psi[\varphi(x)]$; d) $y=\psi[\psi(x)]$.

219. $\varphi(x)=\begin{cases}1, & \text{eger } |x|\le 1;\\ 0, & \text{eger } |x|>1,\end{cases}$ we $\psi(x)=\begin{cases}2-x^2, & \text{eger } |x|\le 2;\\ 2, & \text{eger } |x|>2\end{cases}$ bolsa,

funksiýalar üçin olaryň grafiklerini gurmaly:

a) $y=\varphi[\varphi(x)]$; b) $y=\varphi[\psi(x)]$; ç) $y=\psi[\varphi(x)]$; d) $y=\psi[\psi(x)]$.

220. $x>0$ ýaýlada kesgitlenen $f(x)$ funksiýany $x<0$ ýaýlada alynýan funksiýa: 1) jübüt; 2) täk bolar ýaly dowam etdirmeli:

a) $f(x)=1-x$; ç) $f(x)=\sqrt{x}$; e) $f(x)=e^x$;

b) $f(x)=2x-x^2$; d) $f(x)=\sin x$; ä) $f(x)=\ln x$.

Alnan funksiýalaryň grafiklerini gurmaly.

221. Funksiýalaryň haýsy dik oklara görä simmetrikdigini kesgitlemeli:

a) $y=ax^2+bx+c$; ç) $y=\sqrt{a+x}+\sqrt{b-x}$, $(0<a<b)$;

b) $y=\dfrac{1}{x^2}+\dfrac{1}{(1-x)^2}$; d) $y=a+b\cos x$.

222. Funksiýalaryň haýsy merkeze görä simmetrikdigini kesgitlemeli:

a) $y = ax + b$;

b) $y = \dfrac{ax+b}{cx+d}$;

ç) $y = ax^3 + bx^2 + cx + d$;

d) $y = \dfrac{1}{x-1} + \dfrac{1}{x-2} + \dfrac{1}{x-3}$;

e) $y = 1 + \sqrt[3]{x-2}$.

223. Periodik funksiýalaryň grafiklerini gurmaly:

a) $y = |\sin x|$;

b) $y = \operatorname{sgn}\cos x$;

ç) $y = f(x)$, $\quad f(x) = A\frac{x}{l}\left(2 - \frac{x}{l}\right)$, $\quad 0 \le x \le 2l$, $\; f(x + 2l) \equiv f(x)$;

d) $y = [x] - 2\left[\frac{x}{2}\right]$;

e) $y = (x)$, bu ýerde (x) san x-den oňa ýakyn bolan bitin sana çenli uzaklyk.

224. Eger $y = f(x)$ $(-\infty < x < +\infty)$ funksiýanyň grafigi $x = a$ we $x = b$ $(b > a)$ wertikal oklara görä simmetrik bolsa, onda ol funksiýanyň periodikdigini subut etmeli.

225. Eger $y = f(x)$ $(-\infty < x < +\infty)$ funksiýanyň grafigi $A(a, y_0)$ we $B(b, y_1)$ $(b > a)$ nokatlara görä simmetrik bolsa, onda $f(x)$ funksiýanyň çyzykly we periodik funksiýalaryň jemi bolýandygyny subut etmeli. Eger $y_0 = y_1$ bolsa, onda $f(x)$ periodik funksiýadyr.

226. Eger $y = f(x)$ $(-\infty < x < +\infty)$ funksiýanyň grafigi $A(a, y_0)$ nokada we $x = b$ $(b \ne a)$ göni çyzyga görä simmetrik bolsa, onda $f(x)$ funksiýanyň periodikdigini subut etmeli.

227. $f(x + 1) = 2f(x)$ we $f(x) = x(1 - x)$ $(0 \le x \le 1)$ bolýan $y = f(x)$ $(-\infty < x < +\infty)$ funksiýanyň grafigini gurmaly.

228. $f(x + \pi) = f(x) + \sin x$ we $f(x) = 0$ $(0 \le x \le \pi)$ bolýan $y = f(x)$ $(-\infty < x < +\infty)$ funksiýanyň grafigini gurmaly.

229. $y = y(x)$ funksiýanyň grafigini gurmaly (x-iň aşakdaky bahalary üçin):

a) $x = y - y^3$; $\qquad$ b) $x = \dfrac{1-y}{1+y^2}$; $\qquad$ ç) $x = y - \ln y$; $\qquad$ d) $x^2 = \sin y$.

230. Parametrik görnüşde berlen $y = y(x)$ funksiýalaryň grafiklerini gurmaly:

a) $x = 1 - t$, $y = 1 - t^2$;

b) $x = t + \dfrac{1}{t}$, $\quad y = t + \dfrac{1}{t^2}$;

ç) $x = 10\cos t$, $y = \sin t$ (ellips);

d) $x = \operatorname{ch} t$, $y = \operatorname{sh} t$ (giperbola);

e) $x = 5\cos^2 t$, $y = 3\sin^2 t$;

ä) $x = 2(t - \sin t)$, $y = 2(1 - \cos t)$;

f) $x = \sqrt[t+1]{t}$, $\; y = \sqrt[t]{t+1}$, $(t > 0)$.

231. Anyk däl görnüşde berlen funksiýalaryň grafiklerini gurmaly:

a) $x^2 - xy + y^2 = 1$ (ellips);

b) $x^3 + y^3 - 3xy = 0$ (Dekartyň ýapragy);

ç) $\sqrt{x} + \sqrt{y} = 1$ (parabola);

d) $x^{2/3} + y^{2/3} = 4$ (astroida);

e) $\sin x = \sin y$;

ä) $\cos(\pi x^2) = \cos(\pi y)$;

f) $x^y = y^x$ $(x > 0, y > 0)$;

g) $x - |x| = y - |y|$.

232. Anyk däl görnüşde berlen funksiýalaryň grafiklerini gurmaly:

a) $\min(x, y) = 1$;

b) $\max(x, y) = 1$;

ç) $\max(|x|, |y|) = 1$;

d) $\min(x^2, y) = 1$.

233. Polýar (r, φ) koordinatalarynda berlen $r = r(\varphi)$ funksiýalaryň grafiklerini gurmaly:

a) $r = \varphi$ (Arhimediň spiraly);

b) $r = \dfrac{\pi}{\varphi}$ (giperbolik spiral);

ç) $r = \dfrac{\varphi}{\varphi + 1}$ $(0 \le \varphi < +\infty)$;

d) $r = 2^{\varphi/2\pi}$ (logarifmik spiral);

e) $r = 2(1 + \cos\varphi)$ (kardioida);

ä) $r = 10\sin 3\varphi$ (üç ýaprakly bägül);

f) $r^2 = 36\cos 2\varphi$ (Bernulliniň lemniskatasy);

g) $\varphi = \dfrac{r}{r - 1}$ $(r > 1)$;

h) $\varphi = 2\pi \sin r$.

234. Polýar r we φ koordinatalarynda funksiýalaryň grafiklerini gurmaly:

a) $\varphi = 4r - r^2$;

b) $\varphi = \dfrac{12r}{1 + r^2}$;

ç) $r^2 + \varphi^2 = 100$.

235. Parametrik görnüşde berlen funksiýalaryň ($t \ge 0$ – parametr) polýar r we φ koordinatalarynda grafiklerini gurmaly:

a) $\left.\begin{aligned} \varphi &= t\cos^2 t, \\ r &= t\sin^2 t, \end{aligned}\right\}$

b) $\left.\begin{aligned} \varphi &= 1 - 2^{-t}\sin\frac{\pi t}{2}, \\ r &= 1 - 2^{-t}\cos\frac{\pi t}{2}. \end{aligned}\right\}$

236. $y = x^3 - 3x + 1$ funksiýanyň grafigini gurup, $x^3 - 3x - 1 = 0$ deňlemäniň takmynan çözüwini tapmaly.

Deňlemeleri grafiki usulda çözmeli:

237. $x^3 - 4x - 1 = 0$.

238. $x^4 - 4x + 1 = 0$.

239. $x = 2^{-x}$.

240. $\lg x = 0{,}1x$.

241. $10^x = x^2$.

242. $\lg x = x$ $(0 \le x \le 2\pi)$.

Deňlemeler sistemasyny grafiki usulda çözmeli:

243. $x + y^2 = 1$, $16x^2 + y = 4$.

244. $x^2 + y^2 = 100$, $y = 10(x^2 - x - 2)$.

§ 2. Funksiýanyň predeli

1. ***Funksiýanyň predeliniň kesgitlenişi.*** Goý, f funksiýa käbir X köplükde kesgitlenen bolsun. Ol funksiýanyň a nokatdaky predeli düşünjesi girizilende şol nokadyň X köplüge degişli bolmagy hökman däldir, ýöne bu halda a nokat X köplügiň predel nokady bolmalydyr, ýagny ol nokadyň islendik golaý töwereginde X köplügiň nokatlary bolmalydyr. Şeýle köplüge (a, b) interwal we a nokadyň islendik $\mathring{U}(a)$ sünjülen golaý töweregi mysal bolup biler.

Geýnäniň kesgitlemesi. Eger a sana ýygnanýan $\forall\{x_n\}$ $(x_n \neq a)$ yzygiderlik üçin $\{f(x_n)\}$ yzygiderlik B sana ýygnanýan bolsa, onda B sana f funksiýanyň a nokatdaky (ýa-da $x \to a$ bolandaky) predeli diýilýär.

Koşiniň kesgitlemesi. Eger $\forall\varepsilon > 0$ san üçin $\delta = \delta(\varepsilon) > 0$ san tapylyp, $0<|x-a|<\delta$ şerti kanagatlandyrýan $\forall x$ üçin $|f(x) - B| < \varepsilon$ deňsizlik ýerine ýetse, onda B sana f funksiýanyň a nokatdaky predeli diýilýär.

B sanyň f funksiýanyň a nokatdaky predeli bolýandygy

$$\lim_{x\to a} f(x) = B \quad \text{ýa-da} \quad f(x) \to B\ (x \to a)$$

bilen belgilenýär. Bu ýazgyny simwollary ulanyp, Koşiniň kesgitlemesi esasynda gysgaça şeýle ýazmak bolar:

$$(\forall\varepsilon > 0)\ (\exists\delta > 0)\ (\forall x,\ 0 < |x - a| < \delta) : |f(x) - B| < \varepsilon.$$

Funksiýanyň predeli üçin Geýnäniň we Koşiniň kesgitlemeleri deňgüýçlüdirler.

1-nji mysal. $f(x) = x\sin\frac{1}{x}$ funksiýanyň $a = 0$ nokatdaky predeliniň nola deňdigini subut etmeli.

Ç.B. Eger $\forall\varepsilon > 0$ üçin $\delta = \varepsilon$ alsak, onda $0 < |x| < \delta$ şerti kanagatlandyrýan $\forall x$ üçin $\left|x\sin\frac{1}{x}\right| \leq |x| < \varepsilon$ deňsizlik ýerine ýetýär. Şonuň üçin hem Koşiniň kesgitlemesi esasynda $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$. **Ç.S.**

2-nji mysal. $f(x) = \sin\frac{1}{x}$ funksiýanyň $a = 0$ nokatda predeliniň ýokdugyny subut etmeli.

Ç.B. Bu funksiýa $x \neq 0$ nokatlaryň hemmesinde kesgitlenendir. Goý, $x_n = \dfrac{2}{\pi(2n+1)}$ $(n = 0, 1, 2, \ldots)$ bolsun. Onda $\lim\limits_{x\to\infty} x_n = 0$, ýöne $f(x_n) = (-1)^n$. Şonuň üçin ol hiç bir predele ymtylmaýar. Diýmek, funksiýanyň $a = 0$ nokatda predeli ýokdur. **Ç.S.**

3-nji mysal. Koşiniň kesgitlemesini peýdalanyp, $\lim\limits_{x\to-2}(2x+5)=1$ deňligi subut etmeli we ε sanyň 0,1 we 0,01 bahalaryna degişli δ sany anyklamaly.

Ç.B. Kesgitleme boýunça $\forall\varepsilon>0$ üçin $|x+2|<\delta$ bolýan $\forall x$ üçin $|2x+5-1|=$ $=|2x+4|=2|x+2|<\varepsilon$ deňsizligiň ýerine ýetmegi üçin δ san hökmünde $\delta=\varepsilon/2$ ýa-da ondan kiçi bolan položitel sany almak bolar. Şonda $\varepsilon=0,1$ bolanda $\delta(0,1)=0,05$ we $\varepsilon=0,01$ bolanda $\delta(0,01)=0,005$ bolar. **Ç.S.**

Bellik. Bu mysalda $2|x+2|<\varepsilon$ deňsizligi kanagatlandyrýan x-iň bahalar köplügini, ol deňsizligi çözüp bilmedik sebäbi bize $a=-2$ nokadyň diňe şol deňsizlik ýerine ýetýän golaý töweregini kesgitlemek gyzykly bolup, ol deňsizligiň şol golaý töweregiň daşynda ýerine ýetýändigi ýa-da ýetmeýändigi bizi gyzyklandyrmady. Şeýle ýagdaýda başda a nokadyň käbir golaý töweregini alyp, bahalandyrmany şol golaý töwereginde geçirmek amatly bolýar. Ýöne bu bahalandyrmada alynýan golaý töweregiň başda alnan golaý töwerekden uly bolmazlygyny gazanmalydyr.

4-nji mysal. $\lim\limits_{x\to-2}x^2=4$ deňligi subut etmeli.

Ç.B. Ony subut etmek üçin $|x^2-4|=|x-2||x+2|$ tapawudy bahalandyrmaly. San okunda $|x-2|$ köpeldijiniň çäksiz bolany üçin bahalandyrmany käbir golaý töwerekde, mysal üçin, $a=-2$ nokadyň 1 golaý töwereginde, ýagny $(-3,-1)$ interwalda geçirmek aňsatdyr. Şol interwala degişli ähli x üçin $|x-2|<5$ bolar. Şonuň üçin hem $|x^2-4|<5|x+2|$ deňsizligi alarys. $a=-2$ nokadyň $(-2-\delta,-2+\delta)$ δ – golaý töwereginiň hökman şol nokadyň 1 golaý töwereginiň içinde ýerleşmeli bolany üçin, δ sany $\delta=\min(1,\varepsilon/5)$ deňlikden kesgitläliň. Şonda $0<|x+2|<\delta$ bolanda $|x^2-4|<5|x+2|<\varepsilon$ deňsizlik ýerine ýeter. Şeýlelikde, $\lim\limits_{x\to-2}x^2=4$. **Ç.S.**

5-nji mysal. Dirihläniň

$$D(x)=\begin{cases}0, & \text{eger } x \text{ irrasional san bolsa;}\\ 1, & \text{eger } x \text{ rasional san bolsa,}\end{cases}$$

funksiýasynyň hiç bir nokatda predeliniň ýokdugyny subut etmeli.

Ç.B. Erkin a nokatda funksiýanyň predeliniň ýokdugyny görkezmek üçin şol nokada ýygnanýan rasional sanlaryň $\{x_n\}$ we irrasional sanlaryň $\{x'_n\}$ iki yzygiderligine garalyň. Onda $\forall n$ üçin $D(x_n)=1$ we $D(x'_n)=0$ bolar. Şonuň üçin hem $\lim\limits_{n\to\infty}D(x_n)=1$ we $\lim\limits_{n\to\infty}D(x'_n)=0$. Şoňa görä-de Geýnäniň kesgitlemesi boýunça $D(x)$ funksiýanyň a nokatda predeli ýokdur. **Ç.S.**

f funksiýanyň $x\to\infty$ bolandaky $\lim\limits_{n\to\infty}f(x)=B$ predeli üçin hem deňgüýçli bolan Geýnäniň we Koşiniň kesgitlemelerini getirmek bolar. Şonda x diňe položitel ýa-da diňe otrisatel bahalary alýan halynda $\lim\limits_{n\to+\infty} f(x) = B$ ýa-da $\lim\limits_{n\to-\infty} f(x) = B$ ýazgy ulanylýar.

Eger $\lim\limits_{n\to\infty} f(x) = B$ predel bar bolsa, onda funksiýanyň grafigi x argumentiň bahalary ulaldygyça $y = B$ göni çyzyga ýakynlaşýar. Şeýle ýagdaýda şol göni çyzyga funksiýanyň grafiginiň gorizontal asimptotasy diýilýär.

6-njy mysal. $f(x) = \dfrac{1}{x}$ funksiýanyň $x \to \infty$ bolandaky predelini we gorizontal asimptotasyny tapmaly.

Ç.B. Islendik tükeniksiz uly $\{x_n\}$ yzygiderlik üçin $\{f(x_n)\} = \left\{\dfrac{1}{x_n}\right\}$ yzygiderlik tükeniksiz kiçidir, ýagny onuň predeli nola deňdir. Şonuň üçin hem kesgitleme boýunça $\lim\limits_{x\to\infty}\dfrac{1}{x} = 0$, $x = 0$ göni çyzyk berlen funksiýanyň gorizontal asimptotasydyr. **Ç.S.**

7-nji mysal. $f(x) = \sin x$ funksiýanyň $x \to \infty$ bolanda predeliniň ýokdugyny subut etmeli.

Ç.B. Tükeniksiz uly $x_n = \dfrac{\pi}{2}(2n+1)$, $n \in N$ yzygiderlik üçin $\{\sin x_n\} = \{\cos n\pi\} = \{(-1)^n\}$ yzygiderligiň predeli ýokdur. Şonuň üçin hem Geýnäniň kesgitlemesi boýunça $f(x) = \sin x$ funksiýanyň $x \to \infty$ bolanda predeli ýokdur. **Ç.S.**

Funksiýanyň üýtgeýäniniň tükenikli sana ýa-da tükeniksizlige ymtylandaky predeliniň häsiýetleriniň birmeňzeşdigi sebäpli, golaý töwerek düşünjesini ulanyp, şeýle umumy kesgitleme bermek bolar.

Eger $\forall\varepsilon > 0$ üçin a nokadyň $\mathring{U}(a, \delta)$ sünjülen golaý töweregi tapylyp, $\forall x \in \mathring{U}(a, \delta)$ üçin $|f(x) - B| < \varepsilon$ (ýagny $f(x) \in U(B, \varepsilon)$) bolsa, onda B sana f funksiýanyň a nokatdaky predeli diýilýär.

Goý, f funksiýa (a, c) $((c, a))$ interwalda kesgitlenen bolsun.

Eger Geýnäniň kesgitlemesindäki $\forall\{x_n\}$ $(x_n \neq a)$ üçin diýlen ýazgy $a < x_n < c$ $(c < x_n < a)$ deňsizlikleri kanagatlandyrýan $\forall\{x_n\}$ üçin diýlen ýazgy bilen, şeýle hem Koşiniň kesgitlemesindäki $0 < |x - a| < \delta$ şerti kanagatlandyrýan $\forall x$ üçin diýlen ýazgy $a < x < a + \delta$ $(a - \delta < x < a)$ deňsizlikleri kanagatlandyrýan $\forall x$ üçin diýlen ýazgy bilen çalşyrylsa, onda şol kesgitlemelerdäki B sana f funksiýanyň a nokatdaky **sag (çep)** predeli diýilýär we ol

$$B = \lim_{x\to a+0} f(x) = f(a+0) \quad \left(B = \lim_{x\to a-0} f(x) = f(a-0)\right)$$

görnüşde ýazylýar.

8-nji mysal. $F(x) = \operatorname{sgn} x$ funksiýanyň $x = 0$ nokatdaky sag we çep predellerini tapmaly.

Ç.B. Goý, $\forall n \in N$ üçin $x_n > 0$, $x'_n < 0$, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x'_n = 0$ şertler ýerine ýetsin, onda $\lim\limits_{n\to\infty} \operatorname{sgn} x_n = 1$, $\lim\limits_{n\to\infty} x'_n = -1$.

Şonuň üçin hem kesgitleme esasynda:

$$\lim_{n\to+0} \operatorname{sgn} x = 1, \qquad \lim_{n\to-0} x = -1.$$ **Ç.S.**

Eger f funksiýanyň a nokatda sag we çep predelleri bar bolup, $f(a+0) = = f(a-0) = B$ bolsa, onda $\lim\limits_{n\to a} f(x) = B$ we tersine.

Koşiniň ölçegleri. f funksiýanyň a nokatda predeliniň bolmagy üçin $\forall \varepsilon > 0$ üçin şeýle $\delta > 0$ san tapylyp, $\forall x', x'' \in \mathring{U}(a, \delta)$ üçin

$$|f(x') - f(x'')| < \varepsilon$$

deňsizligiň ýerine ýetmegi zerur we ýeterlikdir.

Eger a sana ýygnanýan käbir $\{x_n\}$ $(x_n \neq a)$ yzygiderlik üçin $\{f(x_n)\}$ yzygiderlik B sana ýygnanýan bolsa, onda B sana f funksiýanyň a nokatdaky bölekleýin predeli diýilýär.

Bölekleýin predelleriň iň ulusyna (iň kiçisine) f funksiýanyň a nokatdaky ýokarky (aşaky) predeli diýilýär we ol

$$\overline{\lim_{x\to a}} f(x) \quad \Big(\underline{\lim_{x\to a}} f(x)\Big)$$

görnüşde belgilenýär.

Eger m we M sanlar tapylyp, $\forall x \in (a, b)$ üçin $m \le f(x) \le M$ bolsa, onda f funksiýa (a, b) interwalda çäkli funksiýa diýilýär.

$$m_0 = \inf_{x\in(a,b)} \{f(x)\} = \max m$$

we

$$M_0 = \sup_{x\in(a,b)} \{f(x)\} = \min M$$

sanlara degişlilikde f funksiýanyň (a, b) interwaldaky takyk aşaky we takyk ýokarky çäkleri diýilýär. $M_0 - m_0$ tapawuda f funksiýanyň (a, b) interwaldaky yrgyldysy diýilýär.

Çäkli bolmadyk funksiýalara çäksiz funksiýalar diýilýär.

2. *Funksiýanyň predeliniň esasy häsiýetleri.* 1) eger funksiýanyň a nokatda tükenikli predeli bar bolsa, onda ol predel ýeke-täkdir we funksiýa a nokadyň käbir $\mathring{U}(a)$ sünjülen golaý töwereginde çäklidir.

2) eger $\lim\limits_{x\to a} f(x) = B \neq 0$ bolsa, onda a nokadyň $\mathring{U}(a)$ sünjülen golaý töweregi tapylyp, $\forall x \in \mathring{U}(a)$ üçin $|f(x)| > |B|/2$ bolar. Has anygy, $B > 0$ bolanda $f(x) > \frac{B}{2}$, $B < 0$ bolanda $f(x) < \frac{B}{2}$.

3) eger f we g funksiýalaryň $\lim\limits_{x\to a} f(x) = A$, $\lim\limits_{x\to a} g(x) = B$ predelleri bar bolup, a nokadyň $\mathring{U}(a)$ sünjülen golaý töwereginde $f(x) > g(x)$ ýa-da $f(x) \geq g(x)$ bolsa, onda $A \geq B$.

4) eger $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = B$ we a nokadyň käbir $\mathring{U}(a)$ sünjülen golaý töwereginde $f(x) \leq \varphi(x) \leq g(x)$ bolsa, onda $\lim\limits_{x\to a} \varphi(x) = B$.

3. *çylşyrymly funksiýanyň predeli.* Eger $\lim\limits_{x\to a} g(x) = b$ we $\lim\limits_{y\to b} f(y) = A$ predeller bar we $c > 0$ san tapylyp, $0 < |x - a| < c$ üçin $|g(x) - b| > 0$ bolsa, onda çylşyrymly funksiýanyň predeli bardyr:

$$\lim_{x\to a} f[g(x)] = A.$$

6) eger $\lim\limits_{x\to a} f(x) = A$, $\lim\limits_{x\to a} \varphi(x) = B$ predeller bar bolsa, onda $f(x) \pm \varphi(x)$, $f(x)\cdot\varphi(x)$ we $\dfrac{f(x)}{\varphi(x)}$ funksiýalaryň hem a nokatda predelleri bardyr we aşakdaky deňlikler dogrudyr:

$$\lim_{x\to a}[f(x) \pm \varphi(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} \varphi(x), \qquad \lim_{x\to a}[f(x)\varphi(x)] = \lim_{x\to a} f(x) \lim_{x\to a} \varphi(x),$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} \varphi(x)} \qquad \left(\lim_{x\to a} \varphi(x) \neq 0\right).$$

7) eger $\forall x \neq a$ üçin $f(x) = g(x)$ we $\lim\limits_{x\to a} g(x) = B$ predel bar bolsa, onda $\lim\limits_{x\to a} f(x) = B$.

$f(x) = C$ – hemişelik we $g(x) = x$ funksiýalar üçin ýerine ýetýän

$$\lim_{x\to a} f(x) = C \quad \text{we} \quad \lim_{x\to a} g(x) = a$$

deňlikleriň esasynda şeýle häsiýetler gelip çykýar:

a) hemişelik köpeldijini predel belgisiniň öňüne çykarmak bolar:

$$\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x).$$

b) eger $f(x)$ funksiýa n derejeli köpagza bolsa, ýagny

$$f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n,$$

onda $\lim\limits_{x\to a} f(x) = f(a)$ bolar.

ç) eger $f(x)$ we $\varphi(x)$ degişlilikde n we m derejeli köpagzalar bolsalar hem-de $\varphi(a) \neq 0$ bolsa, onda

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\frac{f(a)}{\varphi(a)}=\frac{a_0a^n+a_1a^{n-1}+...+a_{n-1}a+a_n}{b_0a^m+b_1a^{m-1}+...+b_{m-1}a+b_m}.$$

Funksiýalaryň predelleri hasaplanylanda, köplenç,

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad \infty-\infty,\quad 0\times\infty,\quad 1^{\infty},\quad 0^0,\quad \infty^0$$

görnüşdäki kesgitsizliklere duş gelinýär. Şeýle bolanda predelleri tapmak üçin, ilki bilen, predelleri tapylýan aňlatmalary dürli usullar arkaly özgerdip, belli bolan formulalary we düzgünleri ulanyp bolar ýaly görnüşlere getirmeli.

9-njy mysal. $f(x)=\dfrac{x^2+3x-4}{x^2-1}$ funksiýanyň $x=1$ nokatdaky predelini hasaplamaly.

Ç.B. Funksiýa $x=1$ nokatda 0/0 görnüşdäki kesgitsizlige öwrülýär. $x\neq 1$ bolanda

$$f(x)=\frac{x^2+3x-4}{x^2-1}=\frac{(x-1)(x+4)}{(x+1)(x-1)}=\frac{x+4}{x+1}=g(x)$$

we $\lim\limits_{x\to 1}g(x)=\lim\limits_{x\to 1}\dfrac{x+4}{x+1}=\dfrac{5}{2}$. Şonuň üçin hem predeliň 7-nji häsiýeti boýunça $\lim\limits_{x\to 1}f(x)=\dfrac{5}{2}$. **Ç.S.**

10-njy mysal. $\lim\limits_{x\to a}\sqrt[n]{x}=\sqrt[n]{a}$ $(a>0)$ deňligi subut etmeli.

Ç.B. Goý, $a>0$ üçin $x=a+t$ bolsun. Onda $x\to a$ bolanda $t\to 0$. Şonuň üçin $|t|<a$ hasap edip,

$$\sqrt[n]{a}\left(1-\frac{|t|}{a}\right)<\sqrt[n]{a+t}=\sqrt[n]{a}\sqrt[n]{1+\frac{t}{a}}<\sqrt[n]{a}\left(1+\frac{|t|}{a}\right)$$

deňsizlikleri alarys. Olardan bolsa 4-nji häsiýet esasynda

$$\lim_{x\to a}\sqrt[n]{x}=\lim_{t\to 0}\sqrt[n]{a+t}=\sqrt[n]{a}.\quad \textbf{Ç.S.}$$

Ajaýyp predeller. Funksiýalaryň predelleri tapylanda ulanylýan

$$\textbf{1. }\lim_{x\to 0}\frac{\sin x}{x}=1;\qquad \textbf{2. }\lim_{x\to 0}(1+x)^{1/x}=e$$

predellere degişlilikde birinji we ikinji ajaýyp predeller diýilýär.

Bu formulalaryň esasynda amalyýetde ulanylýan şeýle formulalar gelip çykýar: eger $\lim\limits_{x\to a}u(x)=0$ bolsa, onda

$$\lim_{x\to a}\frac{\sin u(x)}{u(x)}=1; \qquad (1)$$

$$\lim_{x\to a}(1+u(x))^{\frac{1}{u(x)}}=e. \qquad (2)$$

Şeýle hem funksiýalaryň predelleri tapylanda

3. $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1,$ **4.** $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1;$ **5.** $\lim\limits_{x\to 0}\dfrac{(1+x)^\lambda-1}{x}=\lambda$

formulalar ulanylýar. Olar, köplenç, şeýle görnüşde ulanylýar: eger $\lim\limits_{x\to a}u(x)=0$ bolsa, onda

$$\lim_{x\to a}\frac{\ln(1+u(x))}{u(x)}=1; \quad \lim_{x\to a}\frac{e^{u(x)}-1}{u(x)}=1; \quad \lim_{x\to a}\frac{(1+u(x))^\lambda-1}{u(x)}=\lambda. \qquad (3)$$

11-nji mysal. a) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 7x}$; b) $\lim\limits_{x\to 0}(1+3x)^{1/x}$ predelleri tapmaly.

Ç.B. a) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 7x}=\lim\limits_{x\to 0}\dfrac{\frac{\sin 3x}{3x}}{\frac{\sin 7x}{7x}}\cdot\dfrac{3x}{7x}=\dfrac{\lim\limits_{x\to 0}\frac{\sin 3x}{3x}}{\lim\limits_{x\to 0}\frac{\sin 7x}{7x}}\cdot\dfrac{3}{7}=\dfrac{3}{7};$

b) $\lim\limits_{x\to 0}(1+3x)^{1/x}=\lim\limits_{x\to 0}\left[(1+3x)^{1/3x}\right]^3=\left[\lim\limits_{x\to 0}(1+3x)^{1/3x}\right]^3=e^3$. **Ç.S.**

12-nji mysal. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\ln\sin x}{\cos^2 x}$ predeli hasaplamaly.

Ç.B. Predeli hasaplamak üçin (3) formulany ulanarys:

$$\lim_{x\to\frac{\pi}{2}}\frac{\ln\sin x}{\cos^2 x}=\lim_{x\to\frac{\pi}{2}}\frac{\ln\sqrt{1-\cos^2 x}}{\cos^2 x}=-\frac{1}{2}\lim_{x\to\frac{\pi}{2}}\frac{\ln(1-\cos^2 x)}{-\cos^2 x}=-\frac{1}{2}.$$ **Ç.S.**

13-nji mysal. $\lim\limits_{x\to 2}\dfrac{3^{x^2-x-2}-1}{x^2-5x+6}$ predeli hasaplamaly.

Ç.B. Predeli hasaplamak üçin ilki aňlatmany özgerdeliň we soňra (3) formuladan peýdalanalyň:

$$\lim_{x\to 2}\frac{3^{x^2-x-2}-1}{x^2-5x+6}=\lim_{x\to 2}\left(\frac{3^{x^2-x-2}-1}{x^2-x-2}\cdot\frac{x^2-x-2}{x^2-5x+6}\right)=$$

$$=\lim_{x\to 2}\frac{3^{x^2-x-2}-1}{x^2-x-2}\lim_{x\to 2}\frac{(x-2)(x+1)}{(x-2)(x-3)}=-3\ln 3.$$ **Ç.S.**

4. *Tükeniksiz kiçi we tükeniksiz uly hem-de* o *(o kiçi) we* O *(O uly) funksiýalar.* Eger $\lim\limits_{x\to a}\alpha(x) = 0$ bolsa, onda $\alpha(x)$ funksiýa a nokatda tükeniksiz kiçi funksiýa diýilýär. Eger-de $\lim\limits_{x\to a}f(x) = \infty$, ýagny $\forall K > 0$ üçin $\delta > 0$ tapylyp, $0<|x-a|<\delta$ bolanda $|f(x)| > K$ bolsa, onda $f(x)$ funksiýa a nokatda tükeniksiz uly funksiýa diýilýär. Olaryň özara baglanyşykly şeýle häsiýetleri bar:

1) eger $\alpha(x)$ funksiýa a nokatda tükeniksiz kiçi we $\alpha(x) \neq 0$ bolsa, onda $1/\alpha(x)$ funksiýa a nokatda tükeniksiz uludyr.

2) eger $f(x)$ funksiýa a nokatda tükeniksiz uly bolsa, onda $1/f(x)$ funksiýa a nokatda tükeniksiz kiçidir.

3) eger $\alpha(x)$ funksiýa a nokatda tükeniksiz kiçi we $\alpha(x)\neq 0$ we $\lim\limits_{x\to a}f(x) = B \neq 0$ predel bar bolsa, onda $f(x)/\alpha(x)$ funksiýa a nokatda tükeniksiz uludyr.

4) eger $f(x)$ funksiýa a nokatda tükeniksiz uly we $\alpha(x)$ funksiýanyň a nokatda predeli bar bolsa, onda $\alpha(x)/f(x)$ funksiýa a nokatda tükeniksiz kiçidir.

Her bir tükeniksiz uly funksiýa çäksizdir, ýöne çäksiz funksiýa tükeniksiz uly bolman hem biler.

$\lim\limits_{x\to a}f(x) = B$ predeliň bolmagy üçin

$$f(x) = B + \alpha(x), \qquad \lim_{x\to a}\alpha(x) = 0$$

deňlikleriň ýerine ýetmegi zerur we ýeterlikdir.

Goý, f we g funksiýalar a nokadyň käbir sünjülen golaý töwereginde kesgitlenen bolsun.

1. $\forall\varepsilon > 0$ üçin a nokadyň $\mathring{U}(a)$ golaý töweregi tapylyp, $\forall x \in \mathring{U}(a)$ üçin

$$|f(x)| < \varepsilon|g(x)| \tag{1}$$

deňsizligiň ýerine ýetmegi

$$f(x) = o(g(x)), \quad x \to a \tag{2}$$

görnüşde ýazylýar (okalyşy: $f(x)$ deňdir o kiçi $g(x)$).

Eger şeýle φ funksiýa tapylyp, $\forall x \in \mathring{U}(a)$ üçin

$$f(x) = \varphi(x)g(x), \qquad \lim_{x\to a}\varphi(x) = 0$$

bolsa, şeýle hem eger $\mathring{U}(a)$ golaý töwereginde $g(x) \neq 0$ we $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = 0$ bolsa, onda bu hallarda hem (2) deňlik ulanylýar.

Bellik. «o kiçi» simwoly özünde saklaýan deňlik adaty deňlik däldir. Mysal üçin, $x^2 = o(x)$, $x \to 0$ we $x^3 = o(x)$, $x \to 0$ deňlikler esasynda olaryň çep bölekleri deň diýip bolmaz, çünki $o(x)$ simwol käbir anyk funksiýany aňlatman, ol 0 nokatda x-a görä ýokary tertipli tükeniksiz kiçi bolan islendik funksiýany aňladýar. Şeýle funksiýalar bolsa tükeniksiz köpdür. Mysal üçin, islendik x^p $(p > 1)$ funksiýa $o(x)$, $x \to 0$ funksiýa bolýandyr.

14-nji mysal. a) $x^2 = o(x)$, $x \to 0$, çünki $x^2 = x \cdot x$;

b) $\frac{1}{x} = o\left(\frac{1}{x^2}\right)$, $x \to 0$, çünki $\frac{1}{x} = x\frac{1}{x^2}$ $(x \neq 0)$;

ç) $\frac{1}{x^2} = o\left(\frac{1}{x}\right)$, $x \to \infty$, çünki $\frac{1}{x^2} = \frac{1}{x} \cdot \frac{1}{x}$ $(x \neq 0)$.

15-nji mysal. Eger $f(x) = (x - a)^3$ we $g(x) = \frac{\sin^2 x}{x}$ bolsa, onda $f(x) = o(1)$, $x \to a$ we $g(x) = o(1)$, $x \to 0$.

2. Şeýle $c > 0$ san we $\mathring{U}(a)$ sünjülen golaý töwerek tapylyp, $\forall x \in \mathring{U}(a)$ üçin

$$|f(x)| \leq c|g(x)|$$

deňsizligiň ýerine ýetmegi

$$f(x) = O(g(x)), \quad x \to a \tag{3}$$

görnüşde ýazylýar (okalyşy: $f(x)$ deňdir O uly $g(x)$).

Eger şeýle φ funksiýa we $\mathring{U}(a)$ sünjülen golaý töwerek tapylyp, $\forall x \in \mathring{U}(a)$ üçin

$$f(x) = \varphi(x)g(x) \qquad \lim_{x \to a} \varphi(x) = K \neq \infty$$

bolsa, şeýle hem $\mathring{U}(a)$ golaý töwerekde $g(x) \neq 0$ we $\lim_{x \to a} \frac{f(x)}{g(x)} = B \neq 0$ bolsa, onda bu hallarda hem (3) deňsizlik ulanylýar.

16-njy mysal. a) $\frac{1}{x} = O\left(\frac{1}{x^2}\right)$, $x \to 0$, çünki $|x| \leq 1$ bolanda $\left|\frac{1}{x}\right| \leq \frac{1}{x^2}$.

b) $\frac{1}{x^2} = O\left(\frac{1}{x}\right)$, $x \to \infty$, çünki $|x| \geq 1$ bolanda $\frac{1}{x^2} \leq \left|\frac{1}{x}\right|$.

17-nji mysal. Eger $f(x) = \frac{\text{tg} 5x}{x}$, $p(x) = \frac{1 - x^2}{\sin \pi x}$ bolsa, onda

$$f(x) = O(1), \ x \to 0, \qquad p(x) = O(1), \ x \to 1,$$

çünki $\lim_{x \to 0} f(x) = 5$, $\lim_{x \to 1} p(x) = \frac{2}{\pi}$.

18-nji mysal. Eger $f(x) = 9x^2$ we $g(x) = \sin x^2$ bolsa, onda

$$\lim_{x \to 0} \frac{g(x)}{f(x)} = \lim_{x \to 0} \frac{\sin x^2}{9x^2} = \frac{1}{9} \lim_{x \to 0} \frac{\sin x^2}{x^2} = \frac{1}{9},$$

ýagny $x \to 0$ bolanda $9x^2$ we $g(x) = \sin x^2$ deňtertipli tükeniksiz kiçi funksiýalardyr. Bu halda $\sin x^2$ funksiýa $3x$-a görä ikinji tertipli tükeniksiz kiçi funksiýa hem diýilýär.

3. Eger şeýle φ funksiýa tapylyp, $\forall x \in \mathring{U}(a)$ üçin

$$f(x) = \varphi(x)g(x), \qquad \lim_{x\to a}\varphi(x) = 1 \tag{4}$$

bolsa, onda $x \to a$ bolanda f funksiýa g funksiýa deňgüýçli funksiýa diýilýär we ol $f(x) \sim g(x)$, $x \to a$ ymtylmada aňladylýar.

Mysal üçin, $x \to 0$ bolanda

$$x \sim \sin x \sim \mathrm{tg}x \sim \ln(1+x) \sim e^x - 1 \sim \arcsin x \sim \mathrm{arctg}x.$$

19-njy mysal. Eger $f(x) = \dfrac{x^2}{1+x^4}$ we $g(x) = x^2$ bolsa, onda $f(x) \sim g(x)$, $x \to 0$, çünki $\varphi(x) = \dfrac{1}{1+x^4}$ funksiýa üçin (4) deňlikler ýerine ýetýär:

$$f(x) = \frac{x^2}{1+x^4} = \frac{1}{1+x^4}x^2 = \varphi(x)g(x), \quad \lim_{x\to 0}(x)\lim_{x\to 0}\frac{1}{1+x} = 1.$$

20-nji mysal. Eger $f(x) = \dfrac{x^6}{1+x^4}$ we $g(x) = x^2$ bolsa, onda $f(x) \sim g(x)$, $x \to \infty$, çünki $\varphi(x) = \dfrac{x^4}{1+x^4}$ üçin $\dfrac{x^6}{1+x^4} = \varphi(x)x^2$ we $\lim\limits_{x\to\infty}\dfrac{x^4}{1+x^4} = 1.$

Eger $\alpha(x) \sim \alpha_1(x)$, $x \to a$ we $\beta(x) \sim \beta_1(x)$, $x \to a$ bolsa, onda

$$\lim_{x\to a}\frac{\alpha(x)}{\beta(x)} = \lim_{x\to a}\frac{\alpha_1(x)}{\beta_1(x)}.$$

21-nji mysal. $\lim\limits_{x\to o}\dfrac{\sin 3x}{x+x^2}$ predeli hasaplamaly.

Ç.B. $\sin 3x \sim 3x$, $x \to 0$ we $x + x^2 \sim x$, $x \to 0$ bolýandygy üçin $\lim\limits_{x\to o}\dfrac{\sin 3x}{x+x^2} = \lim\limits_{x\to o}\dfrac{3x}{x} = 3$. **Ç.S.**

o kiçi we O uly funksiýalaryň şeýle häsiýetleri bardyr:

$$o(g) + o(g) = o(g), \qquad o(g) + O(g) = O(g),$$

$$o(g)\cdot o(f) = o(gf), \qquad O(g)\cdot O(f) = O(gf), \qquad O(g) + O(g) = O(g).$$

Funksiýalary deňeşdirmek bilen baglanyşykly ýokarda görkezilen häsiýetlerden we 1– 5 ajaýyp predelleriň esasynda gelip çykýan

$$(1+x)^n \sim (1+nx), \quad x \to 0$$

$$(1+x)^n - (1+nx) \sim \frac{n(n-1)}{2}x^2, \quad x \to 0$$

$$x \sim \sin x \sim \operatorname{tg} x \sim \ln(1+x) \sim e^x - 1 \sim \arcsin x \sim \operatorname{arctg} x, \quad x \to 0$$

formulalar predelleri tapmaklykda giňişleýin ulanylýar.

22-nji mysal. $\lim\limits_{x \to o} \frac{(1+x)^n - 1 - nx}{x^2}$ predeli hasaplamaly.

Ç.B. $(1+x)^n - (1+nx) \sim \frac{n(n-1)}{2}x^2$, $x \to 0$ esasynda

$$\lim_{x \to 0} \frac{(1+x)^n - (1+nx)}{x^2} = \lim_{x \to 0} \frac{n(n-1)x^2}{2x^2} = \frac{n(n-1)}{2}. \quad \textbf{Ç.S.}$$

Gönükmeler

245. Özara ýönekeý m we n sanlar we $n > 0$ üçin $x = m/n$ bolanda $f(x) = n$ we irrasional x üçin $f(x) = 0$ deňlikler boýunça kesgitlenýän funksiýanyň tükenikli, ýöne her bir x nokatda çäkli däldigini (ol nokadyň islendik golaý töwereginde çäkli däldigini) görkezmeli.

246. Eger $f(x)$ funksiýa: a) interwalyň, b) kesimiň her bir nokadynda kesgitlenen we lokal çäkli bolsa, onda ol funksiýa degişlilikde interwalda we kesimde çäkli bolup bilermi?

Degişli mysallary getiriň.

247. $f(x) = \frac{1+x^2}{1+x^4}$ funksiýanyň $-\infty < x < +\infty$ interwalda çäklidigini subut etmeli.

248. $f(x) = \frac{1}{x}\cos\frac{1}{x}$ funksiýanyň $x = 0$ nokadyň islendik golaý töwereginde çäksizdigini, ýöne $x \to 0$ bolanda tükeniksiz uly däldigini subut etmeli.

249. $f(x) = \ln x \cdot \sin^2\frac{\pi}{x}$ funksiýanyň $0<x<\varepsilon$ interwalda çäklidigini derňemeli.

250. $f(x) = \frac{x}{1+x}$ funksiýanyň $0 \le x < +\infty$ ýaýlada takyk aşaky $m = 0$ we takyk ýokarky $M = 1$ çägini alýandygyny subut etmeli.

251. $f(x)$ funksiýa $[a, b]$ kesimde kesgitlenendir we artýandyr. Onuň şol kesimdäki takyk aşaky we takyk ýokarky çäkleri nämä deň?

Funksiýalaryň görkezilen ýaýlalarda takyk aşaky we takyk ýokarky çäklerini tapmaly:

252. $f(x) = x^2$, $[-2, 5)$.

253. $f(x) = \dfrac{1}{1+x^2}$, $(-\infty, +\infty)$.

254. $f(x) = \dfrac{2x}{1+x^2}$, $(0, +\infty)$.

255. $f(x) = x + \dfrac{1}{x}$, $(0, +\infty)$.

256. $f(x) = \sin x$, $(0, +\infty)$.

257. $f(x) = \sin x + \cos x$, $[0, 2\pi]$.

258. $f(x) = 2^x$, $(-1, 2)$.

259. $f(x) = [x]$: a) $(0, 2)$; b) $[0, 2]$.

260. $f(x) = x - [x]$, $[0, 1]$.

261. $f(x) = x^2$ funksiýanyň interwallardaky yrgyldysyny tapmaly:
a) $(1; 3)$; b) $(1{,}9; 2{,}1)$; ç) $(1{,}99; 2{,}01)$; d) $(1{,}999; 2{,}001)$.

262. $f(x) = \operatorname{arctg}(1/x)$ funksiýanyň interwallardaky yrgyldysyny tapmaly:
a) $(-1; 1)$; b) $(-0{,}1; 0{,}1)$; ç) $(-0{,}01; 0{,}01)$; d) $(-0{,}001; 0{,}001)$.

263. Goý, $m[f]$ we $M[f]$ degişlilikde $f(x)$ funksiýanyň (a, b) interwaldaky takyk aşaky we takyk ýokarky çäkleri bolsun. (a, b) interwalda kesgitlenen $f_1(x)$ we $f_2(x)$ funksiýalar üçin

$$m[f_1 + f_2] \geq m[f_1] + m[f_2] \quad \text{we} \quad M[f_1 + f_2] \leq M[f_1] + M[f_2]$$

deňsizlikleri subut etmeli.

Bu deňsizliklerde: a) deňlik bolan mahaly; b) deňsizlik bolan mahaly $f_1(x)$ we $f_2(x)$ funksiýalaryň mysallaryny getiriň.

264. Goý, $f(x)$ funksiýa $[a, +\infty)$ ýaýlada kesgitlenen we her bir $[a, b] \subset [a, +\infty)$ kesimde çäkli bolsun.

$$m(x) = \inf_{a \leq \xi \leq x} f(\xi) \quad \text{we} \quad M(x) = \sup_{a \leq \xi \leq x} f(\xi)$$

funksiýalar üçin $y = m(x)$ we $y = M(x)$ funksiýalaryň grafigini gurmaly:
a) $f(x) = \sin x$; b) $f(x) = \cos x$.

265. «$\varepsilon - \delta$» dilinde $\lim\limits_{x \to 2} x^2 = 4$ deňligi subut etmeli we tablisany doldurmaly:

ε	0,1	0,01	0,001	0,0001	...
δ					

266. «$K - \delta$» dilinde $\lim\limits_{x \to 1} \dfrac{1}{(1-x)^2} = +\infty$ deňligi subut etmeli we tablisany doldurmaly:

K	10	100	1000	10000	...
δ					

267. Aşakdaky tassyklamalary deňsizliklerde aňlatmaly:

a) $\lim\limits_{x \to a} f(x) = b$; b) $\lim\limits_{x \to a-0} f(x) = b$; ç) $\lim\limits_{x \to a+0} f(x) = b$.

Degişli mysallary getirmeli.

Aşakdaky tassyklamalary deňsizliklerde aňlatmaly we degişli mysallary getirmeli:

268. a) $\lim\limits_{x \to \infty} f(x) = b$; b) $\lim\limits_{x \to -\infty} f(x) = b$; ç) $\lim\limits_{x \to +\infty} f(x) = b$.

269.

a) $\lim\limits_{x \to a} f(x) = \infty$; d) $\lim\limits_{x \to a-0} f(x) = \infty$; f) $\lim\limits_{x \to a+0} f(x) = \infty$;

b) $\lim\limits_{x \to a} f(x) = -\infty$; e) $\lim\limits_{x \to a-0} f(x) = -\infty$; g) $\lim\limits_{x \to a+0} f(x) = -\infty$;

ç) $\lim\limits_{x \to a} f(x) = +\infty$; ä) $\lim\limits_{x \to a-0} f(x) = +\infty$; h) $\lim\limits_{x \to a+0} f(x) = +\infty$.

270.

a) $\lim\limits_{x \to \infty} f(x) = \infty$; d) $\lim\limits_{x \to -\infty} f(x) = \infty$; f) $\lim\limits_{x \to +\infty} f(x) = \infty$;

b) $\lim\limits_{x \to \infty} f(x) = -\infty$; e) $\lim\limits_{x \to -\infty} f(x) = -\infty$; g) $\lim\limits_{x \to +\infty} f(x) = -\infty$;

ç) $\lim\limits_{x \to \infty} f(x) = +\infty$; ä) $\lim\limits_{x \to -\infty} f(x) = +\infty$; h) $\lim\limits_{x \to +\infty} f(x) = +\infty$.

271. $y = f(x)$ funksiýa üçin aşakdakylary deňsizliklerde aňlatmaly:

a) $x \to a$ bolanda $y \to b-0$;
b) $x \to a-0$ bolanda $y \to b-0$;
ç) $x \to a+0$ bolanda $y \to b-0$;
d) $x \to a$ bolanda $y \to b+0$;
e) $x \to a-0$ bolanda $y \to b+0$;
ä) $x \to a+0$ bolanda $y \to b+0$;
f) $x \to \infty$ bolanda $y \to b-0$;
g) $x \to -\infty$ bolanda $y \to b-0$;
h) $x \to +\infty$ bolanda $y \to b-0$;
i) $x \to \infty$ bolanda $y \to b+0$;
j) $x \to -\infty$ bolanda $y \to b+0$;
ž) $x \to +\infty$ bolanda $y \to b+0$.

Degişli mysallary getirmeli:

272. Goý, $P(x) = a_0x^n + a_1x^{n-1} + ... + a_n$ bolsun, bu ýerde a_i $(i = 0, 1, ..., n; n \geq 1, a_0 \neq 0)$ hakyky sanlar.

$$\lim_{x \to \infty} |P(x)| = +\infty$$

deňligi subut etmeli.

273. Goý, $R(x) = \dfrac{a_0x^n + a_1x^{n-1} + ... + a_n}{b_0x^m + b_1x^{m-1} + ... + b_m}$ bolsun, bu ýerde $a_0 \neq 0$ we $b_0 \neq 0$.

$$\lim_{x\to\infty} R(x) = \begin{cases} \infty, & \text{eger } n > m; \\ \dfrac{a_0}{b_0}, & \text{eger } n = m; \\ 0, & \text{eger } n < m \text{ bolsa,} \end{cases}$$

deňligi subut etmeli.

274. Goý, $R(x) = \dfrac{P(x)}{Q(x)}$ bolsun, bu ýerde $P(x)$ we $Q(x)$ köpagzalar we $P(a)=Q(a)=0$ bolsun.

$$\lim_{x\to a} \frac{P(x)}{Q(x)}$$

aňlatmanyň bahalary nähili bolup biler?

Predelleri tapmaly:

275.

a) $\lim\limits_{x\to 0} \dfrac{x^2-1}{2x^2-x-1}$; b) $\lim\limits_{x\to 1} \dfrac{x^2-1}{2x^2-x-1}$; ç) $\lim\limits_{x\to \infty} \dfrac{x^2-1}{2x^2-x-1}$.

276. $\lim\limits_{x\to 0} \dfrac{(1+x)(1+2x)(1+3x)-1}{x}$.

277. $\lim\limits_{x\to 0} \dfrac{(1+x)^5-(1+5x)}{x^2+x^5}$.

278. $\lim\limits_{x\to 0} \dfrac{(1+mx)^n-(1+nx)^m}{x^2}$ (m we n – natural sanlar).

279. $\lim\limits_{x\to \infty} \dfrac{(x-1)(x-2)(x-3)(x-4)(x-5)}{(5x-1)^5}$.

280. $\lim\limits_{x\to \infty} \dfrac{(2x-3)^{20}(3x+2)^{30}}{(2x+1)^{50}}$.

281. $\lim\limits_{x\to \infty} \dfrac{(x+1)(x^2+1)\ldots(x^n+1)}{[(nx)^n+1]^{\frac{n+1}{2}}}$.

282. $\lim\limits_{x\to 3} \dfrac{x^2-5x+6}{x^2-8x+15}$.

283. $\lim\limits_{x\to 1} \dfrac{x^3-3x+2}{x^4-4x+3}$.

284. $\lim\limits_{x\to 1} \dfrac{x^4-3x+2}{x^5-4x+3}$.

285. $\lim\limits_{x\to 2} \dfrac{x^3-2x^2-4x+8}{x^4-8x^2+16}$.

286. $\lim\limits_{x\to -1} \dfrac{x^3-2x-1}{x^5-2x-1}$.

287. $\lim\limits_{x\to 2} \dfrac{(x^2-x-2)^{20}}{(x^3-12x+16)^{10}}$.

288. $\lim\limits_{x\to 1} \dfrac{x+x^2+\ldots+x^n-n}{x-1}$.

289. $\lim\limits_{x\to 1} \dfrac{x^{100}-2x+1}{x^{50}-2x+1}$.

290. $\lim\limits_{x\to 1}\dfrac{x^m-1}{x^n-1}$ (m we n – natural sanlar).

291. $\lim\limits_{x\to a}\dfrac{(x^n-a^n)-na^{n-1}(x-a)}{(x-a)^2}$ (n – natural san).

292. $\lim\limits_{x\to 1}\dfrac{x^{n+1}-(n+1)x+n}{(x-1)^2}$ (n – natural san).

293. $\lim\limits_{x\to 1}\left(\dfrac{m}{1-x^m}-\dfrac{n}{1-x^n}\right)$ (m we n – natural san).

294. $\lim\limits_{n\to\infty}\dfrac{1}{n}\left[\left(x+\dfrac{a}{n}\right)+\left(x+\dfrac{2a}{n}\right)+\ldots+\left(x+\dfrac{(n-1)a}{n}\right)\right]$.

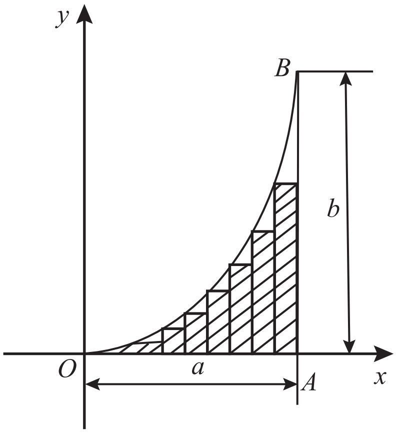
295. $\lim\limits_{n\to\infty}\dfrac{1}{n}\left[\left(x+\dfrac{a}{n}\right)^2+\left(x+\dfrac{2a}{n}\right)^2+\ldots+\left(x+\dfrac{(n-1)a}{n}\right)^2\right]$. (*Görkezme*: *1-nji bölümdäki 29-njy mysala seret*).

296. $\lim\limits_{n\to\infty}\dfrac{1^2+3^2+\ldots+(2n-1)^2}{2^2+4^2+\ldots+(2n)^2}$.

297. $\lim\limits_{n\to\infty}\left(\dfrac{1^3+2^3+\ldots+n^3}{n^3}-\dfrac{n}{4}\right)$. (*Görkezme*: *1-nji bölümdäki 31-nji mysala seret*).

298. $\lim\limits_{n\to\infty}\dfrac{1^3+4^3+7^3+\ldots+(3n-2)^3}{[1+4+7+\ldots+(3n-2)]^2}$.

299. $y = b(x/a)^2$ parabola, Ox oky we $x = a$ göni çyzyk bilen çäklenen OAB (($O(0, 0)$, $B(a, b)$) $A(a, 0)$ egri çyzykly üçburçlugyň (*11-nji surat*) meýdanyny esaslary a/n bolan içinden çyzylan gönüburçluklaryň meýdanlarynyň jeminiň $n\to\infty$ bolandaky predeli hökmünde tapmaly.



11-nji surat

Predelleri tapmaly:

300. $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}$.

301. $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}}{\sqrt{2x+1}}$.

302. $\lim\limits_{x\to 4}\dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2}$.

303. $\lim\limits_{x \to -\infty} \dfrac{\sqrt{1-x}-3}{2+\sqrt[3]{x}}$.

304. $\lim\limits_{x \to a} \dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$ $(a > 0)$.

305. $\lim\limits_{x \to 3} \dfrac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9}$.

306. $\lim\limits_{x \to -2} \dfrac{\sqrt[3]{x-6}+2}{x^3+8}$.

307. $\lim\limits_{x \to 16} \dfrac{\sqrt[4]{x}-2}{\sqrt{x}-4}$.

308. $\lim\limits_{x \to 8} \dfrac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2}$.

309. $\lim\limits_{x \to 0} \dfrac{\sqrt[n]{1+x}-1}{x}$ (n – bitin san).

310. $\lim\limits_{x \to 0} \dfrac{\sqrt{1-2x-x^2}-(1+x)}{x}$.

311. $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{8+3x-x^2}-2}{x+x^2}$.

312. $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{27+x}-\sqrt[3]{27-x}}{x+2\sqrt[3]{x^4}}$.

313. $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[3]{1+x}-\sqrt[3]{1-x}}$.

314. $\lim\limits_{x \to 7} \dfrac{\sqrt{x+2}-\sqrt[3]{x+20}}{\sqrt[4]{x+9}-2}$.

315. $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{1+\frac{x}{3}}-\sqrt[4]{1+\frac{x}{4}}}{1-\sqrt{1-\frac{x}{2}}}$.

316. $\lim\limits_{x \to 0} \dfrac{x^2}{\sqrt[5]{1+5x}-(1+x)}$.

317. $\lim\limits_{x \to 0} \dfrac{\sqrt[m]{1+\alpha x}-\sqrt[n]{1+\beta x}}{x}$ (m we n – bitin sanlar).

318. $\lim\limits_{x \to 0} \dfrac{\sqrt[m]{1+\alpha x}\sqrt[n]{1+\beta x}-1}{x}$ (m we n – bitin sanlar).

319. $P(x)=a_1x+a_2x^2+\ldots+a_nx^n$ (m – bitin san) bolanda $\lim\limits_{x \to 0} \dfrac{\sqrt[m]{1+P(x)}-1}{x} = \dfrac{a_1}{m}$ deňligi subut etmeli.

Predelleri tapmaly:

320. $\lim\limits_{x \to 1} \dfrac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1}$ (m we n – bitin sanlar).

321. $\lim\limits_{x \to 1} \left(\dfrac{3}{1-\sqrt{x}}-\dfrac{2}{1-\sqrt[3]{x}}\right)$.

322. $\lim\limits_{x \to 1} \dfrac{(1-\sqrt{x})(1-\sqrt[3]{x})\ldots(1-\sqrt[n]{x})}{(1-x)^{n-1}}$.

323. $\lim\limits_{x \to +\infty} [\sqrt{(x+a)(x+b)}-x]$.

324. $\lim\limits_{x\to+\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right).$

325. $\lim\limits_{x\to+\infty}x\left(\sqrt{x^2+2x}-2\sqrt{x^2+x}+x\right).$

326. $\lim\limits_{x\to+0}\left(\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}-\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}\right).$

327. $\lim\limits_{x\to\infty}\left(\sqrt[3]{x^3+x^2+1}-\sqrt[3]{x^3-x^2+1}\right).$

328. $\lim\limits_{x\to+\infty}\left(\sqrt[3]{x^3+3x^2}-\sqrt{x^2-2x}\right).$

329. $\lim\limits_{x\to\infty}x^{1/3}[(x+1)^{2/3}-(x-1)^{2/3}].$

330. $\lim\limits_{x\to+\infty}x^{3/2}\left(\sqrt{x+2}-2\sqrt{x+1}+\sqrt{x}\right).$

331. $\lim\limits_{x\to+\infty}\left[\sqrt[n]{(x+a_1)...(x+a_n)}-x\right].$

332. $\lim\limits_{x\to+\infty}\frac{\left(x-\sqrt{x^2-1}\right)^n+\left(x+\sqrt{x^2-1}\right)^n}{x^n}$ (n – natural san).

333. $\lim\limits_{x\to0}\frac{\left(\sqrt{1+x^2}+x\right)^n-\left(\sqrt{1+x^2}-x\right)^n}{x}$ (n – natural san).

334. $ax^2+bx+c=0$ kwadrat deňlemäniň a koeffisiýenti nola ymtylanda, b we c koeffisiýentleri hemişelik, şeýle-de, $b\neq0$ bolanda x_1 we x_2 kökleriniň üýtgeýşini derňemeli.

335. a we b hemişelik sanlaryň $\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=0$ deňligi kanagatlandyrýan bahalaryny tapmaly.

336. a_i we b_i $(i=1, 2)$ hemişelik sanlary

$$\lim_{x\to-\infty}\left(\sqrt{x^2-x+1}-a_1x-b_1\right)=0$$

we

$$\lim_{x\to+\infty}\left(\sqrt{x^2-x+1}-a_2x-b_2\right)=0$$

şertlerden tapmaly.

Predelleri tapmaly:

337. $\lim\limits_{x\to0}\frac{\sin5x}{x}.$

338. $\lim\limits_{x\to\infty}\frac{\sin x}{x}.$

339. $\lim\limits_{x\to\pi}\frac{\sin mx}{\sin nx}$ (m we n – bitin sanlar).

340. $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$.

341. $\lim\limits_{x\to 0}\frac{\mathrm{tg}x}{x}$.

342. $\lim\limits_{x\to 0}x\mathrm{ctg}3x$

343. $\lim\limits_{x\to 0}\frac{\mathrm{tg}x-\sin x}{\sin^3 x}$.

344. $\lim\limits_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin x}$.

345. $\lim\limits_{x\to 0}\frac{\cos x-\cos 3x}{x^2}$.

346. $\lim\limits_{x\to 0}\frac{1+\sin x-\cos x}{1+\sin px-\cos px}$.

347. $\lim\limits_{x\to\frac{\pi}{4}}\mathrm{tg}2x\mathrm{tg}\left(\frac{\pi}{4}-x\right)$

348. $\lim\limits_{x\to 1}(1-x)\mathrm{tg}\frac{\pi x}{2}$.

349. Deňlikleri subut etmeli:

a) $\lim\limits_{x\to a}\sin x=\sin a$;

b) $\lim\limits_{x\to a}\cos x=\cos a$;

ç) $\lim\limits_{x\to a}\mathrm{tg}x=\mathrm{tg}a \quad \left(a\neq\frac{2n-1}{2}\pi;\ n=0,\pm 1,\pm 2,\ldots\right)$.

Predelleri tapmaly:

350. $\lim\limits_{x\to a}\frac{\sin x-\sin a}{x-a}$.

351. $\lim\limits_{x\to a}\frac{\cos x-\cos a}{x-a}$.

352. $\lim\limits_{x\to a}\frac{\mathrm{tg}x-\mathrm{tg}a}{x-a}$.

353. $\lim\limits_{x\to a}\frac{\mathrm{ctg}x-\mathrm{ctg}a}{x-a}$.

354. $\lim\limits_{x\to a}\frac{\sec x-\sec a}{x-a}$.

355. $\lim\limits_{x\to a}\frac{\mathrm{cosec}\,x-\mathrm{cosec}\,a}{x-a}$.

356. $\lim\limits_{x\to 0}\frac{\sin(a+2x)-2\sin(a+x)+\sin a}{x^2}$.

357. $\lim\limits_{x\to 0}\frac{\cos(a+2x)-2\cos(a+x)+\cos a}{x^2}$.

358. $\lim\limits_{x\to 0}\frac{\mathrm{tg}(a+2x)-2\mathrm{tg}(a+x)+\mathrm{tg}a}{x^2}$.

359. $\lim\limits_{x\to 0}\frac{\mathrm{ctg}(a+2x)-2\mathrm{ctg}(a+x)+\mathrm{ctg}a}{x^2}$.

360. $\lim\limits_{x\to 0}\dfrac{\sin(a+x)\sin(a+2x)-\sin^2 a}{x}$.

361. $\lim\limits_{x\to \pi/6}\dfrac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}$.

362. $\lim\limits_{x\to 0}\dfrac{1-\cos x\cos 2x\cos 3x}{1-\cos x}$.

363. $\lim\limits_{x\to \pi/3}\dfrac{\sin\left(x-\frac{\pi}{3}\right)}{1-2\cos x}$.

364. $\lim\limits_{x\to \pi/3}\dfrac{\operatorname{tg}^3 x-3\operatorname{tg}x}{\cos\left(x+\frac{\pi}{6}\right)}$.

365. $\lim\limits_{x\to 0}\dfrac{\operatorname{tg}(a+x)\operatorname{tg}(a-x)-\operatorname{tg}^2 a}{x^2}$.

366. $\lim\limits_{x\to \pi/4}\dfrac{1-\operatorname{ctg}^3 x}{2-\operatorname{ctg}x-\operatorname{ctg}^3 x}$.

367. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\operatorname{tg}x}-\sqrt{1+\sin x}}{x^3}$.

368. $\lim\limits_{x\to 0}\dfrac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}}$.

369. $\lim\limits_{x\to 0}\dfrac{\sqrt{\cos x}-\sqrt[3]{\cos x}}{\sin^2 x}$.

370. $\lim\limits_{x\to 0}\dfrac{\sqrt{1-\cos x^2}}{1-\cos x}$.

371. $\lim\limits_{x\to 0}\dfrac{1-\sqrt{\cos x}}{1-\cos(\sqrt{x})}$.

372. $\lim\limits_{x\to 0}\dfrac{1-\cos x\sqrt{\cos 2x}\sqrt[3]{\cos 3x}}{x^2}$.

373. $\lim\limits_{x\to +\infty}(\sin\sqrt{x+1}-\sin\sqrt{x})$.

374. a) $\lim\limits_{x\to 0}\left(\dfrac{1+x}{2+x}\right)^{(1-\sqrt{x})/(1-x)}$;

b) $\lim\limits_{x\to 1}\left(\dfrac{1+x}{2+x}\right)^{(1-\sqrt{x})/(1-x)}$;

ç) $\lim\limits_{x\to +\infty}\left(\dfrac{1+x}{2+x}\right)^{(1-\sqrt{x})/(1-x)}$.

375. $\lim\limits_{x\to \infty}\left(\dfrac{x+2}{2x-1}\right)^{x^2}$.

376. $\lim\limits_{x\to \infty}\left(\dfrac{3x^2-x+1}{2x^2+x+1}\right)^{x^3/(1-x)}$.

377. $\lim\limits_{n\to \infty}\left(\sin^n\dfrac{2\pi n}{3n+1}\right)$.

378. $\lim\limits_{x\to \pi/4+0}\left[\operatorname{tg}\left(\dfrac{\pi}{8}+x\right)\right]^{\operatorname{tg}2x}$.

379. $\lim\limits_{x\to \infty}\left(\dfrac{x^2-1}{x^2+1}\right)^{\frac{x-1}{x+1}}$.

380. $\lim\limits_{x\to \infty}\left(\dfrac{x^2+1}{x^2-2}\right)^{x^2}$.

381. $\lim\limits_{x\to \infty}\left(\dfrac{x^2+2x-1}{2x^2-3x-2}\right)^{1/x}$.

382. $\lim\limits_{x\to 0}\sqrt[x]{1-2x}$.

383. $\lim\limits_{x\to \infty}\left(\dfrac{x+a}{x-a}\right)^{x}$.

384. $\lim\limits_{x\to +\infty}\left(\dfrac{a_1x+b_1}{a_2x+b_2}\right)$ $(a_1>0,\ a_2>0)$.

385. $\lim\limits_{x\to 0}(1+x^2)^{\operatorname{ctg}^2 x}$.

386. $\lim\limits_{x\to 1}(1+\sin\pi x)^{\operatorname{ctg}\pi x}$.

387. $\lim\limits_{x\to 0}\left(\dfrac{1+\operatorname{tg}x}{1+\sin x}\right)^{1/\sin x}$.

388. $\lim\limits_{x\to 0}\left(\dfrac{1+\operatorname{tg}x}{1+\sin x}\right)^{1/\sin^3 x}$.

389. $\lim\limits_{x\to a}\left(\dfrac{\sin x}{\sin a}\right)^{1/(x-a)}$.

390. $\lim\limits_{x\to 0}\left(\dfrac{\cos x}{\cos 2x}\right)^{1/x^2}$.

391. $\lim\limits_{x\to \pi/4}(\operatorname{tg}x)^{\operatorname{tg}2x}$.

392. $\lim\limits_{x\to \pi/2}(\sin x)^{\operatorname{tg}x}$.

393. $\lim\limits_{x\to 0}\left[\operatorname{tg}\left(\dfrac{\pi}{4}-x\right)\right]^{\operatorname{ctg}x}$.

394. $\lim\limits_{x\to \infty}\left(\sin\dfrac{1}{x}+\cos\dfrac{1}{x}\right)^{x}$.

395. $\lim\limits_{x\to 0}\sqrt[x]{\cos\sqrt{x}}$.

396. $\lim\limits_{n\to \infty}\left(\dfrac{n+x}{n-1}\right)^{n}$.

397. $\lim\limits_{n\to \infty}\cos^n\dfrac{x}{\sqrt{n}}$.

398. $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}$.

399. $\lim\limits_{x\to +\infty}x[\ln(x+1)-\ln x]$.

400. $\lim\limits_{x\to a}\dfrac{\ln x-\ln a}{x-a}\quad(a>0)$.

401. $\lim\limits_{x\to +\infty}[\sin\ln(x+1)-\sin\ln x]$.

402. $\lim\limits_{x\to +\infty}\dfrac{\ln(x^2-x+1)}{\ln(x^{10}+x+1)}$.

403. $\lim\limits_{x\to \infty}\left(\operatorname{tg}\dfrac{100+x^2}{1+100x^2}\right)$.

404. $\lim\limits_{x\to +\infty}\dfrac{\ln(2+e^{3x})}{\ln(3+e^{2x})}$.

405. $\lim\limits_{x\to +\infty}\dfrac{\ln(1+\sqrt{x}+\sqrt[3]{x})}{\ln(1+\sqrt[3]{x}+\sqrt[4]{x})}$.

406. $\lim\limits_{h\to 0}\dfrac{\log(x+h)+\log(x-h)-2\log x}{h^2}\quad(x>0)$.

407. $\lim\limits_{x\to 0}\dfrac{\ln\operatorname{tg}\left(\dfrac{\pi}{4}+ax\right)}{\sin bx}$.

408. $\lim\limits_{x\to 0}\dfrac{\ln\cos ax}{\ln\cos bx}$

409. $\lim\limits_{x\to 0}\left(\ln\dfrac{nx+\sqrt{1-n^2x^2}}{x+\sqrt{1-x^2}}\right)$.

410. $\lim\limits_{x\to 0}\left(\dfrac{\ln(nx+\sqrt{1-n^2x^2})}{\ln(x+\sqrt{1-x^2})}\right)$.

411. $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}\quad(a>0)$.

412. $\lim\limits_{x\to a}\dfrac{a^x-x^a}{x-a}\quad(a>0)$.

413. $\lim\limits_{x\to a}\dfrac{x^x-a^a}{x-a}\quad(a>0)$.

414. $\lim\limits_{x\to 0}(x+e^x)^{1/x}$.

415. $\lim\limits_{x\to 0}\left(\dfrac{1+x\cdot 2^x}{1+x\cdot 3^x}\right)^{\frac{1}{x^2}}$.

416. $\lim\limits_{x\to 0}\left(\dfrac{1+\sin x\cos\alpha x}{1+\sin x\cos\beta x}\right)^{\operatorname{ctg}^3 x}$.

417. $\lim\limits_{x\to 1}\dfrac{\sin(\pi x^{\alpha})}{\sin(\pi x^{\beta})}$.

418. $\lim\limits_{x\to 1}\dfrac{\sin^2(\pi\cdot 2^x)}{\ln[\cos(\pi\cdot 2^x)]}$.

419. $\lim\limits_{x\to\infty}\operatorname{tg}^n\left(\dfrac{\pi}{4}+\dfrac{1}{n}\right)$.

420. $\lim\limits_{x\to 0}\dfrac{e^{\alpha x}-e^{\beta x}}{\sin\alpha x-\sin\beta x}$.

421. $\lim\limits_{x\to a}\dfrac{x^{\alpha}-a^{\alpha}}{x^{\beta}-a^{\beta}}\quad (a>0)$.

422. $\lim\limits_{x\to b}\dfrac{a^x-a^b}{x-b}\quad (a>0)$.

423. $\lim\limits_{h\to 0}\dfrac{a^{x+h}+a^{x-h}-2a^x}{h^2}\quad (a>0)$.

424. $\lim\limits_{x\to\infty}\dfrac{(x+a)^{x+a}(x+b)^{x+b}}{(x+a+b)^{2x+a+b}}$.

425. $\lim\limits_{n\to\infty}n(\sqrt[n]{x}-1)\quad (x>0)$.

426. $\lim\limits_{n\to\infty}n^2(\sqrt[n]{x}-\sqrt[n+1]{x})\quad (x>0)$.

427. $\lim\limits_{n\to\infty}\left(\dfrac{a-1+\sqrt[n]{b}}{a}\right)^n\quad (a>0,\ b>0)$.

428. $\lim\limits_{n\to\infty}\left(\dfrac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n\quad (a>0,\ b>0)$.

429. $\lim\limits_{x\to 0}\left(\dfrac{a^x+b^x+c^x}{3}\right)^{1/x}\quad (a>0,\ b>0,\ c>0)$.

430. $\lim\limits_{x\to 0}\left(\dfrac{a^{x+1}+b^{x+1}+c^{x+1}}{a+b+c}\right)^{1/x}\quad (a>0,\ b>0,\ c>0)$.

431. $\lim\limits_{x\to 0}\left(\dfrac{a^{x^2}+b^{x^2}}{a^x+b^x}\right)^{1/x}\quad (a>0,\ b>0)$.

432. $\lim\limits_{x\to 0}\dfrac{a^{x^2}-b^{x^2}}{(a^x-b^x)^2}\quad (a>0,\ b>0)$.

433. $\lim\limits_{x\to a}\dfrac{a^{a^x}-a^{x^a}}{a^x-x^a}\quad (a>0)$.

434. a) $\lim\limits_{x\to-\infty}\dfrac{\ln(1+3^x)}{\ln(1+2^x)}$;

b) $\lim\limits_{x\to+\infty}\dfrac{\ln(1+3^x)}{\ln(1+2^x)}$.

435. $\lim\limits_{x\to+\infty}\ln(1+2^x)\ln\left(1+\dfrac{3}{x}\right)$.

436. $\lim\limits_{x\to 1}(1-x)\log_x 2$.

Deňlikleri subut etmeli:

437. $\lim_{x\to+\infty}\frac{x^n}{a^x}=0 \quad (a>1,\ n>0).$ **438.** $\lim_{x\to+\infty}\frac{\log_a x}{x^\varepsilon} \quad (a>1,\ \varepsilon>0).$

Predelleri tapmaly:

439. a) $\lim_{x\to0}\frac{\ln(x^2+e^x)}{\ln(x^4+e^{2x})}$; b) $\lim_{x\to+\infty}\frac{\ln(x^2+e^x)}{\ln(x^4+e^{2x})}$.

440. $\lim_{x\to0}\frac{\ln(1+xe^x)}{\ln(x+\sqrt{1+x^2})}$.

441. $\lim_{x\to+\infty}[(x+2)\ln(x+2)-2(x+1)\ln(x+1)+x\ln x]$.

442. $\lim_{x\to+0}\left[\ln(x\ln a)\cdot\ln\left(\frac{\ln ax}{\ln\frac{x}{a}}\right)\right] \quad (a>1).$

443. $\lim_{x\to+\infty}\left(\ln\frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2-1}}\cdot\ln^{-2}\frac{x+1}{x-1}\right)$.

444. $\lim_{x\to0}\frac{\sqrt{1+x\sin x}-1}{e^{x^2}-1}$. **445.** $\lim_{x\to0}\frac{\cos(xe^x)-\cos(xe^{-x})}{x^3}$.

446. $\lim_{x\to0}(2e^{x/(x+1)}-1)^{(x^2+1)/x}$. **447.** $\lim_{x\to1}(2-x)^{\sec(\pi x/2)}$.

448. $\lim_{x\to\pi/2}\frac{1-\sin^{\alpha+\beta}x}{\sqrt{(1-\sin^\alpha x)(1-\sin^\beta x)}} \quad (\alpha>0,\ \beta>0).$

449. (*Görkezme*: *201-nji mysala seret*).

a) $\lim_{x\to0}\frac{\mathrm{sh}x}{x}$; b) $\lim_{x\to0}\frac{\mathrm{ch}x-1}{x^2}$; ç) $\lim_{x\to0}\frac{\mathrm{th}x}{x}$.

450. $\lim_{x\to0}\frac{\mathrm{sh}^2x}{\ln(\mathrm{ch}3x)}$. (*Görkezme*: *201-nji mysala seret*).

451. $\lim_{x\to+\infty}\frac{\mathrm{sh}\sqrt{x^2+x}-\mathrm{sh}\sqrt{x^2-x}}{\mathrm{ch}x}$.

452. a) $\lim_{x\to a}\frac{\mathrm{sh}x-\mathrm{sh}a}{x-a}$; b) $\lim_{x\to a}\frac{\mathrm{ch}x-\mathrm{ch}a}{x-a}$.

453. $\lim_{x\to0}\frac{\ln\mathrm{ch}x}{\ln\cos x}$. **454.** $\lim_{x\to+\infty}(x-\ln\mathrm{ch}x)$.

455. $\lim_{x\to0}\frac{e^{\sin2x}-e^{\sin x}}{\mathrm{th}x}$. **456.** $\lim_{n\to\infty}\left(\frac{\mathrm{ch}\frac{\pi}{n}}{\cos\frac{\pi}{n}}\right)^{n^2}$.

457. $\lim\limits_{x\to\infty}\arcsin\dfrac{1-x}{1+x}$.

458. $\lim\limits_{x\to+\infty}\arccos\left(\sqrt{x^2+x}-x\right)$.

459. $\lim\limits_{x\to2}\operatorname{arctg}\dfrac{x-4}{(x-2)^2}$.

460. $\lim\limits_{x\to-\infty}\operatorname{arctg}\dfrac{x}{\sqrt{1+x^2}}$.

461. $\lim\limits_{h\to0}\dfrac{\operatorname{arctg}(x+h)-\operatorname{arctg}x}{h}$.

462. $\lim\limits_{x\to0}\dfrac{\ln\dfrac{1+x}{1-x}}{\operatorname{arctg}(1+x)-\operatorname{arctg}(1-x)}$.

463. $\lim\limits_{n\to\infty}\left[n\operatorname{arctg}\dfrac{1}{n(x^2+1)+x}\cdot\operatorname{tg}^n\left(\dfrac{\pi}{4}+\dfrac{x}{2n}\right)\right]$.

464. $\lim\limits_{x\to\infty}x\left(\dfrac{\pi}{4}-\operatorname{arctg}\dfrac{x}{x+1}\right)$.

465. $\lim\limits_{x\to+\infty}x\left(\dfrac{\pi}{2}-\arcsin\dfrac{x}{\sqrt{x^2+1}}\right)$.

466. $\lim\limits_{n\to\infty}\left[1+\dfrac{(-1)^n}{n}\right]^{\operatorname{cosec}(\pi\sqrt{1+n^2})}$.

467. $\lim\limits_{x\to0}\dfrac{1}{x^{100}}e^{-1/x^2}$.

468. $\lim\limits_{x\to+0}x\ln x$.

469. a) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+x}-x\right)$;

b) $\lim\limits_{x\to+\infty}\left(\sqrt{x^2+x}-x\right)$.

470.

a) $\lim\limits_{x\to-\infty}\left(\sqrt{1+x+x^2}-\sqrt{1-x+x^2}\right)$;

b) $\lim\limits_{x\to+\infty}\left(\sqrt{1+x+x^2}-\sqrt{1-x+x^2}\right)$.

471. $f(x)=\ln\dfrac{x+\sqrt{x^2+a^2}}{x+\sqrt{x^2+b^2}}$ funksiýa üçin tapmaly: $h=\lim\limits_{x\to+\infty}f(x)-\lim\limits_{x\to-\infty}f(x)$.

472. a) $\lim\limits_{x\to1-0}\operatorname{arctg}\dfrac{1}{1-x}$;

b) $\lim\limits_{x\to1+0}\operatorname{arctg}\dfrac{1}{1-x}$.

473. a) $\lim\limits_{x\to-0}\dfrac{1}{1+e^{1/x}}$;

b) $\lim\limits_{x\to+0}\dfrac{1}{1+e^{1/x}}$.

474. a) $\lim\limits_{x\to-\infty}\dfrac{\ln(1+e^x)}{x}$;

b) $\lim\limits_{x\to+\infty}\dfrac{\ln(1+e^x)}{x}$.

475. Subut etmeli:

a) $x\longrightarrow-\infty$ bolanda $\dfrac{2x}{1+x}\to2+0$;

b) $x\longrightarrow+\infty$ bolanda $\dfrac{2x}{1+x}\to2-0$.

476. Subut etmeli:

a) $x\longrightarrow-0$ bolanda $2^x\longrightarrow1-0$;

b) $x\longrightarrow+0$ bolanda $2^x\longrightarrow1+0$.

477. $f(x) = x + [x^2]$ funksiýanyň $f(1), f(1-0), f(1+0)$ bahalaryny hasaplamaly.

478. $f(x) = \operatorname{sgn}(\sin \pi x)$ üçin $f(n), f(n-0), f(n+0)$ $(n = 0, \pm 1, \ldots)$ bahalary hasaplamaly.

Predelleri tapmaly:

479. $\lim\limits_{x \to 0} x \sqrt{\cos \frac{1}{x}}$.

480. $\lim\limits_{x \to 0} x \left[\frac{1}{x} \right]$.

481. $\lim\limits_{n \to \infty} \sin(\pi \sqrt{n^2 + 1})$.

482. $\lim\limits_{n \to \infty} \sin^2(\pi \sqrt{n^2 + n})$.

483. $\lim\limits_{n \to \infty} \underbrace{\sin \sin \ldots \sin}_{n \ gezek} x$.

484. Eger $\lim\limits_{x \to a} \varphi(x) = A$ we $\lim\limits_{x \to A} \psi(x) = B$ predeller bar bolsa, onda $\lim\limits_{x \to a} \psi(\varphi(x)) = B$ diýmek bolarmy? Aşakdaky mysallary şu görnüşde işlemeli: $x = p/q$ bolanda $\varphi(x) = 1/q$, bu ýerde p we q özara ýönekeý bitin sanlar we irrasional x üçin $\varphi(x) = 0$; $x \neq 0$ bolanda $\psi(x) = 1$ we $x = 0$ bolanda $\psi(x) = 0$; şeýle-de, $x \to 0$.

485. Eger $f(x)$ funksiýa $(a, +\infty)$ interwalda kesgitlenen we her bir tükenikli (a, b) interwalda çäkli bolsa, onda aşakdaky deňlikleriň sagyndaky predeller bar hasap edilende, Koşiniň teoremalaryny subut etmeli:

a) $\lim\limits_{x \to +\infty} \dfrac{f(x)}{x} = \lim\limits_{x \to +\infty} [f(x+1) - f(x)]$;

b) $\lim\limits_{x \to +\infty} [f(x)]^{1/x} = \lim\limits_{x \to +\infty} \dfrac{f(x+1)}{f(x)}$ $\quad (f(x) \geq C > 0)$.

486. Eger a) $f(x)$ funksiýa $x > a$ ýaýlada kesgitlenen; b) her bir tükenikli $a < x < b$ ýaýlada çäkli; ç) $\lim\limits_{x \to +\infty} [f(x+1) - f(x)] = \infty$ bolsa, onda $\lim\limits_{x \to +\infty} \dfrac{f(x)}{x} = \infty$ deňligi subut etmeli.

487. Eger a) $f(x)$ funksiýa $x > a$ ýaýlada kesgitlenen; b) her bir tükenikli $a < x < b$ ýaýlada çäkli; ç) käbir natural n üçin tükenikli ýa-da tükeniksiz

$$\lim_{x \to +\infty} \frac{f(x+1) - f(x)}{x^n} = l$$

predel bar bolsa, onda $\lim\limits_{x \to +\infty} \dfrac{f(x)}{x^{n+1}} = \dfrac{l}{n+1}$ deňligi subut etmeli.

488. Subut etmeli:

a) $\lim\limits_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$;

b) $\lim\limits_{n \to \infty} \left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}\right) = e^x$.

489. $\lim\limits_{n\to\infty} n\sin(2\pi e n!) = 2\pi$. (*Görkezme: 2-nji bölümdäki 32-nji mysaldaky formuladan peýdalanmaly*).

Funksiýalaryň grafiklerini gurmaly:

490. a) $y = 1 - x^{100}$; b) $y = \lim\limits_{n\to\infty}(1 - x^{2n})$ $(-1 \le x \le 1)$.

491. a) $y = \dfrac{x^{100}}{1 + x^{100}}$ $(x \ge 0)$; b) $y = \lim\limits_{n\to\infty}\dfrac{x^n}{1 + x^n}(x \ge 0)$.

492. $y = \lim\limits_{n\to\infty}\dfrac{x^n - x^{-n}}{x^n + x^{-n}}$ $(x \ne 0)$. **493.** $y = \lim\limits_{n\to\infty}\sqrt{x^2 + \dfrac{1}{n^2}}$.

494. $y = \lim\limits_{n\to\infty}\sqrt[n]{1 + x^n}$ $(x \ge 0)$.

495. $y = \lim\limits_{n\to\infty}\sqrt[n]{1 + x^n + \left(\dfrac{x^2}{2}\right)^n}$ $(x \ge 0)$.

496. $y = \lim\limits_{n\to\infty}\dfrac{x^{n+2}}{\sqrt{2^{2n} + x^{2n}}}$ $(x \ge 0)$.

497. a) $y = \sin^{1000}x$; b) $y = \lim\limits_{n\to\infty}\sin^{2n}x$.

498. $y = \lim\limits_{n\to\infty}\dfrac{\ln(2^n + x^n)}{n}$ $(x \ge 0)$. **499.** $y = \lim\limits_{n\to\infty}(x-1)\operatorname{arctg}x^n$.

500. $y = \lim\limits_{n\to\infty}\sqrt[n]{1 + e^{n(x+1)}}$. **501.** $y = \lim\limits_{t\to+\infty}\dfrac{x + e^{tx}}{1 + e^{tx}}$.

502. $y = \lim\limits_{t\to x}\dfrac{1}{t - x}\ln\dfrac{t}{x}$ $(x > 0)$. **503.** $y = \lim\limits_{n\to\infty}\dfrac{x\operatorname{tg}^{2n}\dfrac{\pi x}{4} + \sqrt{x}}{\operatorname{tg}^{2n}\dfrac{\pi x}{4} + 1}$ $(x \ge 0)$.

504. $y = \lim\limits_{n\to\infty} x\operatorname{sgn}|\sin^2(n!\,\pi x)|$.

505. $\lim\limits_{n\to\infty}\sqrt[n]{|x|^n + |y|^n} = 1$ çyzygy gurmaly.

506. Eger $\lim\limits_{x\to\infty}[f(x) - (kx + b)] = 0$ bolsa, onda $y = kx + b$ göni çyzyga $y = f(x)$ funksiýanyň grafiginiň asimptotasy (ýapgytlygy) diýilýär.

Bu deňlemeden peýdalanyp, asimptotanyň bolmagynyň zerur we ýeterlik şertlerini getirip çykarmaly.

507. Aşakdaky funksiýalaryň grafikleriniň asimptotalaryny tapmaly we grafiklerini gurmaly:

a) $y = \dfrac{x^3}{x^2 + x - 2}$; ç) $y = \sqrt[3]{x^2 - x^3}$; e) $y = \ln(1 + e^x)$;

b) $y = \sqrt{x^2 + x}$; d) $y = \dfrac{xe^x}{e^x - 1}$; ä) $y = x + \arccos\dfrac{1}{x}$.

Predelleri tapmaly:

508. $\lim\limits_{n\to\infty}\left[\dfrac{x^{n+1}}{(n+1)!} + \dfrac{x^{n+2}}{(n+2)!} + ... + \dfrac{x^{2n}}{(2n)!}\right]$.

509. $\lim\limits_{n\to\infty}[(1+x)(1+x^2)(1+x^4)...(1+x^{2n})], \quad |x| < 1.$

510. $\lim\limits_{n\to\infty}\left(\cos\dfrac{x}{2}\cos\dfrac{x}{4}...\cos\dfrac{x}{2^n}\right)$.

511. Goý, $\lim\limits_{x\to 0}\dfrac{\varphi(x)}{\psi(x)} = 1$ bolsun, bu ýerde $\psi(x) > 0$ we $n \to \infty$ bolanda $\alpha_{mn} \rightrightarrows 0$ ($m = 1, 2, ...$), ýagny $m = 1, 2, ...$ we $n > n_o(\varepsilon)$ üçin $|\alpha_{mn}| < \varepsilon$ bolsun.

Deňligiň sagyndaky predeli bar hasap edip,

$$\lim_{n\to\infty}[\varphi(\alpha_{1n}) + \varphi(\alpha_{2n}) + ... + \varphi(\alpha_{mn})] = \\ = \lim_{n\to\infty}[\psi(\alpha_{1n}) + \psi(\alpha_{2n}) + ... + \psi(\alpha_{mn})] \tag{1}$$

deňligi subut etmeli.

Bu teoremadan peýdalanyp, predelleri tapmaly:

512. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\sqrt[3]{1+\dfrac{k}{n^2}} - 1\right)$.

513. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\sin\dfrac{ka}{n^2}\right)$.

514. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}(a^{k/n^2} - 1) \quad (a > 0)$.

515. $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\left(1 + \dfrac{k}{n^2}\right)$.

516. $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\cos\dfrac{ka}{n\sqrt{n}}$.

517. $x_1 = \sqrt{a},\ x_2 = \sqrt{a + \sqrt{a}},\ x_3 = \sqrt{a + \sqrt{a + \sqrt{a}}}, ... (a > 0)$ deňlikler boýunça kesgitlenýän $\{x_n\}$ yzygiderlik üçin $\lim\limits_{n\to\infty} x_n$ predeli tapmaly.

518. $x_1 = 0, x_2 = 1,\ x_n = \dfrac{1}{2}(x_{n-1} + x_{n-2})$ ($n = 2, 3, ...$) deňlikler boýunça kesgitlenýän $\{x_n\}$ yzygiderlik üçin $\lim\limits_{n\to\infty} x_n$ predeli tapmaly.

519. $\{y_n\}$ yzygiderlik $\{x_n\}$ yzygiderlik arkaly şeýle kesgitlenýär:

$$y_0 = x_0, \quad y_n = x_n - \alpha x_{n-1} \quad (n = 1, 2, ...), \quad |\alpha| < 1.$$

Eger $\lim\limits_{n\to\infty} y_n = b$ bolsa, $\lim\limits_{n\to\infty} x_n$ predeli tapmaly.

520. $x_0 = 1,\ x_n = \dfrac{1}{1 + x_{n-1}}$ $(n = 1, 2, ...)$ deňlikler boýunça kesgitlenýän $\{x_n\}$ yzygiderlik üçin $\lim\limits_{n\to\infty} x_n$ predeli tapmaly. (*Görkezme*: x_n *bilen* $x = \dfrac{1}{1+x}$ *deňlemäniň kökleriniň tapawudyna seretmeli*).

521. $y_n = y_n(x)$ $(0 \le x \le 1)$ funksiýalaryň yzygiderligi

$$y_1 = \frac{x}{2}, \qquad y_n = \frac{x}{2} - \frac{y_{n-1}^2}{2} \qquad (n = 2, 3, ...)$$

deňlikler boýunça kesgitlenýär. $\lim\limits_{n\to\infty} y_n$ predeli tapmaly.

522. $y_n = y_n(x)$ $(0 \le x \le 1)$ funksiýalaryň yzygiderligi

$$y_1 = \frac{x}{2}, \qquad y_n = \frac{x}{2} + \frac{y_{n-1}^2}{2} \qquad (n = 2, 3, ...)$$

deňlikler boýunça kesgitlenýär. $\lim\limits_{n\to\infty} y_n$ predeli tapmaly.

523. Goý, $x > 0$ we $y_n = y_{n-1}(2 - xy_{n-1})$ $(n = 1, 2, ...)$ bolsun. Eger $y_i > 0$ $(i = 0,1)$ bolsa, $\{y_n\}$ yzygiderligiň predeliniň bardygyny we $\lim\limits_{n\to\infty} y_n = \dfrac{1}{x}$ deňligi subut etmeli. (*Görkezme*: $\dfrac{1}{x} - y_n$ *tapawudy derňemeli*).

524. $x > 0$ üçin $y = \sqrt{x}$ köki tapmak üçin $y_0 > 0$ erkin alnyp,

$$y_n = \frac{1}{2}\left(y_{n-1} + \frac{x}{y_{n-1}}\right) \qquad (n = 1, 2, ...)$$

formuladan peýdalanylýar. Subut etmeli:

$$\lim_{n\to\infty} y_n = \sqrt{x}\,.$$

(*Görkezme*: $\dfrac{y_n - \sqrt{x}}{y_n + \sqrt{x}} = \left(\dfrac{y_{n-1} - \sqrt{x}}{y_{n-1} + \sqrt{x}}\right)^2$ $(n \ge 1)$ *formulany ulanmaly*).

525. Kepleriň

$$x - \varepsilon \sin x = m \qquad (0 < \varepsilon < 1) \tag{1}$$

deňlemesiniň takmynan çözüwini tapmak üçin ýakynlaşma

$$x_0 = m, \quad x_1 = m + \varepsilon \sin x_0, \ ..., \quad x_n = m + \varepsilon \sin x_{n-1}, \ ...$$

deňlikler boýunça gurulýar (yzygiderli ýakynlaşmalar usuly). $\xi = \lim\limits_{n\to\infty} x_n$ predeliň bardygyny we ξ sanyň (1) deňlemäniň ýeke-täk kökүdigini subut etmeli.

526. Eger $\omega_h[f]$ berlen $f(x)$ funksiýanyň $|x-\xi| \le h$ $(h > 0)$ kesimdäki yrgyldysy bolsa, onda

$$\omega_0[f] = \lim_{h\to 0} \omega_h[f]$$

sana $f(x)$ funksiýanyň ξ nokatdaky yrgyldysy diýilýär.

$f(0) = 0$ we $x \neq 0$ bolanda aşakdaky ýaly kesgitlenen $f(x)$ funksiýanyň $x = 0$ nokatdaky yrgyldysyny kesgitlemeli:

a) $f(x) = \sin\frac{1}{x}$;

b) $f(x) = \frac{1}{x^2}\cos^2\frac{1}{x}$;

ç) $f(x) = x\left(2 + \sin\frac{1}{x}\right)$;

d) $f(x) = \frac{1}{\pi}\operatorname{arctg}\frac{1}{x}$;

e) $f(x) = \frac{|\sin x|}{x}$;

ä) $f(x) = \frac{1}{1 + e^{1/x}}$;

f) $f(x) = (1 + |x|)^{1/x}$.

527. Goý, $f(x) = \sin\frac{1}{x}$ bolsun.

$-1 \le \alpha \le 1$ deňsizligi kanagatlandyrýan islendik α üçin $\lim\limits_{n\to\infty} f(x_n) = \alpha$ bolýan $x_n \to 0$ $(n = 1, 2, ...)$ yzygiderligi saýlap bolýandygyny subut etmeli.

528. Aşakdaky funksiýalar üçin

$$l = \varliminf_{x\to 0} f(x) \quad \text{we} \quad L = \varlimsup_{x\to 0} f(x)$$

predelleri tapmaly:

a) $f(x) = \sin^2\frac{1}{x} + \frac{2}{\pi}\operatorname{arctg}\frac{1}{x}$;

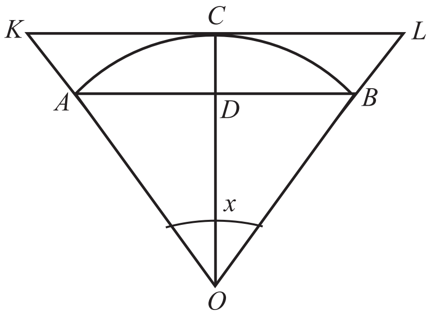
b) $f(x) = (2 - x^2)\cos\frac{1}{x}$;

ç) $f(x) = \left(1 + \cos^2\frac{1}{x}\right)^{\sec^2(1/x)}$.

529. Aşakdaky funksiýalar üçin

$$l = \varliminf_{x\to \infty} f(x) \quad \text{we} \quad L = \varlimsup_{x\to \infty} f(x),$$

predelleri tapmaly:



12-nji surat

a) $f(x) = \sin x$; ç) $f(x) = 2^{\sin x^2}$;

b) $f(x) = x^2\cos^2 x$;

d) $f(x) = \dfrac{x}{1 + x^2\sin^2 x} \quad (x \geq 0)$.

530. Merkezi $AOB = x$ burçy (*12-nji surat*) 1-nji tertipli tükeniksiz kiçi hasap edip, aşakdaky ululyklaryň tükeniksiz kiçilik tertibini kesgitlemeli: a) AB hordanyň; b) CD kesimiň; ç) AOB sektoryň meýdanynyň; d) $ABLK$ trapesiýanyň meýdanynyň; e) ABC segmentiň meýdanynyň.

531. Goý, $o(f(x))$ funksiýa $x \to a$ bolanda $f(x)$-a görä kiçi tertipli erkin funksiýa we $O(f(x))$ funksiýa $x \to a$ bolanda $f(x)$ bilen deň tertipdäki islendik funksiýa bolsun, bu ýerde $f(x) > 0$. Aşakdakylary subut etmeli:

a) $o(o(f(x))) = o(f(x))$;

b) $O(o(f(x))) = o(f(x))$;

ç) $o(O(f(x))) = o(f(x))$;

d) $O(O(f(x))) = O(f(x))$;

e) $O(f(x)) + o(f(x)) = O(f(x))$.

532. Goý, $x \to 0$ we $n > 0$ bolsun. Aşakdakylary subut etmeli:

a) $CO(x^n) = O(x^n)$ ($C \neq 0$ – hemişelik);

b) $O(x^n) + O(x^m) = O(x^n)$ $(n < m)$;

ç) $O(x^n)O(x^m) = O(x^{n+m})$.

533. Goý, $x \to +\infty$ we $n > 0$ bolsun. Aşakdakylary subut etmeli:

a) $CO(x^n) = O(x^n)$;

b) $O(x^n) + O(x^m) = O(x^n)$ $(n > m)$;

ç) $O(x^n)O(x^m) = O(x^{n+m})$.

534. Ekwiwalentligi (deňgüýçlüligi) aňladýan ~ simwolyň aşakdaky häsiýetlerini görkezmeli: 1) reflektiwlik: $\varphi(x) \sim \varphi(x)$; 2) simmetriklik: eger $\varphi(x) \sim \psi(x)$ bolsa, onda $\psi(x) \sim \varphi(x)$; 3) tranzitiwlilik: eger $\varphi(x) \sim \psi(x)$ we $\psi(x) \sim g(x)$ bolsa, onda $\varphi(x) \sim g(x)$.

535. Goý, $x \to 0$ bolsun. Aşakdakylary subut etmeli:

a) $2x - x^2 = O(x)$;

b) $x\sin\sqrt{x} = O(x^{3/2})$;

ç) $x\sin\dfrac{1}{x} = O(|x|)$;

d) $\ln x = o\left(\dfrac{1}{x^{\varepsilon}}\right) \quad (\varepsilon > 0)$;

e) $\sqrt{x + \sqrt{x + \sqrt{x}}} \sim \sqrt[8]{x}$;

ä) $\operatorname{arctg}\dfrac{1}{x} = O(1)$;

f) $(1 + x)^n = 1 + nx + o(x)$.

536. Goý, $x \to +\infty$ bolsun. Aşakdakylary subut etmeli:

a) $2x^3 - 3x^2 + 1 = O(x^3)$;

b) $\dfrac{x+1}{x^2+1} = O\left(\dfrac{1}{x}\right)$;

ç) $x + x^2\sin x = O(x^2)$;

d) $\dfrac{\operatorname{arctg} x}{1+x^2} = O\left(\dfrac{1}{x^2}\right)$;

e) $\ln x = o(x^\varepsilon)$ $(\varepsilon > 0)$;

ä) $x^p e^{-x} = o\left(\dfrac{1}{x^2}\right)$;

f) $x^2 + x\ln^{100}x \sim x^2$;

g) $\sqrt{x + \sqrt{x + \sqrt{x}}} \sim \sqrt{x}$.

537. Ýeterlik uly $x > 0$ üçin deňsizlikleri subut etmeli:

a) $x^2 + 10x + 100 < 0{,}001x^3$; b) $\ln^{1000}x < \sqrt{x}$; ç) $x^{10}e^x < e^{2x}$.

538. $x \to +\infty$ bolanda asimptotik formulany subut etmeli:

$$\sqrt{x^2 + px + q} = x + \frac{p}{2} + O\left(\frac{1}{x}\right).$$

539. Goý, $x \to 0$ bolsun. Aşakdaky funksiýalaryň hemişelik C üçin Cx^n görnüşdäki baş agzasyny görkezmeli we üýtgeýän x ululyga görä kiçilik tertibini kesgitlemeli:

a) $2x - 3x^3 + x^5$;

b) $\sqrt{1+x} - \sqrt{1-x}$;

ç) $\sqrt{1-2x} - \sqrt[3]{1-3x}$;

d) $\operatorname{tg}x - \sin x$.

540. Goý, $x \to 0$ bolsun. Tükeniksiz kiçi a) $f(x) = \dfrac{1}{\ln x}$; b) $f(x) = e^{-1/x^2}$ funksiýalary tükeniksiz kiçi x^n $(n > 0)$ bilen n-iň hiç bir bahasynda deňeşdirip bolmaýandygyny, ýagny $\lim\limits_{x \to 0} \dfrac{f(x)}{x^n} = k$ deňligiň n-iň hiç bir bahasynda ýerine ýetmeýändigini görkezmeli, bu ýerde k noldan tapawutly tükenikli ululykdyr.

541. Goý, $x \to 1$ bolsun. Aşakdaky funksiýalaryň $C(x-1)^n$ görnüşdäki baş agzasyny görkezmeli we $(x-1)$ üýtgeýäne görä kiçilik tertibini kesgitlemeli:

a) $x^3 - 3x + 2$;

b) $\sqrt[3]{1 - \sqrt{x}}$;

ç) $\ln x$;

d) $e^x - e$;

e) $x^x - 1$.

542. Goý, $x \to +\infty$ bolsun. Aşakdaky funksiýalaryň Cx^n görnüşdäki baş agzasyny görkezmeli we tükeniksiz uly üýtgeýän ululyga görä ösüş tertibini kesgitlemeli:

a) $x^2 + 100x + 10000$;

b) $\dfrac{2x^5}{x^3 - 3x + 1}$;

ç) $\sqrt[3]{x^2 - x} + \sqrt{x}$;

d) $\sqrt{1 + \sqrt{1 + \sqrt{x}}}$.

543. Goý, $x \to +\infty$ bolsun. Aşakdaky funksiýalaryň $C\left(\dfrac{1}{x}\right)^n$ görnüşdäki baş agzasyny görkezmeli we tükeniksiz kiçi $\left(\dfrac{1}{x}\right)$ üýtgeýän ululyga görä kiçilik tertibini kesgitlemeli:

a) $\dfrac{x + 1}{x^4 + 1}$;

b) $\sqrt{x + 1} - \sqrt{x}$;

ç) $\sqrt{x + 2} - 2\sqrt{x + 1} + \sqrt{x}$;

d) $\dfrac{1}{x}\sin\dfrac{1}{x}$.

544. Goý, $x \to 1$ bolsun. Aşakdaky funksiýalaryň $C\left(\dfrac{1}{x - 1}\right)^n$ görnüşdäki baş agzasyny görkezmeli we tükeniksiz uly $\left(\dfrac{1}{x - 1}\right)$ üýtgeýän ululyga görä ösüş tertibini kesgitlemeli:

a) $\dfrac{x^2}{x^2 - 1}$;

b) $\sqrt{\dfrac{1 + x}{1 - x}}$;

ç) $\dfrac{x}{\sqrt[3]{1 - x^3}}$;

d) $\dfrac{1}{\sin \pi x}$.

e) $\dfrac{\ln x}{(1 - x)^2}$.

545. Goý, $x \to +\infty$ we $f_n(x) = x^n$ $(n = 1, 2, ...)$ bolsun. Subut etmeli: 1) her bir $f_n(x)$ funksiýanyň öňündäki $f_{n-1}(x)$ funksiýadan çalt artýandygyny; 2) e^x funksiýanyň $f_n(x)$ $(n = 1, 2, ...)$ funksiýanyň her birinden çalt artýandygyny.

546. Goý, $x \to +\infty$ we $f_n = \sqrt[n]{x}$ $(n = 1, 2, ...)$ bolsun. Subut etmeli: 1) her bir $f_n(x)$ funksiýanyň öňündäki $f_{n-1}(x)$ funksiýadan haýal artýandygyny; 2) $f(x) = \ln x$ funksiýanyň $f_n(x)$ $(n = 1, 2, ...)$ funksiýanyň her birinden haýal artýandygyny.

547. Funksiýalaryň $f_1(x), f_2(x), ..., f_n(x), ...$ $(x_0 < x < +\infty)$ yzygiderliginiň nähili bolmagyna garamazdan $x \to +\infty$ bolanda $f_n(x)$ $(n = 1, 2, ...)$ funksiýalaryň her birinden çalt artýan $f(x)$ funksiýany gurup bolýandygyny subut etmeli.

§3. Üznüksiz funksiýalar

1. ***Funksiýanyň üznüksizdiginiň kesgitlenilişi.*** Goý, f funksiýa a nokadyň käbir golaý töwereginde kesgitlenen bolsun. Eger f funksiýanyň a nokatda predeli bar bolup, ol predel funksiýanyň şol nokatdaky bahasyna deň bolsa, ýagny

$$\lim_{x \to a} f(x) = f(a),$$

onda f funksiýa a nokatda üznüksiz funksiýa diýilýär.

$\lim\limits_{x \to a} x = a$ deňlik esasynda funksiýanyň a nokatda üznüksizligini aňladýan deňligi

$$\lim_{x \to a} f(x) = f(\lim_{x \to a} x)$$

görnüşde ýazmak bolar. Ol deňlik üznüksiz funksiýa üçin predeliň «lim» belgisi bilen funksiýany häsiýetlendirýän «f» belginiň ornuny çalşyryp bolýandygyny aňladýar.

Funksiýanyň predeli üçin Geýnäniň we Koşiniň kesgitlemelerini ulanyp, funksiýanyň nokatda üznüksizlik kesgitlemesini giňişleýin şeýle okamak bolar.

Eger a sana ýygnanýan $\forall\{x_n\}$ yzygiderlik üçin $\{f(x_n)\}$ yzygiderlik $f(a)$ sana ýygnanýan bolsa, onda f funksiýa a nokatda üznüksiz funksiýa diýilýär.

Eger $\forall\varepsilon>0$ üçin şeýle $0<\delta=\delta(\varepsilon)$ tapylyp, $|x-a|<\delta$ şerti kanagatlandyrýan $\forall x$ üçin $|f(x)-f(a)|<\varepsilon$ deňsizlik ýerine ýetse, onda f funksiýa a nokatda üznüksiz funksiýa diýilýär.

Bu kesgitlemäni ulanyp, funksiýanyň a nokatda üznüksizligini aňladýan $\lim\limits_{x \to a} f(x) = f(a)$ ýazgyny gysgaça şeýle ýazmak bolar:

$$(\forall\varepsilon>0)\ (\exists\ \delta>0)\ (\forall x,\ 0<|x-a|<\delta) : |f(x)-f(a)|<\varepsilon.$$

Eger $\Delta x = x - a$ we $\Delta f = f(x) - f(a) = f(a + \Delta x) - f(a)$ tapawutlar degişlilikde x-iň we f funksiýanyň a nokatdaky artymlary bolsa, onda funksiýanyň a nokatdaky üznüksizligini $\lim\limits_{\Delta x \to 0} \Delta f = 0$ görnüşde hem ýazmak bolar.

23-nji mysal. $f(x) = \sin x$ funksiýanyň $\forall a \in R$ nokatda üznüksizdigini subut etmeli.

Ç.B. Mälim bolşy ýaly, $0<x<\dfrac{\pi}{2}$ bolanda $|\sin x| \le |x|$. Şonuň esasynda $0<|x|<\dfrac{\pi}{2}$ bolanda hem $|\sin x| = \sin|x| \le |x|$ bolar. Eger $|x| \le \dfrac{\pi}{2}$ bolsa, onda $|\sin x| \le 1 < \dfrac{\pi}{2} \le |x|$. Şeýlelikde, $\forall a \in R$ üçin $\sin|x| \le |x|$. Bu deňsizligiň esasynda:

$$|\Delta f| = |\sin(a+\Delta x) - \sin a| = \left|2\sin\frac{\Delta x}{2}\cos\left(a+\frac{\Delta x}{2}\right)\right| \le$$
$$\le 2\left|\sin\frac{\Delta x}{2}\right| \le 2\frac{|\Delta x|}{2} = |\Delta x|.$$

Şonuň üçin hem $\lim\limits_{\Delta x \to o} \Delta f = 0,$ ýagny $f(x) = \sin x$ funksiýa $\forall a \in R$ nokatda üznüksizdir. **Ç.S.**

2. *Üznüksiz funksiýalaryň esasy häsiýetleri.* Funksiýanyň a nokatda üznüksiz bolmagy üçin onuň şol nokatda predeli bolmalydyr. Şonuň üçin funksiýanyň predeli üçin ýerine ýetýän ähli häsiýetler funksiýanyň üznüksizligi üçin hem ýerine ýetýändir.

Çylşyrymly funksiýanyň üznüksizlik häsiýeti şeýledir:

Eger $u=\varphi(x)$ funksiýa a nokatda üznüksiz, $y=f(u)$ funksiýa $b=\varphi(a)$ nokatda üznüksiz bolsa, onda $F(x)=f[\varphi(x)]$ çylşyrymly funksiýa a nokatda üznüksizdir.

24-nji mysal. $\forall n\in N$ üçin $f(x)=x^n$ funksiýanyň we $P_n(x)=a_o x^n+a_1x^{n-1}+\ldots+$ $+a_{n-1}x+a_n$ bitin rasional funksiýanyň $\forall x\in R$ nokatda üznüksizdigini görkezmeli.

Ç.B. Birinji funksiýanyň $\forall x\in R$ nokatda üznüksizligi $g(x)=x$ funksiýanyň üznüksizliginden, ikinjiniň üznüksizligi bolsa birinjiniň üznüksizliginden, üznüksiz funksiýalaryň esasy häsiýetlerinden gelip çykýar. **Ç.S.**

25-nji mysal. $f(x)=P_n(x)/Q_m(x)$ drob rasional funksiýanyň $Q_m(x)$ köpagzanyň köki bolmadyk $\forall x\in R$ nokatda üznüksizdigini subut etmeli.

Ç.B. Bu funksiýanyň üznüksizligi üznüksiz funksiýalaryň häsiýetlerinden, 24-nji mysaldaky funksiýanyň üznüksizliginden gelip çykýar. **Ç.S.**

26-njy mysal. $f(x)=\cos x$ funksiýanyň $\forall x\in R$ nokatda üznüksizdigini subut etmeli.

Ç.B. Bu funksiýa üznüksiz $u=\frac{\pi}{2}-x$ we $y=\sin u$ funksiýalara görä çylşyrymly $F(x)=\sin\left(\frac{\pi}{2}-x\right)=\cos x$ funksiýa hökmünde üznüksizdir. **Ç.S.**

27-nji mysal. $f(x)=\mathrm{tg}x$ funksiýa $\forall x\neq\pi/2+k\pi,\ k\in Z$ nokatda üznüksizdir.

Ç.B. Bu funksiýanyň üznüksizdigi $\sin x$ we $\cos x$ funksiýalaryň üznüksizdiginden, üznüksiz funksiýalaryň esasy häsiýetinden gelip çykýar. **Ç.S.**

28-nji mysal. $f(x)=a^x$ $(0<a\neq 1)$ görkezijili funksiýa $\forall x\in R$ nokatda üznüksizdir.

Ç.B. Ilki bilen, bu funksiýanyň $x=0$ nokatda üznüksizdigini subut edeliň. Goý, $a>1$ bolsun, onda $-\frac{1}{n}<x<\frac{1}{n}$ bolanda $a^{-1/n}<a^x<a^{1/n}$ deňsizlikler ýerine ýetýär. Bu deňsizliklerden $n\to\infty$ bolanda, $x\to 0$ we $\lim a^{1/n}=1$ bolýandygy sebäpli, $\lim\limits_{x\to o}a^x=1$ gelip çykýar. $a<1$ bolanda hem ol edil şonuň ýaly görkezilýär. Şonda $\forall b\in R$ üçin hem

$$\lim_{x\to b}a^x=\lim_{x\to b}a^b a^{x-b}=a^b\lim_{x\to b}a^{x-b}=a^b.$$

Bu ýerden b nokadyň erkinliginden $f(x)=a^x$ funksiýanyň $\forall x\in R$ nokatda üznüksizdigi gelip çykýar. **Ç.S.**

3. *Funksiýanyň birtaraplaýyn üznüksizdigi we üzülme nokatlary.* Goý, f funksiýa a nokadyň käbir sag (çep) golaý töwereginde, ýagny käbir $[a,c)((c,a])$ aralykda kesgitlenen bolsun.

Eger f funksiýanyň a nokatda sag (çep) predeli bar bolup, ol predel funksiýanyň a nokatdaky bahasyna deň bolsa, ýagny

$$\lim_{x \to a+0} f(x) = f(a+0) = f(a) \quad \left(\lim_{x \to a-0} f(x) = f(a-0) = f(a)\right),$$

onda f funksiýa a nokatda sagdan (çepden) üznüksiz funksiýa diýilýär.

Kesgitlemeden görnüşi ýaly, eger f funksiýa a nokatda hem çepden, hem sagdan üznüksiz bolsa, onda

$$f(a+0) = f(a-0) = f(a) \tag{1}$$

deňlikler ýerine ýeter, ýagny f funksiýa a nokatda üznüksiz bolar. Tersine hem dogrudyr, ýagny funksiýanyň a nokatda üznüksizliginden, onuň şol nokatda hem çepden, hem sagdan üznüksizligi gelip çykýar.

Eger f funksiýa a nokatda kesgitlenmedik bolsa ýa-da kesgitlenen bolup, şol nokatda üznüksiz bolmasa, ýagny $\lim\limits_{x \to a} f(x)$ predel ýok bolsa, ýa-da $\lim\limits_{x \to a} f(x)$ predel bar bolup $\lim\limits_{x \to a} f(x) \neq f(a)$ bolsa, onda a nokada f funksiýanyň üzülme nokady diýilýär.

Eger funksiýanyň a üzülme nokadynda birtaraplaýyn

$$f(a-0) = \lim_{x \to a-0} f(x) \quad \text{we} \quad f(a+0) = \lim_{x \to a+0} f(x) \tag{2}$$

predeller bar bolsa, onda a nokada f funksiýanyň birinji görnüşdäki üzülme nokady diýilýär, $f(a+0) - f(a-0)$ tapawuda bolsa f funksiýanyň a nokatdaky bökmesi diýilýär.

Eger $\lim\limits_{x \to a} f(x)$ predel bar bolup, f funksiýa a nokatda ýa kesgitlenmedik, ýa-da $\lim\limits_{x \to a} f(x) \neq f(a)$ bolsa, onda a nokada f funksiýanyň aýrylýan üzülme nokady diýilýär.

Eger (2) birtaraplaýyn predelleriň iň bolmanda birisi ýok ýa-da ∞-ge deň bolsa, onda a nokada f funksiýanyň ikinji görnüşdäki üzülme nokady diýilýär.

29-njy mysal. Funksiýanyň üzülme nokatlaryny tapmaly we olaryň görnüşlerini kesgitlemeli. Birinji görnüşdäki üzülme nokatlarynda funksiýanyň bökmesini hasaplamaly:

a) $f(x) = \dfrac{|x| - x}{x^2}$; b) $f(x) = \begin{cases} 1/(x-1), & x < 0, \\ (x+1)^2, & 0 \le x \le 2, \\ 1 - x, & 2 < x. \end{cases}$

Ç.B. a) $f(x) = (|x| - x)/x^2$ funksiýa san okunyň ähli $x \neq 0$ nokatlarynda kesgitlenendir. Diýmek, $x = 0$ funksiýanyň üzülme nokadydyr. Berlen funksiýany

$$f(x) = \frac{|x| - x}{x^2} = \begin{cases} 0, & x > 0, \\ -2/x, & x < 0 \end{cases}$$

görnüşde ýazyp, $\lim\limits_{x\to+0} f(x) = 0$, $\lim\limits_{x\to-0} f(x) = +\infty$ predelleri taparys. Bu ýerden görnüşi ýaly, $x = 0$ funksiýanyň ikinji görnüşdäki üzülme nokadydyr.

b) berlen funksiýanyň birtaraplaýyn predellerini tapalyň:

$$\lim_{x\to-0} f(x) = \lim_{x\to-0} \frac{1}{x-1} = -1; \quad \lim_{x\to+0} f(x) = \lim_{x\to+0} (x+1)^2 = 1;$$

$$\lim_{x\to2-0} f(x) = \lim_{x\to2-0} (x+1)^2 = 9; \quad \lim_{x\to2+0} f(x) = \lim_{x\to2+0} (1-x) = -1.$$

Bu ýerden görnüşi ýaly, $x = 0$ we $x = 2$ funksiýanyň birinji görnüşdäki üzülme nokatlarydyr we funksiýanyň şol nokatlardaky bökmesi degişlilikde

$$f(+0) - f(-0) = 1 - (-1) = 2; \quad f(2+0) - f(2-0) = -1 - 9 = -10$$

bolar. **Ç.S.**

4. *Kesimde üznüksiz funksiýalaryň häsiýetleri.* Eger funksiýa $[a, b]$ kesimiň ähli içki nokatlarynda üznüksiz bolup, a nokatda sagdan we b nokatda çepden üznüksiz bolsa, onda oňa $[a, b]$ kesimde üznüksiz funksiýa diýilýär.

Iň bolmanda bir üzülme nokady bolan funksiýa üzülýän (ýa-da üznükli) funksiýa diýilýär.

Eger funksiýa $[a, b]$ kesimiň tükenikli sany birinji görnüşdäki üzülme nokatlaryndan başga ähli nokatlarynda üznüksiz bolsa, onda oňa bölek üznüksiz funksiýa diýilýär.

Eger $\forall\varepsilon > 0$ üçin şeýle $\delta = \delta(\varepsilon) > 0$ tapylyp, $|x - x'| < \delta$ şerti kanagatlandyrýan $\forall x, x' \in X$ üçin $|f(x) - f(x')| < \varepsilon$ deňsizlik ýerine ýetse, onda f funksiýa X köplükde deňölçegli üznüksiz funksiýa diýilýär.

Kesimde üznüksiz funksiýanyň şeýle häsiýetleri bardyr:

1. Eger f funksiýa $[a, b]$ kesimde üznüksiz bolsa, onda ol funksiýa şol kesimde çäklidir we takyk ýokarky we takyk aşaky çäklerini alýar.
2. Eger $[a, b]$ kesimde üznüksiz f funksiýa üçin $f(a) \cdot f(b) < 0$ bolsa, onda onuň (a, b) interwalda iň bolmanda bir noly bardyr.
3. Eger f funksiýa $[a, b]$ kesimde üznüksiz bolup, $A = f(a) \neq f(b) = B$ bolsa, onda ol funksiýa A we B bahalaryň arasyndaky islendik C bahany alar, ýagny $(a, b) \ni c$ tapylyp, $f(c) = C$ bolar.
4. Eger f funksiýa $[a, b]$ kesimde artýan (kemelýän) we üznüksiz bolsa, onda onuň ters funksiýasy uçlary $A = f(a)$ we $B = f(b)$ bolan kesimde kesgitlenen, birbahaly, artýan (kemelýän) we üznüksiz funksiýadyr.
5. Eger f funksiýa $[a, b]$ kesimde üznüksiz bolsa, onda ol funksiýa şol kesimde deňölçegli üznüksizdir.

$[a, b]$ kesimde üznüksiz funksiýalaryň köplügi $C[a, b]$ bilen belgilenýär.

30-njy mysal. $y = \sin x$ funksiýanyň $[-1, 1]$ kesimde artýan we üznüksiz ters funksiýasynyň bardygyny görkezmeli.

Ç.B. $y = \sin x$ funksiýa $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ kesimde artýan we üznüksiz funksiýadyr. Onuň bahalar köplügi $[-1, 1]$ kesimdir. Şonuň üçin hem ol funksiýanyň $[-1, 1]$ kesimde artýan, birbahaly we üznüksiz ters funksiýasy bardyr, ony $x = \arcsin y$ arkaly belgileýärler. **Ç.S.**

Bu mysaldaky x we y ululyklaryň orunlaryny çalşyryp, $y = \arcsin x$ funksiýanyň $[-1, 1]$ kesimde üznüksizdigini alýarys. Beýleki ters trigonometrik funksiýalaryň üznüksizligi hem şular ýaly görkezilýär.

31-nji mysal. $y = a^x$ $(0 < a \neq 1)$ funksiýanyň $(0, +\infty)$ interwalda monoton we üznüksiz ters funksiýasynyň bardygyny subut etmeli.

Ç.B. $y = a^x$ funksiýa $(-\infty, \infty)$ interwalda monoton we üznüksiz, bahalar ýaýlasy $(0, +\infty)$ interwaldyr. Şonuň üçin hem ol funksiýanyň $(0, +\infty)$ interwalda monoton, birbahaly we üznüksiz ters funksiýasy bardyr, ony $x = \log_a y$ bilen belgileýärler. Bu mysalda x we y ululyklaryň orunlaryny çalşyryp, $y = \log_a x$ $(0 < a \neq 1)$ logarifmik funksiýanyň $(0, +\infty)$ interwalda üznüksizdigini alarys. **Ç.S.**

32-nji mysal. $f(x) = ax + b$ funksiýanyň san okunda deňölçegli üznüksizdigini subut etmeli.

Ç.B. Eger $\forall\varepsilon>0$ üçin $\delta=\varepsilon/|a|$ alsak, onda $\forall x, x' \in R$ üçin $|x-x'|<\delta$ bolanda $|f(x) - f(x')| = |a||x - x'| < |a|\delta = \varepsilon$ bolar, ýagny $f(x)=x$ funksiýa san okunda deňölçegli üznüksizdir. **Ç.S.**

33-nji mysal. $f(x) = \sin\frac{1}{x}$ funksiýanyň $(0, 1)$ interwalda deňölçegli üznüksiz däldigini subut etmeli.

Ç.B. Eger $(0, 1)$ interwala degişli $x_k=2/\pi(2k + 1)$ $(k = 0, 1, 2, ...)$ nokatlary alsak, onda $\varepsilon = 1$ üçin $|x_{k+1} - x_k| = 4/\pi(2k + 3)(2k + 1)<\delta$ bolanda,

$$|f(x_{k+1}) - f(x_k)| = \left|\sin\frac{\pi(2k+3)}{2} - \sin\frac{\pi(2k+1)}{2}\right| =$$
$$= |(-1)^{k+1} - (-1)^k| = 2 > \varepsilon$$

deňsizlik ýerine ýetýär, ýagny garalýan funksiýa interwalda deňölçegli üznüksiz däldir. **Ç.S.**

Gönükmeler

548. Üznüksiz $y = f(x)$ funksiýanyň grafigi berlen. Berlen a nokat we $\varepsilon > 0$ üçin $|x - a| < \delta$ bolanda $|f(x) - f(a)| < \varepsilon$ bolýan geometrik taýdan $\delta > 0$ sany görkezmeli.

549. Metaldan tarapy $x_0 = 10$ *sm* bolan kwadrat plastinka ýasamaly. Onuň x tarapy haýsy çäklerde üýtgände $y = x^2$ meýdany göz öňünde tutulan $y_0 = 100\ sm^2$ meýdandan, görkezilen sanlardan, uly bolman tapawutlanar:

a) $\pm 1\ sm^2$; b) $\pm 0{,}1\ sm^2$; ç) $\pm 0{,}01\ sm^2$; d) $\pm \varepsilon\ sm^2$?

550. Kubuň gapyrgasy 2 *m* we 3 *m* aralygynda. Onuň x gapyrgasy haýsy absolýut ýalňyşlyk bilen ölçelende onuň y göwrümini $\varepsilon\ m^3$-dan uly bolmadyk absolýut ýalňyşlyk bilen hasaplamak bolarmy:

a) $\varepsilon = 0{,}1\ m^3$; b) $\varepsilon = 0{,}01\ m^3$; ç) $\varepsilon = 0{,}001\ m^3$?

551. $x_0 = 100$ nokadyň haýsy iň uly golaý töwereginde $y = \sqrt{x}$ funksiýanyň grafiginiň ordinatasy $y_0 = 10$ ordinatadan $\varepsilon = 10^{-n}$ $(n \geq 0)$ sandan nähili kiçilikde tapawutlanar? $n = 0, 1, 2, 3$ sanlar üçin ol golaý töweregiň ölçegini kesgitlemeli.

552. «$\varepsilon - \delta$» dilde $f(x) = x^2$ funksiýanyň $x = 5$ nokatda üznüksizdigini subut etmeli.

Tablisany doldurmaly:

ε	1	0,1	0,01	0,001	...
δ					

553. Goý, $f(x) = \dfrac{1}{x}$ we $\varepsilon = 0{,}001$ bolsun.

$x_0 = 0{,}1; 0{,}01; 0{,}001, \ldots$ sanlar üçin $|x - x_0| < \delta$ bolanda $|f(x)-f(x_0)|<\varepsilon$ bolýan maksimal uly položitel $\delta = \delta(\varepsilon, x_0)$ sanlary tapmaly.

Berlen $\varepsilon = 0{,}001$ üçin $\delta > 0$ sany $(0, 1)$ interwala degişli ähli x_0 üçin bolar ýaly, ýagny islendik $x_0 \in (0, 1)$ üçin $|x - x_0| < \delta$ bolanda $|f(x)-f(x_0)|<\varepsilon$ bolar ýaly saýlap almak bolarmy?

554. «$\varepsilon - \delta$» dilde položitel manysynda aşakdaky tassyklamany okamaly: x_0 nokatda kesgitlenen $f(x)$ funksiýa şol nokatda üznüksiz däldir.

555. Goý, käbir $\varepsilon > 0$ sanlar üçin olara degişli $\delta = \delta(\varepsilon, x_0) > 0$ sanlar tapylyp, $|f(x)-f(x_0)|<\varepsilon$ deňsizlik diňe $|x - x_0| < \delta$ bolanda ýerine ýetýän bolsun.

Aşakdaky şertlerde $f(x)$ funksiýa x_0 nokatda üznüksiz diýmek bolarmy? a) ε sanlar tükenikli köplügi emele getirýär; b) ε sanlar tükeniksiz köp $\varepsilon = \dfrac{1}{2^n}$ $(n=1, 2, \ldots)$ görnüşdäki ikileýin droblary emele getirýär.

556. Goý, $f(x)=x+0{,}001\,[x]$ funksiýa berlen bolsun. Her bir $\varepsilon>0{,}001$ san üçin $|x'-x|<\delta$ bolanda $|f(x')-f(x)|<\varepsilon$ bolýan $\delta=\delta(\varepsilon,x)>0$ sany saýlap bolýandygyny, ýöne $0 < \varepsilon \leq 0{,}001$ bolanda ähli x üçin şol sany saýlap bolmaýandygyny subut etmeli.

557. Goý, her bir ýeterlik kiçi $\delta > 0$ san üçin $\varepsilon = \varepsilon(\delta, x_0) > 0$ tapylyp, $|x - x_0| < \delta$ bolanda $|f(x) - f(x_0)| < \varepsilon$ deňsizlik ýerine ýetsin. Bu ýerden $f(x)$ funksiýanyň $x = x_0$ nokatda üznüksizligi gelip çykýarmy? Bu deňsizlikler bilen $f(x)$ funksiýanyň haýsy häsiýeti aňladylýar?

558. Goý, her bir $\varepsilon > 0$ üçin $\delta = \delta(\varepsilon, x_0) > 0$ san tapylyp, $|f(x) - f(x_0)| < \varepsilon$ bolanda $|x - x_0| < \delta$ deňsizlik ýerine ýetsin. Bu ýerden $f(x)$ funksiýanyň $x = x_0$ nokatda üznüksizligi gelip çykýarmy? Bu deňsizlikler bilen $f(x)$ funksiýanyň haýsy häsiýeti aňladylýar?

559. Goý, her bir $\delta > 0$ san üçin $\varepsilon = \varepsilon(\delta, x_0) > 0$ san tapylyp, $|f(x) - f(x_0)| < \varepsilon$ bolanda $|x - x_0| < \delta$ bolsun. Bu ýerden $f(x)$ funksiýanyň $x = x_0$ nokatda üznüksizligi gelip çykýarmy? Bu deňsizlikler bilen $f(x)$ funksiýanyň haýsy häsiýeti aňladylýar? Aşakdaky mysaly şu görnüşde derňemeli:

$$f(x) = \begin{cases} \operatorname{arctg} x, & \text{eger } x \text{ rasional bolsa,} \\ \pi - \operatorname{arctg} x, & \text{eger } x \text{ irrasional bolsa.} \end{cases}$$

560. «$\varepsilon - \delta$» dilde aşakdaky funksiýalaryň üznüksizdigini subut etmeli:

a) $ax + b$; ç) x^3; e) $\sqrt[3]{x}$; f) $\cos x$;

b) x^2; d) $\sqrt{x}$; ä) $\sin x$; g) $\operatorname{arctg} x$.

Funksiýalaryň üznüksizdigini derňemeli we olaryň grafigini şekillendirmeli:

561. $f(x) = |x|$.

562. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & \text{eger } x \neq 2; \\ A, & \text{eger } x = 2. \end{cases}$

563. $f(x) = \dfrac{1}{(1 + x)^2}$, bu ýerde $x \neq -1$ bolanda we $f(-1)$ erkin san.

564. a) $f_1(x) = \left|\dfrac{\sin x}{|x|}\right|$, bu ýerde $x \neq 0$ bolanda we $f_1(0) = 1$;

b) $f_2(x) = \dfrac{\sin x}{|x|}$, bu ýerde $x \neq 0$ bolanda we $f_2(0) = 1$.

565. $f(x) = \sin\dfrac{1}{x}$, bu ýerde $x \neq 0$ bolanda we $f(0)$ erkin san.

566. $f(x) = x\sin\dfrac{1}{x}$, bu ýerde $x \neq 0$ bolanda we $f(0) = 0$.

567. $f(x) = e^{-1/x^2}$, bu ýerde $x \neq 0$ bolanda we $f(0) = 0$.

568. $f(x) = \dfrac{1}{1 + e^{\frac{1}{x-1}}}$, bu ýerde $x \neq 1$ bolanda we $f(1)$ erkin san.

569. $f(x) = x\ln x^2$, bu ýerde $x \neq 0$ bolanda we $f(0) = a$.

570. $f(x) = \operatorname{sgn} x$.

571. $f(x) = [x]$.

572. $f(x) = \sqrt{x} - [\sqrt{x}\,]$.

Funksiýalaryň üzülme nokatlaryny kesgitlemeli we olaryň görnüşlerini anyklamaly:

573. $y = \dfrac{x}{(1+x)^2}$.

574. $y = \dfrac{1+x}{1+x^3}$.

575. $y = \dfrac{x^2-1}{x^3-3x+2}$.

576. $y = \dfrac{\dfrac{1}{x} - \dfrac{1}{1+x}}{\dfrac{1}{x-1} - \dfrac{1}{x}}$.

577. $y = \dfrac{x}{\sin x}$.

578. $y = \sqrt{\dfrac{1-\cos \pi x}{4-x^2}}$.

579. $y = \cos^2 \dfrac{1}{x}$.

580. $y = \operatorname{sgn}\left(\sin \dfrac{\pi}{x}\right)$.

581. $y = \dfrac{\cos \dfrac{\pi}{x}}{\cos \dfrac{\pi}{x}}$.

582. $y = \operatorname{arctg} \dfrac{1}{x}$.

583. $y = \sqrt{x}\operatorname{arctg} \dfrac{1}{x}$.

584. $y = e^{x+1/x}$.

585. $y = \dfrac{1}{\ln x}$.

586. $y = \dfrac{1}{1-e^{\frac{x}{1-x}}}$.

Funksiýalaryň üznüksizdigini derňemeli we olaryň grafiklerini şekillendirmeli:

587. $y = \operatorname{sgn}(\sin x)$.

588. $y = x - [x]$.

589. $y = x[x]$.

590. $y = [x]\sin \pi x$.

591. $y = x^2 - [x^2]$.

592. $y = \left[\dfrac{1}{x}\right]$.

593. $y = x\left[\dfrac{1}{x}\right]$.

594. $y = \operatorname{sgn}\left(\cos \dfrac{1}{x}\right)$.

595. $y = \left[\dfrac{1}{x^2}\right]\operatorname{sgn}\left(\sin \dfrac{\pi}{x}\right)$.

596. $y = \operatorname{ctg} \dfrac{\pi}{x}$.

597. $y = \sec^2 \dfrac{1}{x}$.

598. $y = (-1)^{[x^2]}$.

599. $y = \operatorname{arctg}\left(\dfrac{1}{x} + \dfrac{1}{x-1} + \dfrac{1}{x-2}\right)$.

600. $y = \dfrac{1}{x^2\sin^2 x}$.

601. $y = \dfrac{1}{\sin(x^2)}$.

602. $y = \ln\dfrac{x^2}{(x+1)(x-3)}$.

603. $y = e^{-1/x}$.

604. $y = 1 - e^{-1/x^2}$.

605. $y = \text{th}\dfrac{2x}{1-x^2}$.

Funksiýalaryň üznüksizdigini derňemeli we olaryň grafiklerini gurmaly:

606. $y = \lim\limits_{n\to\infty}\dfrac{1}{1+x^n} \quad (x \geq 0)$.

607. $y = \lim\limits_{n\to\infty}\dfrac{n^x - n^{-x}}{n^x + n^{-x}}$.

608. $y = \lim\limits_{n\to\infty}\sqrt[n]{1+x^{2n}}$.

609. $y = \lim\limits_{n\to\infty}\cos^{2n} x$.

610. $y = \lim\limits_{n\to\infty}\dfrac{x}{1+(2\sin x)^{2n}}$.

611. $y = \lim\limits_{n\to\infty}[x\,\text{arctg}(n\,\text{ctg}x)]$.

612. $y = \lim\limits_{n\to\infty}\dfrac{x + x^2 e^{nx}}{1+e^{nx}}$.

613. $y = \lim\limits_{t\to+\infty}\dfrac{\ln(1+e^{tx})}{\ln(1+e^t)}$.

614. $y = \lim\limits_{t\to+\infty}(1+x)\,\text{th}\,tx$.

615. $f(x) = \begin{cases} 2x, & \text{eger } 0 \leq x \leq 1, \\ 2-x, & \text{eger } 1 < x \leq 2 \end{cases}$ funksiýa üznüksizmi?

616. $f(x) = \begin{cases} e^x, & \text{eger } x < 0, \\ a+x, & \text{eger } x \geq 0 \end{cases}$ funksiýa a-nyň haýsy bahalarynda üznüksiz bolar?

617. Funksiýalaryň üznüksizdigini derňemeli we olaryň üzülme nokatlarynyň görnüşini anyklamaly:

a) $f(x) = \begin{cases} x^2, & \text{eger } 0 \leq x \leq 1, \\ 2-x, & \text{eger } 1 < x \leq 2. \end{cases}$

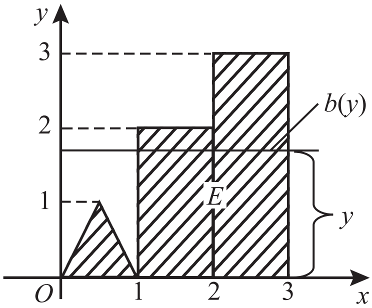
b) $f(x) = \begin{cases} x, & \text{eger } |x| \leq 1, \\ 1, & \text{eger } |x| > 1. \end{cases}$

ç) $f(x) = \begin{cases} \cos\dfrac{\pi x}{2}, & \text{eger } |x| \leq 1, \\ |x-1|, & \text{eger } |x| > 1. \end{cases}$

d) $f(x) = \begin{cases} \text{ctg}^2 \pi x, & \text{eger } x \text{ bitin däl}, \\ 0, & \text{eger } x \text{ bitin}. \end{cases}$

e) $f(x) = \begin{cases} \sin \pi x, & \text{eger } x \text{ rasional}, \\ 0, & \text{eger } x \text{ irrasional}. \end{cases}$

618. $d = d(x)$ funksiýa Ox okunyň x nokadynyň onuň $0 \leq x \leq 1$ we $2 \leq x \leq 3$ kesimlerden düzülen nokatlarynyň köplügine çenli iň ýakyn uzaklygy aňladýar. Ol funksiýanyň analitiki görnüşini tapmaly, grafigini gurmaly we üznüksizdigini derňemeli.



13-nji surat

619. E figura esasy 1 we beýikligi 1 bolan deňýanly üçburçlukdan we her biriniň esaslary 1, beýiklikleri 2 we 3 bolan iki gönüburçluklardan düzülen (*13-nji surat*). $S=S(y)$ $(0\le y<+\infty)$ funksiýa E figuranyň $Y=0$ we $Y=y$ parallel çyzyklaryň arasyndaky böleginiň meýdany $b=b(y)$ $(0\le y<+\infty)$ funksiýa bolsa, $Y=y$ göni çyzygyň E figuradaky kese-kesiginiň uzynlygy. S we b funksiýalaryň analitiki görnüşlerini tapmaly, olaryň grafiklerini gurmaly we üznüksizdigini derňemeli.

620. $D(x) = \lim\limits_{m\to\infty}\left\{\lim\limits_{n\to\infty}\cos^n(\pi m! x)\right\}$ Dirihläniň funksiýasynyň x-iň her bir bahasynda üzülýändigini subut etmeli.

621. Dirihläniň $D(x)$ funksiýasy üçin $f(x) = xD(x)$ funksiýanyň üznüksizdigini derňemeli. Onuň grafigini gurmaly.

622. $f(x) = \begin{cases} \dfrac{1}{n}, & \text{eger } x = \dfrac{m}{n}, \ m \text{ we } n \text{ özara ýönekeý sanlar,} \\ 0, & \text{eger } x \quad \text{irrasional san.} \end{cases}$

Rimanyň funksiýasynyň x-iň her bir rasional bahasynda üzülýändigini we irrasional bahasynda üznüksizdigini subut etmeli. Onuň grafigini gurmaly.

623. Aşakdaky ýaly kesgitlenen $f(x)$ funksiýanyň üznüksizdigini derňemeli: $f(x) = \dfrac{nx}{n+1}$, eger x gysgalmaýan $\dfrac{m}{n}$ $(n \ge 1)$ rasional drob bolsa, $f(x) = |x|$ eger x irrasional san bolsa, onuň grafigini gurmaly.

624. $f(x) = \dfrac{1-\cos x}{x^2}$ funksiýa $x = 0$-dan başga ähli x üçin kesgitlenen. $f(x)$ funksiýa $x = 0$ nokatda nähili kesgitlenende ol funksiýa şol nokatda üznüksiz bolar?

625. $f(1)$ san islendik saýlanyp alnanda hem $x = 1$ nokadyň $f(x) = \dfrac{1}{1-x}$ funksiýanyň üzülme nokady bolýandygyny subut etmeli.

626. $f(x)$ funksiýanyň $x = 0$ nokatda manysy ýok. Aşakdaky funksiýalar $x = 0$ nokatda $f(x)$ üznüksiz bolar ýaly $f(0)$ sany nähili kesgitlemeli:

a) $f(x) = \dfrac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}$;

e) $f(x) = \dfrac{1}{x^2}e^{-1/x2}$;

b) $f(x) = \dfrac{\operatorname{tg} 2x}{x}$;

ä) $f(x) = x^x$ $(x > 0)$;

ç) $f(x) = \sin x \sin\frac{1}{x}$; f) $f(x) = x\ln^2 x$?

d) $f(x) = (1+x)^{1/x}$;

627. Eger $x = x_0$ nokatda: a) $f(x)$ funksiýa üznüksiz, $g(x)$ funksiýa üzülýän bolsa; b) $f(x)$ we $g(x)$ funksiýalaryň ikisi hem üzülýän bolsalar, onda ol funksiýalaryň $f(x) + g(x)$ jemi $x = x_0$ nokatda hökman üzülýän funksiýa bolarmy? Degişli mysallary getiriň.

628. Eger $x = x_0$ nokatda: a) $f(x)$ funksiýa üznüksiz, $g(x)$ funksiýa üzülýän bolsa; b) $f(x)$ we $g(x)$ funksiýalaryň ikisi hem üzülýän bolsalar, onda ol funksiýalaryň $f(x) \cdot g(x)$ köpeltmek hasyly $x = x_0$ nokatda hökman üzülýän funksiýa bolarmy? Degişli mysallary getiriň.

629. Üzülýän funksiýanyň kwadraty üzülýän funksiýadyr diýip tassyklamak bolarmy?

Hemme nokatlarda üzülýän bolup, kwadraty üznüksiz funksiýa degişli mysal getiriň.

630. Berlen funksiýalar boýunça çylşyrymly $f[g(x)]$ we $g[f(x)]$ funksiýalaryň üznüksizdigini derňemeli:

a) $f(x) = \operatorname{sgn} x$ we $g(x) = 1 + x^2$;
b) $f(x) = \operatorname{sgn} x$ we $g(x) = x(1 - x^2)$;
ç) $f(x) = \operatorname{sgn} x$ we $g(x) = 1 + x - [x]$.

631. Berlen

$$f(u) = \begin{cases} u, & 0 < u \le 1; \\ 2-u, & 1 < u < 2 \end{cases} \quad \text{we} \quad \varphi(x) = \begin{cases} x, & x - \text{rasional}; \\ 2-x, & x - \text{irrasional} \end{cases}$$

funksiýalar üçin çylşyrymly $y = f(u)$, $u = \varphi(x)$ $(0 < x < 1)$ funksiýanyň üznüksizdigini derňemeli.

632. Üznüksiz $f(x)$ funksiýa üçin $F(x) = |f(x)|$ funksiýanyň üznüksizdigini subut etmeli.

633. Eger $f(x)$ funksiýa üznüksiz bolsa, onda islendik položitel c san üçin

$$f_c(x) = \begin{cases} -c, & \text{eger } f(x) < -c; \\ f(x), & \text{eger } |f(x)| \le c; \\ c, & \text{eger } f(x) > c \end{cases}$$

bolsa, funksiýanyň üznüksizdigini subut etmeli.

634. Eger $f(x)$ funksiýa $[a, b]$ kesimde üznüksiz bolsa, onda

$$m(x) = \inf_{a \le \xi \le x} \{f(\xi)\} \quad \text{we} \quad M(x) = \sup_{a \le \xi \le x} \{f(\xi)\}$$

funksiýalaryň hem şol kesimde üznüksizdigini subut etmeli.

635. Eger $f(x)$ we $g(x)$ funksiýalar üznüksiz bolsalar, onda

$$\varphi(x) = \min[f(x), g(x)] \quad \text{we} \quad \psi(x) = \max[f(x), g(x)]$$

funksiýalaryň hem üznüksizdigini subut etmeli.

636. Goý, $f(x)$ funksiýa $[a, b]$ kesimde kesgitlenen we çäkli bolsun, onda

$$m(x) = \inf_{a \le \xi < x} \{f(\xi)\} \quad \text{we} \quad M(x) = \sup_{a \le \xi < x} \{f(\xi)\}$$

funksiýalaryň $[a, b]$ kesimde çepden üznüksizdigini subut etmeli.

637. Eger $f(x)$ funksiýa $a \le x < +\infty$ aralykda üznüksiz bolsa we tükenikli $\lim\limits_{x \to +\infty} f(x)$ predel bar bolsa, onda ol funksiýanyň şol aralykda çäklidigini subut etmeli.

638. Goý, $f(x)$ funksiýa $(x_0, +\infty)$ interwalda üznüksiz we çäkli bolsun. Onda islendik T san üçin şeýle $x_n \to +\infty$ yzygiderlik tapylyp, $\lim\limits_{x \to \infty}[f(x_n + T) - f(x_n)] = 0$ deňligiň ýerine ýetýändigini subut etmeli.

639. Goý, $\varphi(x)$ we $\psi(x)$ funksiýalar $-\infty < x < +\infty$ interwalda kesgitlenen üznüksiz periodik funksiýalar we

$$\lim_{x \to +\infty} [\varphi(x) - \psi(x)] = 0$$

bolsun. Subut etmeli: $\varphi(x) \equiv \psi(x)$.

640. Çäkli monoton funksiýalaryň ähli üzülme nokatlarynyň 1-nji görnüşdäki üzülme nokatlarydygyny subut etmeli.

641. Eger $f(x)$ funksiýanyň şeýle häsiýetleri bar bolsa: 1) $[a, b]$ kesimde kesgitlenen we monoton; 2) $f(a)$ we $f(b)$ bahalaryň arasyndaky ähli bahalary alýan bolsa, onda ol funksiýanyň $[a, b]$ kesimde üznüksizdigini subut etmeli.

642. $f(x) = \sin\dfrac{1}{x - a}$, $x \ne a$ we $f(a) = 0$ deňlikler boýunça kesgitlenýän funksiýanyň islendik $[a, b]$ kesimde $f(a)$ we $f(b)$ sanlaryň arasyndaky ähli bahalary alýandygyny, ýöne $[a, b]$ kesimde üznüksiz däldigini subut etmeli.

643. Eger $f(x)$ funksiýa (a, b) interwalda üznüksiz bolsa we $x_1, x_2, ..., x_n$ şol interwalyň islendik bahalary bolsa, onda olaryň arasynda şeýle ξ tapylyp,

$$f(\xi) = \frac{1}{n}[f(x_1) + f(x_2) + ... + f(x_n)]$$

deňligiň ýerine ýetýändigini subut etmeli.

644. Goý, $f(x)$ funksiýa (a, b) interwalda üznüksiz we

$$l = \varliminf_{x \to a} f(x) \quad \text{we} \quad L = \varlimsup_{x \to a} f(x)$$

bolsun. $l \leq \lambda \leq L$ şerti kanagatlandyrýan λ sanyň nähili bolmagyna garamazdan $x_n \to a$ ($n = 1, 2, ...$) yzygiderlik tapylyp, $\lim\limits_{n\to\infty} f(x_n) = \lambda$ bolýandygyny subut etmeli.

645. Drob çyzykly $y = \dfrac{ax+b}{cx+d}$ $(ad - bc \neq 0)$ funksiýanyň ters funksiýasyny tapmaly. Haýsy halda ters funksiýa berlen funksiýa bilen gabat gelýär?

646. $y = x + [x]$ funksiýanyň $x = x(y)$ ters funksiýasyny tapmaly.

647. Kepleriň

$$y - \varepsilon \sin y = x \qquad (o \leq \varepsilon < 1)$$

deňlemesini kanagatlandyrýan ýeke-täk $y = y(x)$ $(-\infty < x < +\infty)$ üznüksiz funksiýanyň bardygyny subut etmeli.

648. Her bir hakyky k $(-\infty < k < +\infty)$ san üçin

$$\operatorname{ctg} x = kx$$

deňlemäniň $0 < x < \pi$ interwalda ýeke-täk üznüksiz $x = x(k)$ kökünüň bardygyny subut etmeli.

649. Monoton däl $y = f(x)$ $(-\infty < x < +\infty)$ funksiýanyň birbahaly ters funksiýasy bolup bilermi? Mysala seret:

$$y = \begin{cases} x, & \text{eger } x \text{ rasional;} \\ -x, & \text{eger } x \text{ irrasional bolsa.} \end{cases}$$

650. Haýsy halda $y = f(x)$ funksiýa we onuň $x = f^{-1}(y)$ ters funksiýasy şol bir funksiýadyr?

651. Üzülýän $y = (1 + x^2) \operatorname{sgn} x$ funksiýanyň ters funksiýasynyň üznüksiz funksiýa bolýandygyny subut etmeli.

652. Eger $f(x)$ funksiýa $[a, b]$ kesimde kesgitlenen we berk monoton bolsa we

$$\lim_{n\to\infty} f(x_n) = f(a) \qquad (a \leq x_n \leq b)$$

bolsa, onda $\lim\limits_{n\to\infty} x_n = a$ deňligi subut etmeli.

Funksiýalaryň ters funksiýalarynyň birbahaly üznüksiz şahalaryny kesgitlemeli:

653. $y = x^2$. **654.** $y = 2x - x^2$. **655.** $y = \dfrac{2x}{1+x^2}$.

656. $y = \sin x$. **657.** $y = \cos x$. **658.** $y = \operatorname{tg} x$.

659. $y = 1 + \sin x$ üznüksiz funksiýanyň $(0 < x < 2\pi)$ interwala degişli bahalar köplüginiň kesim bolýandygyny subut etmeli.

Deňlikleri subut etmeli:

660. $\arcsin x + \arccos x = \dfrac{\pi}{2}$.

661. $\operatorname{arctg} x + \operatorname{arctg}\dfrac{1}{x} = \dfrac{\pi}{2}\operatorname{sgn} x \ (x \neq 0)$.

662. Arktangensleri goşmagyň teoremasyny subut etmeli:

$$\operatorname{arctg} x + \operatorname{arctg} y = \operatorname{arctg}\frac{x+y}{1-xy} + \varepsilon\pi,$$

bu ýerde $\varepsilon = \varepsilon(x, y)$ funksiýa 0, 1, –1 bahalaryň haýsy-da bolsa birini alýandyr.

Berlen x üçin y-iň haýsy bahalarynda ε funksiýa üzülýän funksiýa bolup biler? Oxy tekizlikde ε funksiýanyň üznüksizlik ýaýlasyny gurmaly we funksiýanyň şol ýaýladaky bahasyny kesgitlemeli.

663. Arksinuslaryň goşmak teoremasyny subut etmeli:

$$\arcsin x + \arcsin y = (-1)^{\varepsilon} \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) + \varepsilon\pi$$

$$(|x| \leq 1,\ |y| \leq 1),$$

bu ýerde

$$xy \leq 0 \quad \text{ýa-da} \quad x^2 + y^2 \leq 1 \quad \text{bolanda} \quad \varepsilon = 0$$

we

$$xy > 0 \quad \text{ýa-da} \quad x^2 + y^2 > 1 \quad \text{bolanda} \quad \varepsilon = \operatorname{sgn} x.$$

664. Arkkosinuslary goşmagyň teoremasyny subut etmeli:

$$\arccos x + \arccos y = (-1)^{\varepsilon} \arccos\left(xy - \sqrt{1-y^2}\sqrt{1-x^2}\right) + 2\varepsilon\pi$$

$$(|x| \leq 1,\ |y| \leq 1),$$

bu ýerde

$$x + y \geq 0 \quad \text{bolanda} \quad \varepsilon = 0$$

$$x + y < 0 \quad \text{bolanda} \quad \varepsilon = 1.$$

665. Funksiýalaryň grafiklerini gurmaly:

a) $y = \arcsin x - \arcsin\sqrt{1-x^2}$;

b) $y = \arcsin\left(2x\sqrt{1-x^2}\right) - 2\arcsin x$.

666. $x = \operatorname{arctg} t$, $y = \operatorname{arcctg} t$ $(-\infty < t < +\infty)$ deňlemeler bilen berlen $y = y(x)$ funksiýany tapmaly. Ol funksiýa haýsy ýaýlada kesgitlenen?

667. Goý, $x = \operatorname{ch} t$, $y = \operatorname{sh} t$ $(-\infty < t < +\infty)$ bolsun. t parametr haýsy ýaýlada üýtgände y ululyga x-iň birbahaly funksiýasy hökmünde seretmek bolar?

668. $x = \varphi(t)$, $y = \psi(t)$ $(\alpha < t < \beta)$ deňlemeler sistemasynyň üýtgeýän y ululygy x-iň birbahaly funksiýasy hökmünde kesgitlenmeginiň zerur we ýeterlik şertleri haýsy bolar? Aşakdaky mysaldaky funksiýalar üçin zerur we ýeterlik şertlerini tapyň: $x = \sin^2 t$, $y = \cos^2 t$.

669. Haýsy şertlerde iki

$$x = \varphi(t), \quad y = \psi(t) \quad (a < t < b)$$

we

$$x = \varphi(g(\tau), y = \psi(g(\tau)) \quad (\alpha < \tau < \beta))$$

deňlemeler sistemasy şol bir $y = y(x)$ funksiýany kesgitleýär?

670. Goý, $\varphi(x)$ we $\psi(x)$ funksiýalar (a, b) interwalda kesgitlenen we

$$A = \inf_{a<x<b} \varphi(x), \quad B = \sup_{a<x<b} \varphi(x)$$

bolsun. Haýsy halda (A, B) interwalda kesgitlenen birbahaly we $a<x<b$ bolanda $\psi(x) = f(\varphi(x))$ bolýan $f(x)$ funksiýa bar?

671. Zawodyň sehinde x taraplary 1-den 10 *sm*-e çenli bahalary alyp bilýän kwadrat plastinka ýasalýar. Plastinkalaryň taraplary nähili δ gyşarmada ýasalanda (uzynlygyna bagly bolman), olaryň y meýdany taslamadakydan tapawutlanmalary ε-den kiçi bolar?

Aşakdakylar üçin san hasaplamalaryny geçirmeli:

a) $\varepsilon = 1$ sm^2; b) $\varepsilon = 0{,}01$ sm^2; ç) $\varepsilon = 0{,}0001$ sm^2.

672. Ini ε we uzynlygy δ bolan silindr şekilli mufta $y = \sqrt[3]{x}$ egri çyzyga geýdirilen we şol boýunça muftanyň oky Ox okuna parallel hereket edýär. $-10 \le x \le 10$ deňsizlik boýunça kesgitlenýän uçastogy muftanyň erkin geçmegi üçin ε-niň aşakdaky bahalarynda δ näçä deň bolmaly?

a) $\varepsilon = 1$; b) $\varepsilon = 0{,}1$; ç) $\varepsilon = 0{,}01$; d) ε ýeterlik kiçi.

673. Položitel manysynda «$\varepsilon - \delta$» dilinde şeýle tassyklamany ýazmaly: $f(x)$ funksiýa käbir köplükde (interwalda, kesimde we ş.m.) üznüksiz, ýöne şol köplükde deňölçegli üznüksiz däl.

674. $f(x) = 1/x$ funksiýanyň $(0, 1)$ interwalda üznüksizdigini, ýöne şol interwalda deňölçegli üznüksiz däldigini subut etmeli.

675. $f(x) = \sin(\pi/x)$ funksiýanyň $(0, 1)$ interwalda üznüksiz we çäklidigini, ýöne şol interwalda deňölçegli üznüksiz däldigini subut etmeli.

676. $f(x) = \sin x^2$ funksiýanyň tükeniksiz $-\infty < x < +\infty$ interwalda üznüksiz we çäklidigini, ýöne şol interwalda deňölçegli üznüksiz däldigini subut etmeli.

677. Eger $f(x)$ funksiýa $a \le x < +\infty$ ýaýlada kesgitlenen hem-de üznüksiz bolsa we $\lim\limits_{x\to+\infty} f(x)$ predel bar bolsa, onda $f(x)$ funksiýanyň şol ýaýlada deňölçegli üznüksizdigini subut etmeli.

678. Çäksiz $f(x) = x + \sin x$ funksiýanyň $-\infty < x < +\infty$ san okunda deňölçegli üznüksizdigini subut etmeli.

679. $f(x) = x^2$ funksiýa aşakdaky köplüklerde deňölçegli üznüksizmi? a) $(-l, l)$ interwalda, bu ýerde l islendik uly položitel san; b) $(-\infty, +\infty)$ interwalda.

Funksiýalaryň berlen ýaýlalarda deňölçegli üznüksizdigini derňemeli:

680. $f(x) = \dfrac{x}{4 - x^2}$ $(-1 \le x \le 1)$. **681.** $f(x) = \ln x$ $(0 < x < 1)$.

682. $f(x) = \dfrac{\sin x}{x}$ $(0 < x < \pi)$. **683.** $f(x) = e^x \cos\dfrac{1}{x}$ $(0 < x < 1)$.

684. $f(x) = \operatorname{arctg} x$ $(-\infty < x < +\infty)$. **685.** $f(x) = \sqrt{x}$ $(1 \le x < +\infty)$.

686. $f(x) = x \sin x$ $(0 \le x < +\infty)$.

687. $f(x) = \dfrac{|\sin x|}{x}$ funksiýanyň

$$J_1 = (-1 < x < 0) \quad \text{we} \quad J_2 = (0 < x < 1)$$

köplükleriň her birinde, aýratynlykda, deňölçegli üznüksizdigini, ýöne olaryň $J_1 + J_2 = \{0 < |x| < 1\}$ jeminde deňölçegli üznüksiz däldigini subut etmeli.

688. Eger $f(x)$ funksiýa $[a, c]$ we $[c, b]$ kesimleriň her birinde deňölçegli üznüksiz bolsa, onda olaryň jemi bolan $[a, b]$ kesimde hem deňölçegli üznüksizdigini subut etmeli.

689. Aşakdaky funksiýalaryň berlen aralyklarda deňölçegli üznüksiz bolmak şertini kanagatlandyrýan $\varepsilon > 0$ üçin käbir $\delta = \delta(\varepsilon)$ tapmaly:

a) $f(x) = 5x - 3$ $(-\infty < x < +\infty)$; d) $f(x) = \sqrt{x}$ $(0 \le x < +\infty)$;

b) $f(x) = x^2 - 2x - 1$ $(-2 \le x \le 5)$; e) $f(x) = 2\sin x - \cos x$ $(-\infty < x < +\infty)$;

ç) $f(x) = \dfrac{1}{x}$ $(0{,}1 \le x \le 1)$; ä) $f(x) = x \sin\dfrac{1}{x}$ $(x \ne 0)$ we $f(0)=0$ $(0 \le x \le \pi)$.

690. $[1, 10]$ kesimiň her bir böleginde $f(x) = x^2$ funksiýanyň yrgyldysy 0,0001 sandan kiçi bolar ýaly, ol kesimi näçe deň böleklere bölmek ýeterlik bolar?

691. (a, b) interwalda deňölçegli üznüksiz bolan tükenikli sany funksiýalaryň jeminiň we köpeltmek hasylynyň şol interwalda deňölçegli üznüksizdigini subut etmeli.

692. Eger çäkli monoton $f(x)$ funksiýa tükenikli ýa-da tükeniksiz (a, b) interwalda üznüksiz bolsa, onda ol funksiýanyň (a, b) interwalda deňölçegli üznüksizdigini subut etmeli.

693. Eger $f(x)$ funksiýa tükenikli (a, b) interwalda deňölçegli üznüksiz bolsa, onda

$$A = \lim_{x \to a+0} f(x) \quad \text{we} \quad B = \lim_{x \to b-0} f(x)$$

predelleriň bardygyny subut etmeli.

Bu teorema tükeniksiz (a, b) interwal üçin dogrumy?

694. Tükenikli (a, b) interwalda kesgitlenen we üznüksiz $f(x)$ funksiýany $[a, b]$ kesime üznüksiz dowam etdirmek üçin $f(x)$ funksiýanyň (a, b) interwalda deňölçegli üznüksiz bolmagynyň zerur we ýeterlikdigini subut etmeli.

695. (a, b) interwalyň $|x_1 - x_2| \le \delta$ şerti kanagatlandyrýan islendik iki x_1 we x_2 nokatlary üçin

$$\omega_f(\delta) = \sup|f(x_1) - f(x_2)|$$

funksiýa $f(x)$ funksiýanyň (a, b) interwaldaky üznüksizlik moduly diýilýär.

$f(x)$ funksiýanyň (a, b) interwalda deňölçegli üznüksiz bolmagy üçin

$$\lim_{\delta \to +0} \omega_f(\delta) = 0$$

deňligiň ýerine ýetmeginiň zerur we ýeterlikdigini subut etmeli.

696. Hemişelik C we α üçin aşakdaky funksiýalaryň üznüksizlik modulynyň

$$\omega_f(\delta) \le C\delta^\alpha$$

görnüşdäki bahalandyrmalaryny subut etmeli:

a) $f(x) = x^3 \quad (0 \le x \le 1)$;

b) $f(x) = \sqrt{x} \quad (0 \le x \le a)$ we $(a < x < +\infty)$;

ç) $f(x) = \sin x + \cos x \quad (0 \le x \le 2\pi)$.

697. x-iň we y-iň ähli hakyky bahalary üçin

$$f(x + y) = f(x) + f(y) \tag{1}$$

deňlemäni kanagatlandyrýan ýeke-täk üznüksiz $f(x)$ $(-\infty < x < +\infty)$ funksiýanyň çyzykly, birjynsly

$$f(x) = ax$$

funksiýadygyny subut etmeli, bu ýerde $a = f(1)$ erkin hemişelik.

698. Monoton we (1) deňlemäni kanagatlandyrýan $f(x)$ funksiýanyň çyzykly, birjynsly funksiýadygyny subut etmeli.

699. Ýeterlik kiçi $(-\varepsilon, \varepsilon)$ interwalda çäkli we (1) deňlemäni kanagatlandyrýan $f(x)$ funksiýanyň çyzykly, birjynsly funksiýadygyny subut etmeli.

700. x-iň we y-iň ähli bahalary üçin

$$f(x + y) = f(x) f(y) \tag{2}$$

deňlemäni kanagatlandyrýan toždestwolaýyn nola deň bolmadyk ýeke-täk üznüksiz $f(x)$ $(-\infty < x < +\infty)$ funksiýanyň görkezijili

$$f(x) = a^x$$

funksiýadygyny subut etmeli, bu ýerde $a = f(1)$ položitel hemişelik.

701. Toždestwolaýyn nola deň bolmadyk, $(0, \varepsilon)$ interwalda çäkli we (2) deňlemäni kanagatlandyrýan $f(x)$ funksiýanyň görkezijili funksiýadygyny subut etmeli.

702. x-iň we y-iň ähli položitel bahalary üçin

$$f(xy) = f(x) + f(y)$$

deňlemäni kanagatlandyrýan ýeke-täk toždestwolaýyn nola deň bolmadyk üznüksiz $f(x)$ $(-\infty < x < +\infty)$ funksiýanyň logarifm

$$f(x) = \log_a x \quad (0 < a \neq 1)$$

funksiýadygyny subut etmeli.

703. x-iň we y-iň ähli položitel bahalary üçin

$$f(xy) = f(x)\, f(y) \qquad (3)$$

deňlemäni kanagatlandyrýan ýeke-täk toždestwolaýyn nola deň bolmadyk üznüksiz $f(x)$ $(-\infty < x < +\infty)$ funksiýanyň derejeli

$$f(x) = x^a$$

funksiýadygyny subut etmeli, bu ýerde a hemişelik san.

704. x-iň we y-iň ähli hakyky bahalary üçin (3) deňlemäni kanagatlandyrýan ähli üznüksiz $f(x)$ $(-\infty < x < +\infty)$ funksiýalary tapmaly.

705. Üzülýän $f(x) = \operatorname{sgn} x$ funksiýanyň (3) deňlemäni kanagatlandyrýandygyny subut etmeli.

706. x-iň we y-iň ähli hakyky bahalary üçin

$$f(x + y) + f(x - y) = 2\, f(x)\, f(y)$$

deňlemäni kanagatlandyrýan ähli üznüksiz $f(x)$ $(-\infty < x < +\infty)$ funksiýalary tapmaly.

707. x-iň we y-iň ähli hakyky bahalary üçin

$$f(x + y) = f(x)\, f(y) - g(x)g(y), \qquad g(x + y) = f(x)\, g(y) + f(y)g(x)$$

deňlemeler sistemasyny we $f(0) = 1$ we $g(0) = 0$ şertleri kanagatlandyrýan ähli üznüksiz çäkli $f(x)$ we $g(x)$ $(-\infty < x < +\infty)$ funksiýalary tapmaly. (*Görkezme*: $F(x) = f^2(x) + g^2(x)$ *funksiýa seretmeli*).

708. Goý,

$$\Delta f(x) = f(x + \Delta x) - f(x) \qquad \text{we} \qquad \Delta^2 f(x) = \Delta\{\Delta f(x)\}$$

$f(x)$ funksiýanyň degişlilikde birinji we ikinji tertipli tükenikli tapawutlary bolsun. Eger $f(x)$ $(-\infty < x < +\infty)$ funksiýa üznüksiz we $\Delta^2 f(x) \equiv 0$ bolsa, onda ol funksiýanyň çyzyklydygyny, ýagny $f(x) = ax + b$ bolýandygyny subut etmeli, bu ýerde a we b hemişelik sanlar.

IV. FUNKSIÝANYŇ ÖNÜMI WE DIFFERENSIALY

§1. Funksiýanyň önümi düşünjesi

1. ***Funksiýanyň önüminiň kesgitlenişi.*** Eger x nokadyň käbir $U(x)$ golaý töwereginde kesgitlenen $y = f(x)$ funksiýanyň x nokatdaky $\Delta y = f(x+\Delta x) - f(x)$ artymynyň üýtgeýän x ululygyň Δx artymyna bolan

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad (1)$$

gatnaşygynyň $\Delta x \to 0$ bolanda predeli bar bolsa, onda şol predele $y = f(x)$ funksiýanyň x nokatdaky önümi diýilýär we ol $f'(x)$ ýa-da gysgaça y' bilen belgilenýär:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}. \qquad (2)$$

Kesgitlemeden peýdalanyp, mysal hökmünde, käbir ýönekeý funksiýalaryň önümlerini tapalyň.

1-nji mysal. $y = C$ hemişelik we $y = x$ funksiýalaryň önümlerini tapmaly.

Ç.B. Islendik x we Δx we $f(x) = C$ we $g(x) = x$ üçin

$$\Delta y = f(x+\Delta x) - f(x) = C - C = 0,$$
$$\Delta y = g(x+\Delta x) - g(x) = x + \Delta x - x = \Delta x.$$

Onda (2) formula esasynda

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0, \qquad C' = 0,$$

$$g'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1, \qquad x' = 1.$$ **Ç.S.**

2-nji mysal. $f(x) = \sin x$ we $g(x) = \cos x$ funksiýalaryň önümlerini tapmaly.

Ç.B. $y = \sin x$ funksiýa üçin

$$\Delta y = \sin(x+\Delta x) - \sin x = 2\sin\frac{\Delta x}{2}\cos\left(x + \frac{\Delta x}{2}\right).$$

Şonuň üçin hem (2) formula we 1-nji ajaýyp predel esasynda

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) = \cos x,$$

ýagny $(\sin x)' = \cos x$. Edil şuňa meňzeşlikde $(\cos x)' = -\sin x$. **Ç.S.**

3-nji mysal. $f(x) = \log_a x$ $(0 < a \neq 1,\ x > 0)$ funksiýanyň önümini tapmaly.

Ç.B. $y = \log_a x$ funksiýa üçin

$$\Delta y = \log_a(x + \Delta x) - \log_a x = \log_a\left(1 + \frac{\Delta x}{x}\right).$$

Şoňa görä

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\log_a\left(1 + \frac{\Delta x}{x}\right)}{\Delta x} = \frac{1}{x} \lim_{\Delta x \to 0} \frac{\log_a\left(1 + \frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}} = \frac{1}{x} \log_a e$$

deňligi alarys, ýagny $(\log_a x)' = \frac{1}{x} \log_a e = \frac{1}{x \ln a}$. **Ç.S.**

Bu mysaldan $a = e$ bolanda $(\ln x)' = \frac{1}{x}$ formula alynýar.

Eger (1) formulanyň $\Delta x \to +0$ $(\Delta x \to -0)$ bolanda predeli bar bolsa, onda ol predele $y = f(x)$ funksiýanyň x nokatdaky *sag (çep)* önümi diýilýär we ol $f'_+(x)$ $(f'_-(x))$ ýa-da $f'(x + 0)$ $(f'(x - 0))$ bilen belgilenýär:

$$f_+'(x) = \lim_{\Delta x \to +0} \frac{\Delta y}{\Delta x} \quad \left(f_-'(x) = \lim_{\Delta x \to -0} \frac{\Delta y}{\Delta x}\right). \tag{3}$$

Bu önümlere birtaraplaýyn önümler diýilýär. $f'_+(x) = f'_-(x)$ deňligiň ýerine ýetmegi f funksiýanyň x nokatda differensirlenmegine deňgüýçlüdir, şonda $f' = f'_+(x) = f'_-(x)$ deňlik ýerine ýetýär.

4-nji mysal. $f(x) = |x|$ funksiýanyň önümini tapmaly.

Ç.B. Bu funksiýa üçin

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{|x + \Delta x| - |x|}{\Delta x}.$$

Eger $x > 0$ bolsa, onda ýeterlik kiçi $|\Delta x|$ üçin $x + \Delta x > 0$ we

$$\frac{\Delta y}{\Delta x} = \frac{x + \Delta x - x}{\Delta x} = \frac{\Delta x}{\Delta x} = 1.$$

Eger $x < 0$ bolsa, onda ýeterlik kiçi $|\Delta x|$ üçin $x + \Delta x < 0$ we

$$\frac{\Delta y}{\Delta x} = \frac{-(x + \Delta x) - (-x)}{\Delta x} = -\frac{\Delta x}{\Delta x} = -1.$$

Şonuň üçin hem kesgitleme boýunça $x \neq 0$ bolanda

$$f'(x) = |x|' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \begin{cases} 1, & \text{eger } x > 0 \text{ bolsa;} \\ -1, & \text{eger } x < 0 \text{ bolsa.} \end{cases} \tag{4}$$

Eger-de $x = 0$ bolsa, onda

$$\frac{\Delta y}{\Delta x} = \frac{f(0+\Delta x) - f(0)}{\Delta x} = \frac{|\Delta x|}{\Delta x}.$$

Şonuň üçin hem

$$\lim_{\Delta x \to +0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to +0} \frac{\Delta x}{\Delta x} = 1; \qquad \lim_{\Delta x \to -0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to -0} \frac{(-\Delta x)}{\Delta x} = -1.$$

Diýmek, $f(x) = |x|$ funksiýanyň $x = 0$ nokatdaky sag önümi 1 we çep önümi –1 bolýar. Şonuň üçin ol funksiýanyň $x = 0$ nokatda önümi ýokdur.

2. *Önümiň fiziki we geometrik manysy.* Goý, material nokat göni çyzyk boýunça hereket edýän bolup, $y = f(x)$ şol nokadyň hereketiniň düzgünini, ýagny $t = 0$ wagtdan $t = x$ wagt aralygynda geçen ýoluny aňlatsyn. Onda $f'(x) = \vartheta$, ýagny onuň önümi material nokadyň x pursatdaky tizligini aňladýar.

Goý, $y = f(x)$ funksiýa x_0 nokadyň käbir golaý töwereginde kesgitlenen we üznüksiz bolsun. Onda ol funksiýanyň grafigine $A(x_0, y_0)$ $(y_0 = f(x_0))$ nokatda geçirilen galtaşma çyzygynyň $k = \mathrm{tg}\alpha$ burç koeffisiýenti şol funksiýanyň x_0 nokatdaky önümine deňdir: $\mathrm{tg}\alpha = f'(x_0)$ we ol funksiýanyň x_0 nokatdaky önüminiň geometrik manysyny aňladýar.

3. *Funksiýanyň differensirlenmegi we differensirlemegiň esasy düzgünleri.* Eger funksiýanyň x nokatda önümi bar bolsa, onda ol funksiýa şol nokatda differensirlenýändir. Nokatda differensirlenýän funksiýa şol nokatda üznüksizdir. Funksiýanyň önümini tapmaklyga differensirleme diýilýär.

Eger $u = u(x)$ we $v = v(x)$ funksiýalaryň x nokatda önümleri bar bolsa, onda şol nokatda $u \pm v$, $u \cdot v$ we u/v ($v(x) \neq 0$ bolanda) funksiýalaryň hem önümleri bardyr hem-de

$$(u \pm v)' = u' \pm v'; \tag{5}$$

$$(uv)' = u'v + uv'; \tag{6}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \tag{7}$$

formulalar dogrudyr.

Hususan-da, hemişelik c san üçin

$$[cv(x)]' = cv'(x), \qquad \left[\frac{c}{v(x)}\right]' = -\frac{cv'(x)}{v^2(x)} \qquad (v(x) \neq 0).$$

5-nji mysal. $f(x) = \mathrm{tg}x$, $(x \neq \pi/2 + k\pi,\ k \in Z)$ funksiýanyň önümini tapmaly.

Ç.B. Onuň üçin $\mathrm{tg}x = \sin x/\cos x$ formuladan, $\sin x$, $\cos x$ funksiýalaryň önümleriniň formulalaryndan we (7) formuladan peýdalanarys:

$$(\mathrm{tg}x)' = \left[\frac{\sin x}{\cos x}\right]' = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x} = \frac{1}{\cos^2 x}. \quad \textbf{Ç.S.}$$

Şuňa meňzeşlikde, $x \neq k\pi,\ k \in Z$ üçin

$$(\mathrm{ctg}\ x)' = \left[\frac{\cos x}{\sin x}\right]' = \frac{(\cos x)'\sin x - \cos x(\sin x)'}{\sin^2 x} = -\frac{1}{\sin^2 x}.$$

4. *Çylşyrymly, anyk däl we ters funksiýanyň önümi.* Eger $u=\varphi(x)$ funksiýanyň x nokatda, $y = f(u)$ funksiýanyň bolsa u nokatda önümi bar bolsa, onda çylşyrymly $y = f[\varphi(x)]$ funksiýanyň x nokatda önümi bardyr we ol önüm üçin şeýle formula dogrudyr:

$$y'(x) = f'(u) \cdot u'(x) \qquad (y'_x = f'_u \cdot u'_x). \tag{8}$$

6-njy mysal. $y = \mathrm{tg}(8x - 7)$ funksiýanyň önümini tapmaly.

Ç.B. $u = 8x - 7$ we $y = \mathrm{tg}u$ funksiýalaryň önümleriniň barlygyndan peýdalanyp, (8) formulanyň esasynda taparys:

$$y'(x) = (\mathrm{tg}u)' \cdot (8x - 7)' = \frac{1}{\cos^2 u} \cdot 8 = \frac{8}{\cos^2(8x - 7)}. \quad \textbf{Ç.S.}$$

Eger $y=y(x)$ funksiýa $F(x, y)=0$ deňleme arkaly anyk däl görnüşde berlen bolsa, onda $F(x, y)$ funksiýa x-a görä çylşyrymly funksiýa hökmünde garap, $y'=y'(x)$ önümi $[F(x, y)]'_x = 0$ deňlikden tapmak bolar.

7-nji mysal. $x^2y^3 + \cos y = 0$ deňleme bilen anyk däl görnüşde berlen $y=y(x)$ funksiýanyň $y'=y'(x)$ önümini tapmaly.

Ç.B. Deňlemäniň çep bölegine x ululyga görä çylşyrymly funksiýa hökmünde garap, (5) we (6) düzgünleri we (8) formulany ulanyp taparys:

$$2xy^3 + 3x^2y^2y' - \sin y \cdot y' = 0, \qquad y' = \frac{2xy^3}{\sin y - 3x^2y^2}. \quad \textbf{Ç.S.}$$

Goý, $y = f(x)$ funksiýa x_0 nokadyň käbir golaý töwereginde üznüksiz we artýan (ýa-da kemelýän) funksiýa bolsun. Ondan başga-da ol funksiýanyň x_0 nokatda noldan tapawutly önümi bar bolsun, onda $y = f(x)$ funksiýa ters bolan $x = f^{-1}(y)$ funksiýanyň $y_0 = f(x_0)$ nokatda önümi bardyr we ol önüm

$$[f^{-1}(y_0)]' = \frac{1}{f'(x_0)} \tag{9}$$

formula boýunça tapylýar.

8-nji mysal. $y = a^x$ $(0 < a \neq 1)$ görkezijili funksiýanyň önümini tapmaly.

Ç.B. San okunda kesgitlenen bu görkezijili funksiýa $(0, +\infty)$ interwalda kesgitlenen $x = \log_a y$ funksiýanyň ters funksiýasydyr. Şonuň üçin (9) formula we 3-nji mysal esasynda taparys:

$$\left(a^{x}\right)' = \frac{1}{\left(\log_a y\right)'} = \frac{1}{\frac{1}{y}\log_a e} = \frac{y}{\log_a e} = \frac{a^{x}}{\log_a e} = a^{x}\ln a. \quad \textbf{Ç.S.}$$

Bu formuladan $a = e$ bolanda $(e^x)' = e^x$ formulany alarys.

9-njy mysal. $y = \arcsin x$ funksiýanyň önümini tapmaly.

Ç.B. $(-1,\ 1)$ interwalda kesgitlenen $y = \arcsin x$ funksiýa $(-\pi/2, \pi/2)$ interwalda kesgitlenen $x = \sin y$ funksiýanyň ters funksiýasydyr. Şonuň üçin (9) formula we 2-nji mysal esasynda

$$(\arcsin x)' = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}. \quad \textbf{Ç.S.}$$

Edil şuňa meňzeşlikde beýleki ters trigonometrik funksiýalaryň önümleri tapylýar:

$$(\arccos x)' = \frac{1}{(\cos y)'} = \frac{1}{-\sin y} = \frac{1}{-\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}};$$

$$(\operatorname{arctg} x)' = \frac{1}{(\operatorname{tg} y)'} = \frac{1}{1+\operatorname{tg}^2 y} = \frac{1}{1+x^2};$$

$$(\operatorname{arcctg} x)' = \frac{1}{(\operatorname{ctg} y)'} = \frac{1}{-(1+\operatorname{ctg}^2 y)} = -\frac{1}{1+x^2}.$$

5. *Käbir elementar funksiýalaryň önümleri.* Ýokarda getirilen düzgünlerden, formulalardan we mysallardan peýdalanyp, käbir elementar funksiýalaryň önümleriniň tapylyşyny subut etmeli.

10-njy mysal. $y = x^{\alpha}$ derejeli funksiýanyň önümini tapmaly.

Ç.B. Funksiýany $y = x^{\alpha} = e^{\alpha \ln x}$ görnüşde ýazyp, oňa çylşyrymly funksiýa hökmünde garalyň. Onda $u = \alpha \ln x$ we $y = e^u$ funksiýalaryň önümleriniň barlygyndan peýdalanyp, (8) formula esasynda derejeli funksiýanyň önümini taparys:

$$y' = \left(e^{\alpha \ln x}\right)' = \left(e^{u}\right)'(\alpha \ln x)' = e^{u}\frac{\alpha}{x} = e^{\alpha \ln x}\frac{\alpha}{x} = x^{\alpha}\frac{\alpha}{x} = \alpha x^{\alpha-1},$$

$$(x^{\alpha})' = \alpha x^{\alpha-1}. \quad \textbf{Ç.S.}$$

Bu formuladan $\alpha = -1$, $\alpha = 1/2$ we $\alpha = m/n$ bolanda, $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$, $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$ we $\left(x^{m/n}\right)' = \left(\sqrt[n]{x^m}\right)' = \frac{m}{n}x^{\frac{m}{n}-1} = \frac{m\sqrt[n]{x^{m-n}}}{n}$ formulalar alynýar.

11-nji mysal. $y = u(x)^{v(x)}$ dereje görkezijili funksiýanyň önümini tapmaly.

Ç.B. Eger $u(x)$ we $v(x)$ funksiýalaryň x nokatda önümleri bar bolsa, onda $u^v = e^{v\ln u}$ funksiýanyň hem önümi bardyr we çylşyrymly funksiýanyň önüminiň formulasy esasynda

$$(u^v)' = (e^{v\ln u})' = e^{v\ln u}(v\ln u)' = u^v\left(v'\ln u + v\frac{u'}{u}\right).$$ **Ç.S.**

12-nji mysal. $y = \text{sh}x$ funksiýanyň önümini tapmaly.

Ç.B. $(e^x)' = e^x$ we (9) formula hem-de çylşyrymly funksiýanyň önüminiň formulasy boýunça taparys:

$$(\text{sh}x)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{1}{2}[(e^x)' - (e^{-x})'] = \frac{1}{2}(e^x + e^{-x}) = \text{ch}x.$$ **Ç.S.**

Şuňa meňzeşlikde beýleki giperbolik funksiýalaryň önümleri tapylýar:

$$(\text{ch}x)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{1}{2}[(e^x)' + (e^{-x})'] = \frac{1}{2}(e^x - e^{-x}) = \text{sh}x;$$

$$(\text{th}x)' = \left(\frac{\text{sh}x}{\text{ch}x}\right)' = \frac{(\text{sh}x)'\text{ch}x - \text{sh}x(\text{ch}x)'}{\text{ch}^2 x} = \frac{1}{\text{ch}^2 x};$$

$$(\text{cth}x)' = \left(\frac{\text{ch}x}{\text{sh}x}\right)' = \frac{(\text{ch}x)'\text{sh}x - \text{ch}x(\sin x)'}{\text{sh}^2 x} = -\frac{1}{\text{sh}^2 x}.$$

6. *Logarifmik funksiýanyň önüminiň tapylyşy.* Ilki $y = \log_a|x|$ funksiýanyň islendik $x \neq 0$ nokatdaky önümini tapalyň. $u = |x|$ funksiýanyň islendik $x \neq 0$ nokatdaky we $y = \log_a u$ funksiýanyň islendik u ($u > 0$) nokatdaky önümleriniň formulasy esasynda

$$y' = \left(\log_a|x|\right)' = (\log_a u)'\left(|x|\right)' = \frac{1}{u\ln a}|x|' = \frac{|x|'}{|x|\ln a} = \frac{1}{x\ln a},$$

çünki $x > 0$ bolanda, $|x|' = 1$ we $x < 0$ bolanda, $|x|' = -1$.

Hususan-da, $y' = (\ln|x|)' = 1/x$. Bu formulany ulanyp, çylşyrymly $y = \ln|f(x)|$ funksiýanyň önümini tapalyň:

$$y' = \left(\ln|f(x)|\right)' = \left(\ln|u|\right)' f'(x) = \frac{f'(x)}{f(x)}.$$

Oňa funksiýanyň logarifmik önümi diýilýär we ol $(\ln f(x))' = \dfrac{f'(x)}{f(x)}$ görnüşde ýazylýar.

13-nji mysal. $y = (x^2 + 1)^{\sin x}$ funksiýanyň önümini tapmaly.

Ç.B. Logarifmik önümiň formulasy boýunça

$$\frac{y'}{y} = [\ln(x^2+1)^{\sin x}]' = [\sin x \ln(x^2+1)]'$$

bolar. Onda ýokarda görkezilen mysallary we düzgünleri ulanyp,

$$[\sin x \ln(x^2+1)]' = \cos x \ln(x^2+1) + \frac{2x\sin x}{x^2+1}$$

önümi taparys. Şonuň üçin hem berlen funksiýanyň önümi şeýle tapylýar:

$$y' = y[\ln(x^2+1)^{\sin x}]' = (x^2+1)^{\sin x}\left[\cos x \ln(x^2+1) + \frac{2x\sin x}{x^2+1}\right].$$ **Ç.S.**

Käbir funksiýalaryň önümleri tapylanda ýokarda getirilen düzgünleri we mysallary ulanyp, önümiň kesgitlemesinden hem peýdalanmaly bolýar. Beýle ýagdaý, köplenç, funksiýa birnäçe formula arkaly berlende bolýar.

14-nji mysal. $f(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0 \end{cases}$ funksiýanyň önümini tapmaly.

Ç.B. Eger $x \neq 0$ bolsa, onda funksiýanyň önümi şeýle tapylýar:

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}.$$

Funksiýanyň $x = 0$ nokatdaky önümini tapmak üçin bolsa önümiň kesgitlemesinden peýdalanylýar:

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} x\sin\frac{1}{x} = 0.$$ **Ç.S.**

7. ***Funksiýalaryň önümleriniň tablisasy.*** Funksiýalaryň önümleri tapylan ýokardaky mysallary bir ýere toplap, funksiýalaryň önümlerini tapmak üçin aşakdakylary alarys:

1. $(C)' = 0,\ C = const.$
2. $(x^p)' = px^{p-1},\ p \in R,\ x > 0.$
   $(x^n)' = nx^{n-1},\ n \in N,\ x \in R.$
3. $(a^x)' = a^x\ln a,\ 0 < a \neq 1,\ x \in R;\ (e^x)' = e^x.$
4. $(\log_a x)' = \frac{1}{x\ln a},\ 0 < a \neq 1,\ x > 0.$

   $(\log_a |x|)' = \frac{1}{x\ln a},\ 0 < a \neq 1,\ x \neq 0.$

   $(\ln x)' = \frac{1}{x},\ x > 0.$
5. $(\sin x)' = \cos x,\ x \in R.$

6. $(\cos x)' = -\sin x, \ \ x \in R.$

7. $(\mathrm{tg}x)' = \dfrac{1}{\cos^2 x}, \ \ x \neq \dfrac{\pi}{2} + \pi n, \ \ n \in Z.$

8. $(\mathrm{ctg}x)' = -\dfrac{1}{\sin^2 x}, \ \ x \neq \pi n, \ \ n \in Z.$

9. $(\arcsin x)' = \dfrac{1}{\sqrt{1 - x^2}}, \ \ |x| < 1.$

10. $(\arccos x)' = -\dfrac{1}{\sqrt{1 - x^2}}, \ \ |x| < 1.$

11. $(\mathrm{arctg}x)' = \dfrac{1}{1 + x^2}, \ \ x \in R.$

12. $(\mathrm{arcctg}x)' = -\dfrac{1}{1 + x^2}, \ \ x \in R.$

13. $(\mathrm{sh}x)' = \mathrm{ch}x, \ \ x \in R.$

14. $(\mathrm{ch}x)' = \mathrm{sh}x, \ \ x \in R.$

15. $(\mathrm{th}x)' = \dfrac{1}{\mathrm{ch}^2 x}, \ \ x \in R.$

16. $(\mathrm{cth}x)' = -\dfrac{1}{\mathrm{sh}^2 x}, \ \ x \neq 0.$

Bellik. Çylşyrymly funksiýanyň önüminiň formulasyny peýdalanyp, önümleriň ýokardaky ýaly görnüşlerini $f(x)$ funksiýanyň ornunda çylşyrymly $f[u(x)]$ funksiýa bolanda hem ýazmak bolar. Şonda $f(x)$ funksiýanyň ornunda $f[u(x)]$ funksiýa, $f'(x)$ önümiň ornunda bolsa $f'[u(x)]u'(x)$ bolar.

8. *Parametrik görnüşdäki funksiýanyň önümi.* Goý, x we y ululyklar t parametriň funksiýasy hökmünde t_0 nokadyň käbir golaý töwereginde kesgitlenen

$$x = \varphi(t), \qquad y = \psi(t)$$

parametrik görnüşdäki funksiýalar arkaly berlen bolsun. Eger ol funksiýalaryň şol golaý töwereginde önümleri bar bolup, $x = \varphi(t)$ funksiýanyň $x_0 = \varphi(t_0)$ nokadyň golaý töwereginde kesgitlenen ters funksiýasy we $\varphi'(t_0) \neq 0$ önümi bar bolsa, onda $y = y(x)$ funksiýanyň $x_0 = \varphi(t_0)$ nokatda önümi bardyr we ol aşakdaky ýaly tapylýar:

$$y'(x_0) = \frac{\psi'(t_0)}{\varphi'(t_0)}. \tag{10}$$

15-nji mysal. Parametrik görnüşde berlen

$$x = \ln \sin\frac{1}{2}t, \qquad y = \ln \sin t, \qquad 0 < t < \pi$$

funksiýanyň $y'(x)$ önümini tapmaly.

Ç.B. Ilki bilen funksiýalaryň t görä

$$x'(t) = \frac{1}{2}\frac{\cos(t/2)}{\sin(t/2)}, \qquad y'(t) = \frac{\cos t}{\sin t}$$

önümlerini tapyp, formula boýunça $y'(x)$ önümi tapmak bolar:

$$y'(x) = 2\frac{\cos t \sin(t/2)}{\sin t \cos(t/2)} = \frac{\cos t}{\cos^2(t/2)} = \frac{2\cos t}{1+\cos t}. \quad \textbf{Ç.S.}$$

Gönükmeler

1. x ululyk 1-den 1000-e çenli üýtgände $y = \lg x$ funksiýanyň argumentiniň Δx artymyny we şoňa degişli Δy artymyny kesgitlemeli.

2. x ululyk 0,01-den 0,001-e çenli üýtgände $y = \dfrac{1}{x^2}$ funksiýanyň argumentiniň Δx artymyny we şoňa degişli Δy artymyny kesgitlemeli.

3. Üýtgeýän x ululyk Δx artymy alýar. Aşakdaky fuksiýalaryň Δy artymyny kesgitlemeli:

a) $y = ax + b$; b) $y = ax^2 + bx + c$; ç) $y = a^x$.

4. Subut etmeli:

a) $\Delta[f(x) + g(x)] = \Delta f(x) + \Delta g(x)$;

b) $\Delta[f(x)g(x)] = g(x + \Delta x)\Delta f(x) + f(x)\Delta g(x)$.

5. $y = x^2$ egri çyzygyň $A(2, 4)$ we $A'[2 + \Delta x, 4 + \Delta y]$ nokatlary arkaly AA' kesiji çyzyk geçirilen. Aşakdaky hallarda ol kesiji çyzygyň burç koeffisiýentini tapmaly:

a) $\Delta x = 1$; b) $\Delta x = 0{,}1$; ç) $\Delta x = 0{,}01$; d) Δx erkin kiçi.

Berlen egri çyzyga A nokatda geçirilen galtaşma çyzygynyň burç koeffisiýenti nämä deň?

6. Ox okunyň $1 \leq x \leq 1 + h$ kesimi $y = x^2$ funksiýa bilen Oy okuna şöhlelendirilýär. Süýndürmegiň orta koeffisiýentini kesgitlemeli we aşakdaky hallar üçin san hasaplamalary geçirmeli:

a) $h = 0{,}1$; b) $h = 0{,}01$; ç) $h = 0{,}001$.

Şol şöhlelendirmede $x = 1$ näçä deň bolar?

7. Nokadyň Ox oky boýunça hereket düzgüni $x = 10t + 5t^2$ formula boýunça berilýär, bu ýerde t sekuntdaky wagt, x bolsa metrdäki uzaklyk. $20 \leq t \leq 20 + \Delta t$ wagt aralygyndaky hereketiň ortaça tizligini tapmaly we aşakdakylar üçin san hasaplamalaryny geçirmeli:

a) $\Delta t = 1$; b) $\Delta t = 0{,}1$; ç) $\Delta t = 0{,}01$.

Hereketiň tizligi t = 20 pursadynda nämä deň?

8. Funksiýanyň önüminiň kesgitlemesinden peýdalanyp, aşakdaky funksiýalaryň önümlerini tapmaly:

a) x^2; ç) $\frac{1}{x}$; e) $\sqrt[3]{x}$; f) ctg$x$; h) arccos$x$;

b) x^3; d) $\sqrt{x}$; ä) tg$x$; g) arcsin$x$; i) arctg$x$.

9. $f(x)=(x-1)(x-2)^2(x-3)^3$ funksiyanyň $f'(1), f'(2), f'(3)$ önümlerini tapmaly.

10. $f(x)=x^2\sin(x-2)$ funksiýanyň $f'(2)$ önümini tapmaly.

11. $f(x) = x + (x-1)\arcsin\sqrt{\frac{x}{x+1}}$ funksiýanyň $f'(x)$ önümini tapmaly.

12. a nokatda differensirlenýän $f(x)$ üçin $\lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a}$ predeli tapmaly.

13. Differensirlenýän $f(x)$ funksiýa we natural n san üçin, $\lim\limits_{x\to\infty} n\left[f\left(x+\frac{1}{n}\right)-f(x)\right]=f'(x)$ deňligi subut etmeli.

Funksiýalaryň önümleriniň düzgünlerinden peýdalanyp, aşakdaky funksiýalaryň önümlerini tapmaly:

14. $y = 2 + x - x^2$; $y'(0)$, $y'\left(\frac{1}{2}\right)$, $y'(1)$, $y'(-10)$ önümler nämä deň?

15. $y = \frac{x^3}{3} + \frac{x^2}{2} - 2x$. ($x$-iň haýsy bahalarynda)

a) $y'(x) = 0$; b) $y'(x) = -2$; ç) $y'(x) = 10$?

16. $y = a^5 + 5a^3x^2 - x^5$.

17. $y = \frac{ax+b}{a+b}$.

18. $y = (x-a)(x-b)$.

19. $y = (x+1)(x+2)^2(x+3)^3$.

20. $y = (x\sin\alpha + \cos\alpha)(x\cos\alpha - \sin\alpha)$.

21. $y = (1 + nx^m)(1 + mx^n)$.

22. $y = (1-x)(1-x^2)^2(1-x^3)^3$.

23. $y = (5+2x)^{10}(3-4x)^{20}$.

24. $y = \frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3}$.

25. $\left(\frac{ax+b}{cx+d}\right)' = \frac{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}{(cx+d)^2}$ formulany subut etmeli.

26. $y = \dfrac{2x}{1 - x^2}$.

27. $y = \dfrac{1 + x - x^2}{1 - x + x^2}$.

28. $y = \dfrac{x}{(1 - x)^2 (1 + x)^3}$.

29. $y = \dfrac{(2 - x^2)(3 - x^3)}{(1 - x)^2}$.

30. $y = \dfrac{(1 - x)^p}{(1 + x)^q}$.

31. $y = \dfrac{x^p (1 - x)^q}{1 + x}$.

32. $y = x + \sqrt{x} + \sqrt[3]{x}$.

33. $y = \dfrac{1}{x} + \dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt[3]{x}}$.

34. $y = \sqrt[3]{x^2} - \dfrac{2}{\sqrt{x}}$.

35. $y = x\sqrt{1 + x^2}$.

36. $y = (1 + x)\sqrt{2 + x^2}\sqrt[3]{3 + x^3}$.

37. $y = \sqrt[m+n]{(1 - x)^m (1 + x)^n}$.

38. $y = \dfrac{x}{\sqrt{a^2 - x^2}}$.

39. $y = \sqrt[3]{\dfrac{1 + x^3}{1 - x^3}}$.

40. $y = \dfrac{1}{\sqrt{1 + x^2}\left(x + \sqrt{1 + x^2}\right)}$.

41. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$.

42. $y = \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{x}}}$.

43. $y = \cos 2x - 2 \sin x$.

44. $y = (2 - x^2)\cos x + 2x \sin x$.

45. $y = \sin(\cos^2 x)\cos(\sin^2 x)$.

46. $y = \sin^n x \cos nx$.

47. $y = \sin[\sin(\sin x)]$.

48. $y = \dfrac{\sin^2 x}{\sin x^2}$.

49. $y = \dfrac{\cos x}{2\sin^2 x}$.

50. $y = \dfrac{1}{\cos^n x}$.

51. $y = \dfrac{\sin x - x\cos x}{\cos x + x\sin x}$.

52. $y = \mathrm{tg}\dfrac{x}{2} - \mathrm{ctg}\dfrac{x}{2}$.

53. $y = \mathrm{tg} x - \dfrac{1}{3}\mathrm{tg}^3 x + \dfrac{1}{5}\mathrm{tg}^5 x$.

54. $y = 4\sqrt[3]{\mathrm{ctg}^2 x} + \sqrt[3]{\mathrm{ctg}^8 x}$.

55. $y = \sec^2 \dfrac{x}{a} + \mathrm{cosec}^2 \dfrac{x}{a}$.

56. $y = \sin[\cos^2(\mathrm{tg}^3 x)]$.

57. $y = e^{-x^2}$.

58. $y = 2^{\mathrm{tg}\frac{1}{x}}$.

59. $y = e^x(x^2 - 2x + 2)$.

60. $y = \left[\dfrac{1 - x^2}{2}\sin x - \dfrac{(1 + x)^2}{2}\cos x\right]e^{-x}$.

61. $y = e^x\left(1 + \mathrm{ctg}\dfrac{x}{2}\right)$.

62. $y = \dfrac{\ln 3 \sin x + \cos x}{3^x}$.

63. $y = e^{ax}\dfrac{a\sin bx - b\cos bx}{\sqrt{a^2+b^2}}$.

64. $y = e^x + e^{e^x} + e^{e^{e^x}}$.

65. $y = \left(\dfrac{a}{b}\right)^x\left(\dfrac{b}{x}\right)^a\left(\dfrac{x}{a}\right)^b$ $(a > 0, b > 0)$.

66. $y = x^{a^a} + a^{x^a} + a^{a^x}$ $(a > 0)$.

67. $y = \lg^3 x^2$.

68. $y = \ln(\ln(\ln x))$.

69. $y = \ln(\ln^2(\ln^3 x))$.

70. $y = \dfrac{1}{2}\ln(1+x) - \dfrac{1}{4}\ln(1+x^2) - \dfrac{1}{2(1+x)}$.

71. $y = \dfrac{1}{4}\ln\dfrac{x^2-1}{x^2+1}$.

72. $y = \dfrac{1}{4(1+x^4)} + \dfrac{1}{4}\ln\dfrac{x^4}{1+x^4}$.

73. $y = \dfrac{1}{2\sqrt{6}}\ln\dfrac{x\sqrt{3}-\sqrt{2}}{x\sqrt{3}+\sqrt{2}}$.

74. $y = \dfrac{1}{1-k}\ln\dfrac{1+x}{1-x} + \dfrac{\sqrt{k}}{1-k}\ln\dfrac{1+x\sqrt{k}}{1-x\sqrt{k}}$, $(0 < k < 1)$.

75. $y = \sqrt{x+1} - \ln(1+\sqrt{x+1})$.

76. $y = \ln(x+\sqrt{x^2+1})$.

77. $y = x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2}$.

78. $y = x\ln^2(x+\sqrt{1+x^2}) - 2\sqrt{1+x^2}\,\ln(x+\sqrt{1+x^2}) + 2x$.

79. $y = \dfrac{x}{2}\sqrt{x^2+a^2} + \dfrac{a^2}{2}\ln(x+\sqrt{x^2+a^2})$.

80. $y = \dfrac{1}{2\sqrt{ab}}\ln\dfrac{\sqrt{a}+x\sqrt{b}}{\sqrt{a}-x\sqrt{b}}$ $(a > 0, b > 0)$.

81. $y = \dfrac{2+3x^2}{x^4}\sqrt{1-x^2} + 3\ln\dfrac{1+\sqrt{1-x^2}}{x}$.

82. $y = \ln\operatorname{tg}\dfrac{x}{2}$.

83. $y = \ln\operatorname{tg}\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)$.

84. $y = \dfrac{1}{2}\operatorname{ctg}^2 x + \ln\sin x$.

85. $y = \ln\sqrt{\dfrac{1-\sin x}{1+\sin x}}$.

86. $y = -\dfrac{\cos x}{2\sin^2 x} + \ln\sqrt{\dfrac{1+\cos x}{\sin x}}$.

87. $y = \ln\dfrac{b+a\cos x+\sqrt{b^2-a^2}\sin x}{a+b\cos x}$ $(0 \le |a| < |b|)$.

88. $y=\frac{1}{x}(\ln^3 x+3\ln^2 x+6\ln x+6)$.

89. $y=\frac{1}{4x^4}\ln\frac{1}{x}-\frac{1}{16x^4}$.

90. $y=\frac{3}{2}(1-\sqrt[3]{1+x^2})^2+3\ln(1+\sqrt[3]{1+x^2})$.

91. $y=\ln\left[\frac{1}{x}+\ln\left(\frac{1}{x}+\ln\frac{1}{x}\right)\right]$.

92. $y=x[\sin(\ln x)-\cos(\ln x)]$.

93. $y=\ln\operatorname{tg}\frac{x}{2}-\cos x\cdot\ln\operatorname{tg}x$.

94. $y=\arcsin\frac{x}{2}$.

95. $y=\arccos\frac{1-x}{\sqrt{2}}$.

96. $y=\operatorname{arctg}\frac{x^2}{a}$.

97. $y=\frac{1}{\sqrt{2}}\operatorname{arcctg}\frac{\sqrt{2}}{x}$.

98. $y=\sqrt{x}-\operatorname{arctg}\sqrt{x}$.

99. $y=x+\sqrt{1-x^2}\cdot\arccos x$.

100. $y=x\arcsin\sqrt{\frac{x}{1+x}}+\operatorname{arctg}\sqrt{x}-\sqrt{x}$.

101. $y=\arccos\frac{1}{x}$.

102. $y=\arcsin(\sin x)$.

103. $y=\arccos(\cos^2 x)$.

104. $y=\arcsin(\sin x-\cos x)$.

105. $y=\arccos\sqrt{1-x^2}$.

106. $y=\operatorname{arctg}\frac{1+x}{1-x}$.

107. $y=\operatorname{arcctg}\left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)$.

108. $y=\frac{2}{\sqrt{a^2-b^2}}\operatorname{arctg}\left(\sqrt{\frac{a-b}{a+b}}\operatorname{tg}\frac{x}{2}\right)$ $(a>b\geq 0)$.

109. $y=\arcsin\frac{1-x^2}{1+x^2}$.

110. $y=\frac{1}{\arccos^2(x^2)}$.

111. $y=\operatorname{arctg}x+\frac{1}{3}\operatorname{arctg}(x^3)$.

112. $y=\ln(1+\sin^2 x)-2\sin x\cdot\operatorname{arctg}(\sin x)$.

113. $y=\ln\left(\arccos\frac{1}{\sqrt{x}}\right)$.

114. $y = \ln\dfrac{x+a}{\sqrt{x^2+b^2}} + \dfrac{a}{b}\operatorname{arctg}\dfrac{x}{b}$ $(b \neq 0)$.

115. $y = \dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{a^2}{2}\arcsin\dfrac{x}{a}$ $(a > 0)$.

116. $y = \dfrac{1}{6}\ln\dfrac{(x+1)^2}{x^2-x+1} + \dfrac{1}{\sqrt{3}}\operatorname{arctg}\dfrac{2x-1}{\sqrt{3}}$.

117. $y = \dfrac{1}{4\sqrt{2}}\ln\dfrac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1} - \dfrac{1}{2\sqrt{2}}\operatorname{arctg}\dfrac{x\sqrt{2}}{x^2-1}$.

118. $y = x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x$.

119. $y = \dfrac{\arccos x}{x} + \dfrac{1}{2}\ln\dfrac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}$.

120. $y = \operatorname{arctg}\sqrt{x^2-1} - \dfrac{\ln x}{\sqrt{x^2-1}}$.

121. $y = \dfrac{\arcsin x}{\sqrt{1-x^2}} + \dfrac{1}{2}\ln\dfrac{1-x}{1+x}$.

122. $y = \dfrac{1}{12}\ln\dfrac{x^4-x^2+1}{(x^2+1)^2} - \dfrac{1}{2\sqrt{3}}\operatorname{arctg}\dfrac{\sqrt{3}}{2x^2-1}$.

123. $y = \dfrac{x^6}{1+x^{12}} - \operatorname{arcctg} x^6$.

124. $y = \ln\dfrac{1-\sqrt[3]{x}}{\sqrt{1+\sqrt[3]{x}+\sqrt[3]{x^2}}} + \sqrt{3}\operatorname{arctg}\dfrac{1+2\sqrt[3]{x}}{\sqrt{3}}$.

125. $y = \operatorname{arctg}\dfrac{x}{1+\sqrt{1-x^2}}$.

126. $y = \operatorname{arcctg}\dfrac{a-2x}{2\sqrt{ax-x^2}}$ $(a > 0)$.

127. $y = \dfrac{3-x}{2}\sqrt{1-2x-x^2} + 2\arcsin\dfrac{1+x}{\sqrt{2}}$.

128. $y = \dfrac{1}{4}\ln\dfrac{\sqrt[4]{1+x^4}+x}{\sqrt[4]{1+x^4}-x} - \dfrac{1}{2}\operatorname{arctg}\dfrac{\sqrt[4]{1+x^4}}{x}$.

129. $y = \operatorname{arctg}(\operatorname{tg}^2 x)$.

130. $y = \sqrt{1-x^2}\cdot\ln\sqrt{\dfrac{1-x}{1+x}} + \dfrac{1}{2}\ln\dfrac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} + \sqrt{1-x^2} + \arcsin x$.

131. $y = x\operatorname{arctg} x - \dfrac{1}{2}\ln(1+x^2) - \dfrac{1}{2}(\operatorname{arctg} x)^2$.

132. $y=\ln\left(e^{x}+\sqrt{1+e^{2x}}\right)$.

133. $y=\operatorname{arctg}\left(x+\sqrt{1+x^{2}}\right)$.

134. $y=\arcsin\left(\dfrac{\sin a\sin x}{1-\cos a\cos x}\right)$.

135. $y=\dfrac{1}{4\sqrt{3}}\ln\dfrac{\sqrt{x^{2}+2}-x\sqrt{3}}{\sqrt{x^{2}+2}+x\sqrt{3}}+\dfrac{1}{2}\operatorname{arctg}\dfrac{\sqrt{x^{2}+2}}{x}$.

136. $y=\dfrac{1}{2\sqrt{2}}\operatorname{arctg}\dfrac{x\sqrt{2}}{\sqrt{1+x^{4}}}-\dfrac{1}{4\sqrt{2}}\ln\dfrac{\sqrt{1+x^{4}}-x\sqrt{2}}{\sqrt{1+x^{4}}+x\sqrt{2}}$.

137. $y=\dfrac{x\sqrt{1-x^{2}}}{1+x^{2}}-\dfrac{3}{\sqrt{2}}\operatorname{arcctg}\dfrac{x\sqrt{2}}{\sqrt{1-x^{2}}}$.

138. $y=\arccos(\sin x^{2}-\cos x^{2})$.

139. $y=\arcsin(\sin x^{2})+\arccos(\cos x^{2})$.

140. $y=e^{m\arcsin x}[\cos(m\arcsin x)+\sin(m\arcsin x)]$.

141. $y=\operatorname{arctg}e^{x}-\ln\sqrt{\dfrac{e^{2x}}{e^{2x}+1}}$.

142. $y=\sqrt{1+\sqrt[3]{1+\sqrt[4]{1+x^{4}}}}$.

143. $y=\operatorname{arcctg}\dfrac{1}{\sqrt{\operatorname{ctg}(1/x^{2})}}$.

144. $y=\ln^{2}\left(\sec 2^{\sqrt[3]{x}}\right)$.

145. $y=x+x^{x}+x^{x^{x}}$ $(x>0)$.

146. $y=x^{x^{a}}+x^{a^{x}}+a^{x^{x}}$ $(a>0,x>0)$.

147. $y=\sqrt[x]{x}$ $(x>0)$.

148. $y=(\sin x)^{\cos x}+(\cos x)^{\sin x}$.

149. $y=(\ln x)^{x}:x^{\ln x}$.

150. $y=\left[\dfrac{\arcsin(\sin^{2}x)}{\arccos(\cos^{2}x)}\right]^{\operatorname{arctg}^{2}x}$.

151. $y=\log_{x}e$.

152. $y=\ln(\operatorname{ch}x)+\dfrac{1}{2\operatorname{ch}^{2}x}$.

153. $y=\dfrac{\operatorname{ch}x}{\operatorname{sh}^{2}x}-\ln\left(\operatorname{cth}\dfrac{x}{2}\right)$.

154. $y=\operatorname{arctg}(\operatorname{th}x)$.

155. $y=\arccos\left(\dfrac{1}{\operatorname{ch}x}\right)$.

156. $y=\dfrac{b}{a}x+\dfrac{2\sqrt{a^{2}-b^{2}}}{a}\operatorname{arctg}\left(\sqrt{\dfrac{a-b}{a+b}}\operatorname{th}\dfrac{x}{2}\right)$ $(0\le|b|<a)$.

157. Aralyk $u=\cos^{2}x$ üýtgeýän ululygy girizip, $y=\ln\left(\cos^{2}x+\sqrt{1+\cos^{4}x}\right)$ funksiýanyň önümini tapmaly.

157-nji mysalda görkezilen usuly ulanyp, funksiýalaryň önümlerini tapmaly:

158. $y = (\arccos x)^2\left[\ln^2(\arccos x) - \ln(\arccos x) + \frac{1}{2}\right].$

159. $y = \frac{1}{2}\operatorname{arctg}\left(\sqrt[4]{1+x^4}\right) + \frac{1}{4}\ln\frac{\sqrt[4]{1+x^4}+1}{\sqrt[4]{1+x^4}-1}.$

160. $y = \frac{e^{-x^2}\arcsin\left(e^{-x^2}\right)}{\sqrt{1-e^{-2x^2}}} + \frac{1}{2}\ln\left(1-e^{-2x^2}\right).$

161. $y = \frac{a^x}{1+a^{2x}} - \frac{1-a^{2x}}{1+a^{2x}}\operatorname{arcctg} a^{-x}.$

162. Funksiýalaryň önümlerini tapmaly we funksiýalaryň hem-de olaryň önümleriniň grafiklerini gurmaly:

a) $y = |x|$; b) $y = x|x|$; ç) $y = \ln|x|$.

163. Funksiýalaryň önümlerini tapmaly:

a) $y = |(x-1)^2(x+1)^3|$; ç) $y = \arccos\frac{1}{|x|}$;

b) $y = |\sin^3 x|$; d) $y = [x]\sin^2\pi x$.

Funksiýalaryň önümlerini tapmaly we funksiýalaryň hem-de olaryň önümleriniň grafiklerini gurmaly:

164. $\begin{cases} 1-x, & -\infty < x < 1; \\ (1-x)(2-x), & 1 \le x \le 2; \\ -(2-x), & 2 < x < +\infty. \end{cases}$

165. $y = \begin{cases} (x-a)^2(x-b)^2, & a \le x \le b; \\ 0, & [a,b] \text{ kesimiň daşynda.} \end{cases}$

166. $y = \begin{cases} x, & x < 0; \\ \ln(1+x), & x \ge 0. \end{cases}$

167. $y = \begin{cases} \operatorname{arctg} x, & |x| \le 1; \\ \frac{\pi}{4}\operatorname{sgn} x + \frac{x-1}{2}, & |x| > 1. \end{cases}$

168. $y = \begin{cases} x^2 e^{-x^2}, & |x| \le 1; \\ \frac{1}{e}, & |x| > 1. \end{cases}$

169. Funksiýalaryň logarifmik önümlerini tapmaly:

a) $y = x\sqrt{\frac{1-x}{1+x}}$; ç) $y = (x-a_1)^{\alpha_1}(x-a_2)^{\alpha_2}\ldots(x-a_n)^{\alpha_n}$;

b) $y = \frac{x^2}{1-x}\sqrt[3]{\frac{3-x}{(3+x)^2}}$; d) $y = \left(x+\sqrt{1+x^2}\right)^n$.

170. Differensirlenýän $\varphi(x)$ we $\psi(x)$ funksiýalar üçin y funksiýalaryň önümlerini tapmaly:

a) $y = \sqrt{\varphi^2(x) + \psi^2(x)}$;

b) $y = \operatorname{arctg}\dfrac{\varphi(x)}{\psi(x)}$;

ç) $y = \sqrt[\varphi(x)]{\psi(x)}$ $(\varphi(x) \neq 0;\ \psi(x) > 0)$;

d) $y = \log_{\varphi(x)}\psi(x)$ $(\varphi(x) > 0;\ \psi(x) > 0)$.

171. Differensirlenýän $f(u)$ funksiýa üçin y' önümi tapmaly:

a) $y = f(x^2)$;

b) $y = f(\sin^2 x) + f(\cos^2 x)$;

ç) $y = f(e^x)\, e^{f(x)}$;

d) $y = f\{f[f(x)]\}$.

172. $f(x) = x(x-1)(x-2)...(x-1000)$ funksiýanyň $f'(0)$ önümini tapmaly.

173. n tertipli kesgitleýjiniň önümini tapmak üçin

$$\begin{vmatrix} f_{11}(x) f_{12}(x) .. f_{1n}(x) \\ .... \\ f_{k1}(x) f_{k2}(x) .. f_{kn}(x) \\ .... \\ f_{n1}(x) f_{n2}(x) .. f_{nn}(x) \end{vmatrix}' = \sum_{k=1}^{n} \begin{vmatrix} f_{11}(x) f_{12}(x) .. f_{1n}(x) \\ .... \\ f_{k1}'(x) f_{k2}'(x) .. f_{kn}'(x) \\ .... \\ f_{n1}(x) f_{n2}(x) .. f_{nn}(x) \end{vmatrix}$$

formulany subut etmeli.

174. $F(x) = \begin{vmatrix} x-1 & 1 & 2 \\ -3 & x & 3 \\ -2 & -3 & x+1 \end{vmatrix}$ üçin $F'(x)$ önümi tapmaly.

175. $F(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$ üçin $F'(x)$ önümi tapmaly.

176. Funksiýanyň berlen grafigi boýunça onuň önüminiň takmynan grafigini gurmaly.

177. $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x \neq 0; \\ 0, & x = 0 \end{cases}$ funksiýanyň önüminiň üzülýändigini subut etmeli.

178. $f(x) = x^n \sin\dfrac{1}{x}$ $(x \neq 0)$ we $f(0) = 0$ funksiýa haýsy şertlerde

a) $x = 0$ nokatda üznüksiz;

b) $x = 0$ nokatda differensirlenýär;

ç) $x = 0$ nokatda üznüksiz önüme eýedir?

179. $f(x) = |x|^n \sin\dfrac{1}{|x|^m}$ $(x \neq 0)$ we $f(0) = 0$ $(m > 0)$ funksiýanyň haýsy şertlerde

a) koordinatalar başlangyjynyň golaý töwereginde önümi çäkli bolýar;
b) şol golaý töwereginde önümi çäksiz bolýar?

180. $x = a$ nokatda üznüksiz $\varphi(x)$ funksiýa üçin

$$f(x) = (x - a)\,\varphi(x)$$

funksiýanyň $f'(x)$ önümini tapmaly.

181. Üznüksiz $\varphi(x)$ we $\varphi(a) \neq 0$ funksiýa üçin

$$f(x) = |x - a|\,\varphi(x)$$

funksiýanyň a nokatda önüminiň ýokdugyny subut etmeli.
Birtaraplaýyn $f'_-(a)$ we $f'_+(a)$ önümler nämä deň?

182. Berlen $a_1, a_2, ..., a_n$ nokatlarda üznüksiz, önümi ýok funksiýa degişli mysal düzmeli.

183. $f(x) = x^2\left|\cos\frac{\pi}{x}\right|$ $(x \neq 0)$ we $f(0) = 0$ funksiýanyň $x = 0$ nokadyň islendik golaý töwereginde differensirlenmeýän nokadynyň bardygyny, ýöne $x = 0$ nokatda differensirlenýändigini subut etmeli.

184. $f(x) = \begin{cases} x^2, & x - \text{rasional}; \\ 0, & x - \text{irrasional} \end{cases}$ funksiýanyň diňe $x = 0$ nokatda önüminiň bardygyny subut etmeli.

185. Funksiýalaryň differensirlenmegini derňemeli:

a) $y = |(x-1)(x-2)^2(x-3)^3|$;

b) $y = |\cos x|$;

ç) $y = |\pi^2 - x^2|\sin^2 x$;

d) $y = \arcsin(\cos x)$;

e) $y = \begin{cases} \dfrac{x-1}{4}(x+1)^2 & |x| \leq 1; \\ |x| - 1 & |x| > 1. \end{cases}$

Berlen $f(x)$ funksiýanyň $f'_-(x)$ we $f'_-(x)$ birtaraplaýyn önümlerini kesgitlemeli:

186. $f(x) = |x|$.

187. $f(x) = [x]\sin\pi x$.

188. $f(x) = x\left|\cos\frac{\pi}{x}\right|$ $(x \neq 0)$, $f(0) = 0$.

189. $f(x) = \sqrt{\sin x^2}$.

190. $f(x) = \dfrac{x}{1 + e^{1/x}}$ $(x \neq 0)$, $f(0) = 0$.

191. $f(x) = \sqrt{1 - e^{-x^2}}$.

192. $f(x) = |\ln|x||$ $(x \neq 0)$.

193. $f(x) = \arcsin\dfrac{2x}{1 + x^2}$.

194. $f(x) = (x-2)\operatorname{arctg}\dfrac{1}{x-2}$ $\quad (x \neq 2), \quad f(2) = 0.$

195. $f(x) = x\sin\dfrac{1}{x}$, $(x \neq 0)$ we $f(0) = 0$ funksiýanyň $x = 0$ nokatda üznüksizdigini, ýöne şol nokatda onuň çep önüminiň hem, sag önüminiň hem ýokdugyny subut etmeli.

196. Goý, x_0 nokat $f(x)$ funksiýanyň 1-nji görnüşdäki üzülme nokady bolsun.

$$f_-'(x_0) = \lim_{h \to -0} \frac{f(x_0 + h) - f(x_0 - 0)}{h}$$

we

$$f_+'(x_0) = \lim_{h \to +0} \frac{f(x_0 + h) - f(x_0 + 0)}{h}$$

aňlatmalara $f(x)$ funksiýanyň degişlilikde x_0 nokatdaky umumylaşdyrylan birtaraplaýyn (degişlilikde çep we sag) önümleri diýilýär. Berlen $f(x)$ funksiýanyň üzülme x_0 nokatdaky $f_-'(x_0)$ we $f_+'(x_0)$ önümlerini tapmaly:

a) $f(x) = \sqrt{\dfrac{x^2 + x^3}{x}}$; $\qquad$ b) $f(x) = \operatorname{arctg}\dfrac{1+x}{1-x}$; $\qquad$ ç) $f(x) = \dfrac{1}{1 + e^{\frac{1}{x}}}$.

197. Goý,

$$f(x) = \begin{cases} x^2, & x \leq x_0; \\ ax + b, & x > x_0 \end{cases}$$

bolsun. a we b koeffisiýentleri nähili saýlanyňda $f(x)$ funksiýa $x = x_0$ nokatda üznüksiz we differensirlenýän bolar?

198. Goý,

$$F(x) = \begin{cases} f(x), & x \leq x_0; \\ ax + b, & x > x_0 \end{cases}$$

bolsun, bu ýerde $f(x)$ funksiýanyň $x = x_0$ nokatda çep önümi bar. a we b koeffisiýentleri nähili saýlanyňda $F(x)$ funksiýa $x = x_0$ nokatda üznüksiz we differensirlenýän bolar?

199. $y = A(x-a)(x-b)(x-c)$ kubiki parabolanyň kömegi bilen $[a, b]$ kesimde iki $y = k_1(x-a)$ $(-\infty < x < a)$ we $y = k_2(x-b)$ $(b < x < +\infty)$ ýarym göni çyzyklaryň çatyrymyny gurmaly (bu ýerde A we c kesgitlenilmeli parametrler).

200. $y = \dfrac{m^2}{|x|}$ $(|x| > c)$ egri çyzygyň bölegini $y = a + bx^2$ $(|x| \leq c)$ parabola bilen endigan egri çyzyk alnar ýaly doldurmaly (bu ýerde a we b näbelli parametrler).

201. Eger $x = x_0$ nokatda: a) $f(x)$ funksiýanyň önümi bar bolsa, $g(x)$ funksiýanyň önümi ýok bolsa; b) $f(x)$ we $g(x)$ funksiýalaryň ikisiniň hem önümi ýok bolsa, onda olaryň $F(x) = f(x) + g(x)$ jeminiň $x = x_0$ nokatda önümi ýok diýip tassyklamak bolarmy?

202. Eger $x = x_0$ nokatda: a) $f(x)$ funksiýanyň önümi bar bolsa, $g(x)$ funksiýanyň önümi ýok bolsa; b) $f(x)$ we $g(x)$ funksiýalaryň ikisiniň hem önümi ýok bolsa, onda olaryň $F(x) = f(x) \cdot g(x)$ köpeltmek hasylynyň $x = x_0$ nokatda önümi ýok diýip tassyklamak bolarmy? $x_0 = 0$ diýip, aşakdaky mysallary derňemeli:

a) $f(x) = x$, $g(x) = |x|$; b) $f(x) = |x|$, $g(x) = |x|$.

203. Eger: a) $f(x)$ funksiýanyň $x = g(x_0)$ nokatda önümi bar bolsa, $g(x)$ funksiýanyň x_0 nokatda önümi ýok bolsa; b) $f(x)$ funksiýanyň $x = g(x_0)$ nokatda önümi ýok bolsa, $g(x)$ funksiýanyň x_0 nokatda önümi bar bolsa; ç) $f(x)$ funksiýanyň $x = g(x_0)$ nokatda önümi ýok bolsa, $g(x)$ funksiýanyň x_0 nokatda önümi ýok bolsa, onda $F(x) = f(g(x))$ funksiýany $x = x_0$ nokatda differensirlenmegi barada näme aýtmak bolar? $x_0 = 0$ diýip aşakdaky mysallary derňemeli:

a) $f(x) = x^2$, $g(x) = |x|$;

b) $f(x) = |x|$, $g(x) = x^2$;

ç) $f(x) = 2x + |x|$, $g(x) = \frac{2}{3}x - \frac{1}{3}|x|$.

204. Haýsy nokatlarda

$$y = x + \sqrt[3]{\sin x}$$

funksiýanyň grafiginiň dik asimptotasy bar? Şol grafigi gurmaly.

205. Üzülme nokadynda $f(x)$ funksiýanyň: a) tükenikli önümi; b) tükeniksiz önümi bolup bilermi? Aşakdaky mysalda funksiýanyň üzülme nokadynda tükenikli ýa-da tükeniksiz önüminiň bardygyny derňemeli: $f(x) = \operatorname{sgn} x$.

206. Eger $f(x)$ funksiýa çäkli (a, b) interwalda differensirlenip,

$$\lim_{x \to a} f(x) = \infty$$

bolsa, onda aşakdaky deňlikler hökman ýerine ýetermi?

a) $\lim\limits_{x \to a} f'(x) = \infty$; b) $\overline{\lim\limits_{x \to a}} |f'(x)| = +\infty$.

Bu formulalardan peýdalanyp, aşakdaky mysaly işlemeli:

$$f(x) = \frac{1}{x} + \cos\frac{1}{x}, \quad x \to 0 \text{ bolanda.}$$

207. Eger $f(x)$ funksiýa çäkli interwalda differensirlenip,

$$\lim_{x\to a} f'(x) = \infty \text{ bolsa,}$$

onda

$$\lim_{x\to a} f(x) = \infty$$

bolmagy hökmanmy? Bu formuladan peýdalanyp, mysaly işlemeli:

$$f(x) = \sqrt[3]{x}, \qquad x \to 0.$$

208. Goý, $f(x)$ funksiýa $(x_0, +\infty)$ interwalda differensirlenip, $\lim\limits_{x\to+\infty} f(x)$ predel hem bar bolsun. Bu ýerden $\lim\limits_{x\to+\infty} f'(x)$ predeliň bardygy gelip çykarmy? Bu formuladan peýdalanyp, mysaly işlemeli:

$$f(x) = \frac{\sin(x^2)}{x}.$$

209. Goý, çäkli $f(x)$ funksiýa $(x_0, +\infty)$ interwalda differensirlenip, $\lim\limits_{x\to+\infty} f'(x)$ predel bar bolsun. Bu ýerden tükenikli ýa-da tükeniksiz $\lim\limits_{x\to+\infty} f(x)$ predeliň bardygy gelip çykarmy?

210. Funksiýalaryň deňsizligini agzama-agza differensirläp bolarmy?

211. $P_n(x) = 1 + 2x + 3x^2 + ... + nx^{n-1}$ we
$Q_n(x) = 1^2 + 2^2x + 3^2x^2 + ... + n^2x^{n-1}$
jemler üçin formulalary getirip çykarmaly. (*Görkezme*: $(x + x^2 + ... + x^n)'$ *peýdalanmaly*).

212. $S_n = \sin x + \sin 2x + ... + \sin nx$ we $T_n = \cos x + 2\cos 2x + ... + n\cos nx$ jemler üçin formulalary getirip çykarmaly.

213. $S_n = \mathrm{ch}x + 2\mathrm{ch}2x + ... + n\mathrm{ch}nx$ jem üçin formulany getirip çykarmaly. (*Görkezme*: $S_n = (\mathrm{sh}x + \mathrm{sh}2x + ... + \mathrm{sh}nx)'$ *peýdalanmaly*).

214. $\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}...\cos\frac{x}{2^n} = \dfrac{\sin x}{2^n \sin\frac{x}{2^n}}$ toždestwodan peýdalanyp,

$$S_n = \frac{1}{2}\mathrm{tg}\frac{x}{2} + \frac{1}{2^2}\mathrm{tg}\frac{x}{2^2} + ... + \frac{1}{2^n}\mathrm{tg}\frac{x}{2^n}$$

jem üçin formulany getirip çykarmaly.

215. Differensirlenýän jübüt funksiýanyň önüminiň täk funksiýa, differensirlenýän täk funksiýanyň önüminiň jübüt funksiýa bolýandygyny subut etmeli.

Bu tassyklama geometrik taýdan düşündiriş bermeli.

216. Differensirlenýän periodik funksiýanyň önüminiň ýene-de şol periodly periodik funksiýa bolýandygyny subut etmeli.

217. Radiusy $R = 10$ *sm* bolan tegelegiň radiusy 2 *sm/s* tizlik bilen deňölçegli ösen pursadynda onuň meýdany nähili tizlik bilen artar?

218. Bir tarapy $x = 20$ *m*, beýleki tarapy $y = 15$ *m* bolan gönüburçlugyň birinji tarapynyň 1 *m/s* tizlik bilen kiçelen, ikinji tarapynyň 2 *m/s* tizlik bilen artan pursadynda gönüburçlugyň meýdany we diagonaly nähili tizlik bilen üýtgär?

219. Şol bir duralgadan bir wagtda demirgazyk tarapa A gämi 30 *km/sag* tizlik bilen we günorta tarapa B gämi 40 *km/sag* tizlik bilen ugrady. Olaryň arasyndaky uzaklyk nähili tizlik bilen artar?

220. Goý,

$$f(x) = \begin{cases} x, & 0 \le x \le 2; \\ 2x - 2, & 2 < x < +\infty \end{cases}$$

we $S(x)$ funksiýa berlen $y = f(x)$ funksiýanyň çyzygy, Ox oky we x $(x \ge 0)$ nokatda Ox okuna geçirilen perpendikulýar bilen çäklenen figuranyň meýdany bolsun.

$S(x)$ funksiýany derňew etme görnüşde aňlatmaly, onuň $S'(x)$ önümini tapmaly we $y = S'(x)$ funksiýanyň grafigini gurmaly.

221. $S(x)$ funksiýa $y = \sqrt{a^2 - x^2}$ töweregiň dugasy, Ox oky we Ox okuna 0 we x $(|x| \le a)$ nokatlarda geçirilen iki perpendikulýarlar bilen çäklenen meýdany aňladýar.

$S(x)$ funksiýany derňew etme görnüşde aňlatmaly, onuň $S'(x)$ önümini tapmaly we $y = S'(x)$ funksiýanyň grafigini gurmaly.

222. $y^3 + 3y = x$ deňleme bilen kesgitlenýän birbahaly $y = y(x)$ funksiýanyň bardygyny subut etmeli we onuň y_x' önümini tapmaly.

223. $y - \varepsilon \sin y = x$ $(0 \le \varepsilon < 1)$ deňleme bilen kesgitlenýän birbahaly $y = y(x)$ funksiýanyň bardygyny subut etmeli we onuň y_x' önümini tapmaly.

224. Berlen funksiýalaryň $x = x(y)$ ters funksiýalarynyň barlyk ýaýlalaryny kesgitlemeli we olaryň önümlerini tapmaly:

a) $y = x + \ln x$ $(x > 0)$;   ç) $y = \mathrm{sh}x$;

b) $y = x + e^x$;   d) $y = \mathrm{th}x$.

225. Berlen funksiýalaryň $x = x(y)$ ters funksiýalarynyň birbahaly üznüksiz şahalaryny görkezip, olaryň önümlerini tapmaly we grafiklerini gurmaly:

a) $y = 2x^2 - x^4$;   b) $y = \dfrac{x^2}{1 + x^2}$;   ç) $y = 2e^{-x} - e^{-2x}$.

226. $x=-1+2t-t^2$, $y=2-3t+t^3$ deňlemeler bilen berlen $y=y(x)$ funksiýanyň grafigini gurmaly we y'_x önümi tapmaly. $x=0$ we $x=-1$ nokatlarda $y'_x(x)$ önüm näçä deň? Haýsy $M(x, y)$ nokatda $y'_x(x)=0$ bolar?

Parametr görnüşde berlen funksiýalaryň y'_x önümini tapmaly (parametrler položitel):

227. $x=\sqrt[3]{1-\sqrt{t}}$, $y=\sqrt{1-\sqrt[3]{t}}$. **228.** $x=\sin^2 t$, $y=\cos^2 t$.

229. $x=a\cos t$, $y=b\sin t$. **230.** $x=a\,\mathrm{ch}\,t$, $y=b\,\mathrm{sh}\,t$.

231. $x=a\cos^3 t$, $y=a\sin^3 t$. **232.** $x=a(t-\sin t)$, $y=a(1-\cos t)$.

233. $x=e^{2t}\cos^2 t$, $y=e^{2t}\sin^2 t$.

234. $x=\arcsin\dfrac{t}{\sqrt{1+t^2}}$; $y=\arccos\dfrac{1}{\sqrt{1+t^2}}$.

235. $x=2t+|t|$, $y=5t^2+4t|t|$ deňlemeler sistemasy bilen kesgitlenen $y=y(x)$ funksiýanyň $t=0$ nokatda differensirlenýändigini, ýöne onuň önümini berlen nokatda adaty formula boýunça tapyp bolmaýandygyny subut etmeli.

Anyk däl görnüşde berlen funksiýalaryň y'_x önümini tapmaly:

236. $x^2+2xy-y^2=2x$. $x=2$ we $y=4$ bolanda y' önüm nämä deň?

237. $y^2=2px$ (parabola). **238.** $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ (ellips).

239. $\sqrt{x}+\sqrt{y}=\sqrt{a}$ (parabola). **240.** $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ (astroida).

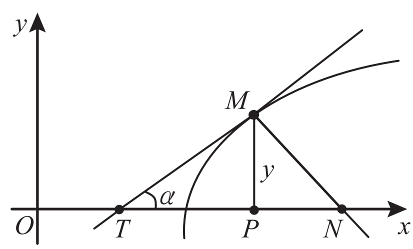
241. $\mathrm{arctg}\dfrac{y}{x}=\ln\sqrt{x^2+y^2}$ (logarifmik spiral).

242. Polýar koordinatalarynda berlen funksiýalaryň y'_x önümini tapmaly:
a) $r=a\varphi$ (Arhimediň spiraly);
b) $r=a(1+\cos\varphi)$ (kardioida);
ç) $r=ae^{m\varphi}$ (logarifmik spiral).

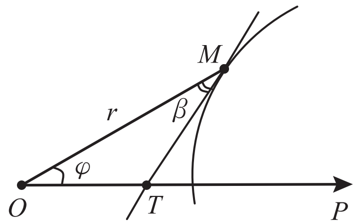
Bu ýerde $r=\sqrt{x^2+y^2}$ we $\varphi=\mathrm{arctg}\dfrac{y}{x}$ polýar koordinatalary.

§2. Funksiýanyň öňüminiň geometrik manysy

Galtaşýan göni çyzygyň we normalyň deňlemesi. $y=f(x)$ funksiýanyň grafigine onuň $M(x,y)$ nokadynda geçirilen galtaşýan göni çyzygyň we normalyň deňlemesi (*14-nji surat*) degişlilikde şeýle görnüşdedir:



14-nji surat



15-nji surat

$$Y - y = y'(X - x) \quad \text{we} \quad Y - y = -\frac{1}{y'}(X - x).$$

Bu deňlemelerde X we Y galtaşýan göni çyzygyň we normalyň üýtgeýän koordinatalarydyr we $y' = f'(x)$ bolsa önümiň galtaşýan nokatdaky bahasydyr. 14-nji suratdaky MT galtaşýan göni çyzyk we MN normal üçin PT galtaşma asty, PN normal asty bolýar. Galtaşýan göni çyzygyň geometrik manysy boýunça $y' = \text{tg}\alpha$ dogrudygy üçin şeýle deňlikleri alarys:

$$PT = \left|\frac{y}{y'}\right|, \quad PN = |yy'|, \quad MT = \left|\frac{y}{y'}\right|\sqrt{1 + y'^2}, \quad MN = |y|\sqrt{1 + y'^2}.$$

Eger $r = r(\varphi)$ çyzygyň polýar koordinatalaryndaky deňlemesi bolsa, onda galtaşýan göni çyzyk bilen M nokatda geçirilen OM radius wektoryň arasyndaky β burç üçin $\text{tg}\beta = \frac{r}{r'}$ formula alynýar (*15-nji surat*).

Gönükmeler

243. $y = (x + 1)\sqrt[3]{3 - x}$ çyzyga a) $A(-1, 0)$; b) $B(2, 3)$; ç) $C(3, 0)$ nokatlarda geçirilen galtaşýan göni çyzygyň we normalyň deňlemelerini ýazmaly.

244. $y = 2 + x - x^2$ çyzygyň haýsy nokatlarynda oňa geçirilen galtaşýan göni çyzyk: a) Ox okuna parallel? b) birinji koordinatalar burçunyň bissektrisasyna parallel?

245. $y = a(x - x_1)(x - x_2)$ $(a \neq 0, x_1 < x_2)$ parabolanyň Ox okuny deň bolan α we β $\left(0 < \alpha < \frac{\pi}{2};\ 0 < \beta < \frac{\pi}{2}\right)$ burçlar boýunça kesişýändigini subut etmeli.

246. $y = 2\sin x$ $(-\pi \leq x \leq \pi)$ çyzykda «çyzygyň ýapgytlygynyň» (ýagny $|y'|$-iň) birden uly bolan aralygyny kesgitlemeli.

247. $y = x$ we $y_1 = x + 0{,}01\sin 1000\pi x$ funksiýalar biri-birinden 0,01-den uly bolmadyk ýagdaýynda tapawutlanýar. Olaryň önümleriniň tapawudynyň maksimal bahasy barada näme aýtmak bolar?

Degişli grafikleri gurmaly.

248. $y = \ln x$ çyzyk Ox oky bilen haýsy burç boýunça kesişýär?

249. $y = x^2$ we $x = y^2$ çyzyklar haýsy burç boýunça kesişýär?

250. $y = \sin x$ we $y = \cos x$ çyzyklar haýsy burç boýunça kesişýär?

251. n parametri nähili saýlanyňda

$$y = \operatorname{arctg} nx \quad (n > 0)$$

çyzyk Ox okuny 89º-dan uly burç boýunça keser?

252. $y = |x|^{\alpha}$ çyzygyň: a) $0 < \alpha < 1$ bolanda Oy okuna galtaşýandygyny; b) $1 < \alpha < +\infty$ bolanda Ox okuna galtaşýandygyny subut etmeli.

253. $y = \begin{cases} |x|^{\alpha}, & \alpha \neq 0,\ x \neq 0, \\ 1, & x = 0 \end{cases}$

funksiýanyň grafigi üçin $A(0,\ 1)$ nokat arkaly geçýän kesiji çyzygyň predel ýagdaýynyň Oy oky bolýandygyny subut etmeli.

254. Berlen çyzyklara görkezilen nokatlarda geçirilen çepden we sagdan galtaşýan göni çyzygyň arasyndaky burçy kesgitlemeli:

a) $y = \sqrt{1 - e^{-a^2x^2}}$, $x = 0$; b) $y = \arcsin \dfrac{2x}{1 + x^2}$, $x = 1$.

255. $r = ae^{m\varphi}$ logarifmik spirala geçirilen galtaşýan göni çyzygyň galtaşma nokadynyň radius-wektory bilen hemişelik burçy emele getirýändigini subut etmeli (a we m – hemişelik sanlar).

256. $y= ax^n$ egri çyzygyň galtaşma astynyň uzynlygyny kesgitlemeli, ol egri çyzyga galtaşýan göni çyzygy geçirmegiň usulyny görkezmeli.

257. $y^2 = 2px$ parabolanyň

a) galtaşma astynyň galtaşma nokadynyň absissasynyň iki essesine deňdigini;

b) normal astynyň hemişelikdigini subut etmeli.

Parabola galtaşýan göni çyzygy geçirmegiň usulyny görkezmeli.

258. $y = a^x$ $(a > 0)$ görkezijili egri çyzygyň galtaşma astynyň hemişelikdigini subut etmeli. Görkezijili egri çyzyga galtaşýan göni çyzygy geçirmegiň usulyny görkezmeli.

259. $y = a\operatorname{ch}\dfrac{x}{a}$ zynjyr çyzygyň islendik $M(x_0,\ y_0)$ nokadyndaky normalynyň uzynlygyny kesgitlemeli.

260. $x^{2/3} + y^{2/3} = a^{2/3}$ $(a > 0)$ astroidanyň galtaşýan göni çyzygyň koordinatalar oklarynyň arasyndaky kesiminiň hemişelik ululykdygyny subut etmeli.

261. a, b we c koeffisiýentler nähili gatnaşykda bolanda $y = ax^2 + bx + c$ parabola Ox okuna galtaşar?

262. Haýsy şertde $y = x^3 + px + q$ kubiki parabola Ox okuna galtaşýar?

263. a parametriň haýsy bahasynda $y = ax^2$ parabola $y = \ln x$ egri çyzyga galtaşar?

264. $y = f(x)$ $(f(x) > 0)$ $y = f(x)\sin\alpha x$ egri çyzyklaryň (bu ýerde $f(x)$ – differensirlenýän funksiýa) üçin umumy nokatlarda galtaşýandygyny subut etmeli.

265. $x^2 - y^2 = a$ we $xy = b$ giperbolalaryň ortogonal tory emele getirýändigini, ýagny ol çyzyklaryň göni burçlar boýunça kesişýändigini subut etmeli.

266. $y^2 = 4a(a - x)$ $(a > 0)$ we $y^2 = 4b(b + x)$ $(b > 0)$ parabolalaryň ortogonal tory emele getirýändigini subut etmeli.

267. $x = 2t - t^2$, $y = 3t - t^3$ çyzyga a) $t = 0$, b) $t = 1$ nokatlarda geçirilen galtaşýan göni çyzyklaryň deňlemelerini ýazmaly.

268. $x = \dfrac{2t + t^2}{1 + t^3}$, $y = \dfrac{2t - t^2}{1 + t^3}$ çyzyga a) $t = 0$, b) $t = 1$, ç) $t = \infty$ nokatlarda geçirilen galtaşýan göni çyzygyň we normalyň deňlemelerini ýazmaly.

269. $x = a(t - \sin t)$, $y = a(1 - \cos t)$ çyzyga erkin $t = t_0$ nokatda geçirilen galtaşýan göni çyzygyň deňlemesini ýazmaly.

270. $x = a\left(\ln \operatorname{tg}\dfrac{t}{2} + \cos t\right)$, $y = a\sin t$ $(a > 0,\ 0 < t < \pi)$ traktrisanyň hemişelik uzynlykly galtaşýan çyzygynyň bardygyny subut etmeli.

Aşakda görkezilen egri çyzyklara berlen nokatlarda geçirilen galtaşýan çyzyklaryň deňlemelerini ýazmaly:

271. $\dfrac{x^2}{100} + \dfrac{y^2}{64} = 1$, $M(6,\ 6{,}4)$. **272.** $xy + \ln y = 1$, $M(1;\ 1)$.

§3. Funksiýanyň differensialy

1. *Differensial düşünjesi.* Goý, $y = f(x)$ funksiýa x nokatda differensirlenýän bolsun, ýagny onuň şol nokatdaky Δy artymy

$$\Delta y = f'(x)\Delta x + \alpha(\Delta x)\Delta x, \qquad \lim_{\Delta x \to 0}(\Delta x) = 0 \tag{1}$$

görnüşde aňladylýan bolsun. Bu aňlatmanyň baş agzasy bolan birinji goşulyja $y = f(x)$ funksiýanyň x nokatdaky differensialy diýilýär we ol dy ýa-da df bilen belgilenýär:

$$dy = f'(x)\Delta x = f'(x)dx. \tag{2}$$

Bu formulanyň we funksiýanyň önümini tapmagyň düzgünleri esasynda differensial üçin esasy düzgünleri ýazyp bileris:

$$d(u \pm v) = (u \pm v)'dx = (u' \pm v')dx = u'dx \pm v'dx = du \pm dv,$$

$$d(u \cdot v) = (u \cdot v)'dx = (u'v + uv')dx = vu'dx + uv'dx = vdu + udv,$$

$$d\left(\frac{u}{v}\right) = \left(\frac{u}{v}\right)' dx = \frac{u'v - uv'}{v^2} dx = \frac{vu'dx - uv'dx}{v^2} = \frac{vdu - udv}{v^2}.$$

Hususan-da, hemişelik $u = c$ funksiýa üçin

$$d(cv) = cdv, \qquad d\left(\frac{c}{v}\right) = -\frac{cdv}{v^2}.$$

16-njy mysal. $y = \sqrt{x}\sin x$ funksiýanyň differensialyny tapmaly.

Ç.B. Differensialyň düzgünlerinden we (2) formuladan peýdalanyp, differensialy taparys:

$$dy = \sqrt{x}\,d(\sin x) + \sin x d(\sqrt{x}) = \sqrt{x}(\sin x)' dx + \sin x(\sqrt{x})' dx =$$

$$= \sqrt{x}\cos x dx + \sin x \frac{1}{2\sqrt{x}} dx \ \textbf{Ç.S.}$$

2. ***Takmyny hasaplamalarda differensialyň ulanylyşy.*** (1) we (2) deňliklerden $\alpha(\Delta x)\Delta x = o(\Delta x)$, $\Delta x \to 0$ bolýandygy sebäpli,

$$\Delta y \approx dy \quad \text{ýa-da} \quad f(x + \Delta x) - f(x) \approx f'(x)\Delta x$$

takmyny deňligi ýazmak bolar. Ony

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x \quad \text{ýa-da} \quad f(x) \approx f(a) + f'(a)(x - a) \tag{3}$$

görnüşde ýazyp, ol formulany üýtgeýän x ululyga ýakyn bolan bahalar üçin funksiýanyň takmyny bahalaryny tapmakda ulanmak bolar.

17-nji mysal. $f(x) = (1 + x)^\alpha$ funksiýanyň $x = 0$ nokadyň golaý töweregindäki takmyny bahasyny tapmaly.

Ç.B. (3) formulany $f(x) = (1 + x)^\alpha$ funksiýa we $a = 0$ üçin ulanalyň:

$$(1 + x)^\alpha \approx f(0) + f'(0)x,$$

$$f'(x) = \alpha(1 + x)^{\alpha-1}, \quad f'(0) = \alpha, \quad f(0) = 1.$$

Şeýlelikde,

$$(1 + x)^\alpha \approx 1 + \alpha x, \quad \textbf{Ç.S.} \tag{4}$$

18-nji mysal. $\sqrt[3]{27{,}027}$ aňlatmanyň takmyny bahasyny tapmaly.

Ç.B. Ilki bilen ony $\sqrt[3]{27{,}027} = \sqrt[3]{27 + 0{,}027} = 3\sqrt[3]{1 + 0{,}001}$ görnüşde ýazyp, soňra $\sqrt[3]{1 + 0{,}001} = (1 + 0{,}001)^{1/3}$ aňlatmany hasaplalyň. Onuň üçin (4) formulada x = 0,001 we α = 1/3 goýup, $(1 + 0{,}001)^{1/3} \approx 1 + \frac{1}{3} \cdot 0{,}001 = \frac{3{,}001}{3}$ takmyny deňligi alarys. Şonuň üçin $\sqrt[3]{27{,}027} = 3(1 + 0{,}001)^{1/3} \approx 3{,}001$. **Ç.S.**

19-njy mysal. $\sin 29^{\circ}57'$ aňlatmanyň takmyny bahasyny tapmaly.

Ç.B. Bu aňlatmany tapmak üçin (3) formulany ulanarys. Onuň üçin şol formulada $f(x)$ funksiýanyň ornunda goýup,

$$\sin(x + \Delta x) \approx \sin x + \cos x \cdot \Delta x$$

formulany alarys. Bu formulada $x = 30^{\circ}$ we $\Delta x = -3' = -\pi/3600$ alsak, onda

$$\sin 29^{\circ}57' = \sin\left(30^{\circ} - \frac{\pi}{3600}\right) \approx \sin 30^{\circ} - \cos 30^{\circ} \cdot \frac{\pi}{3600} =$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}\frac{\pi}{3600} = 0,5 - \frac{\pi\sqrt{3}}{7200} = 0,499237.$$ **Ç.S.**

Gönükmeler

273. $f(x) = x^3 - 2x + 1$ funksiýa üçin 1) $\Delta f(1)$; 2) $df(1)$ kesgitlemeli we olary: a) $\Delta x = 1$; b) $\Delta x = 0,1$; ç) $\Delta x = 0,01$ bahalar üçin deňeşdirmeli.

274. Hereketiň deňlemesi $x = 5t^2$ formula boýunça berilýär, bu ýerde t sekuntda x metrde ölçenýär.

Wagtyň $t = 2$ s pursady üçin ýoluň Δx artymyny we dx differensialyny kesgitlemeli we olary:

a) $\Delta t = 1$ s; b) $\Delta t = 0,1$ s; ç) $\Delta t = 0,01$ s

bahalar üçin deňeşdirmeli.

Berlen y funksiýalaryň differensiallaryny tapmaly:

275. $y = \dfrac{1}{x}$.

276. $y = \dfrac{1}{a}\operatorname{arctg}\dfrac{x}{a}$ $(a \neq 0)$.

277. $y = \dfrac{1}{2a}\ln\left|\dfrac{x-a}{x+a}\right|$.

278. $y = \ln\left|x + \sqrt{x^2 + a}\right|$.

279. $y = \arcsin\dfrac{x}{a}$ $(a \neq 0)$.

280. Tapmaly:

a) $d(xe^x)$; d) $d\left(\dfrac{\ln x}{\sqrt{x}}\right)$; f) $d\ln(1 - x^2)$;

b) $d(\sin x - x\cos x)$; e) $d(\sqrt{a^2 + x^2})$; g) $d\left(\arccos\dfrac{1}{|x|}\right)$;

ç) $d\left(\dfrac{1}{x^3}\right)$; ä) $d\left(\dfrac{x}{\sqrt{1-x^2}}\right)$; h) $d\left[\dfrac{\sin x}{2\cos^2 x} + \dfrac{1}{2}\ln\left|\operatorname{tg}\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)\right|\right]$.

Goý, u, v, ω funksiýalar üýtgeýän x ululyga görä differensirlenýän bolsun. Berlen y funksiýalaryň differensiallaryny tapmaly:

281. $y = u\upsilon\omega$.

282. $y = \dfrac{u}{\upsilon^2}$.

283. $y = \dfrac{1}{\sqrt{u^2 + \upsilon^2}}$.

284. $y = \operatorname{arctg}\dfrac{u}{\upsilon}$.

285. $y = \ln\sqrt{u^2 + \upsilon^2}$.

286. Tapmaly:

a) $\dfrac{d}{d(x^3)}(x^3 - 2x^6 - x^9)$;

b) $\dfrac{d}{d(x^2)}\left(\dfrac{\sin x}{x}\right)$;

ç) $\dfrac{d(\sin x)}{d(\cos x)}$;

d) $\dfrac{d(\operatorname{tg}x)}{d(\operatorname{ctg}x)}$;

e) $\dfrac{d(\arcsin x)}{d(\arccos x)}$.

287. Tegelek sektoryň radiusy $R = 100$ *sm* we merkezi burçy $\alpha = 60^\circ$. Eger a) R radius 1 *sm* ulaldylsa; b) α burç $30'$ kiçeldilse, onda sektoryň meýdany nähili üýtgär?

Takyk we takmyny çözüwini tapmaly.

288. Maýatnigiň (sekuntdaky) yrgyldysynyň periody $T = 2\pi\sqrt{\dfrac{l}{g}}$ formula boýunça kesgitlenýär, bu ýerde l – maýatnigiň santimetr hasabyndaky uzynlygy we $g = 981$ *sm/s*^{2} agyrlyk güýjüň tizlenmesi.

Maýatnigiň $l = 20$ *sm* uzynlygyny näçe üýtgedeniňde T periody 0,05 *s* ulalar?

Funksiýanyň artymyny differensialy bilen çalşyryp, aşakdaky aňlatmalaryň takmyny bahalaryny tapmaly:

289. $\sqrt[3]{1,02}$.

290. $\sin 29^\circ$.

291. $\cos 151^\circ$.

292. $\operatorname{arctg} 1,05$.

293. $\lg 11$.

294. $\sqrt{a^2 + x} \approx a + \dfrac{x}{2a}$ $(a > 0)$ takmyny formulany subut etmeli, bu ýerde $|x| << a$ (polozitel A we B sanlaryň arasyndaky $A << B$ ýazgy A-nyň B-den has kiçidigini görkezýär).

Bu formuladan peýdalanyp, aşakdakylaryň takmyny bahalaryny hasaplamaly we tablisada berlen bahalary bilen deňeşdirmeli:

a) $\sqrt{5}$; b) $\sqrt{34}$; ç) $\sqrt{120}$.

295. $\sqrt{a^2 + x} = a + \dfrac{x}{2a} - r$ $(a > 0, x > 0)$ formulany subut etmeli, bu ýerde $0 < r < \dfrac{x^2}{8a^3}$.

296. $\sqrt[n]{a^n + x} \approx a + \dfrac{x}{na^{n-1}}$ $(a > 0)$ formulany subut etmeli, bu ýerde $|x| << a$.

Bu subut edilen formuladan peýdalanyp, aşakdakylaryň takmyny bahalaryny hasaplamaly:

a) $\sqrt[3]{9}$; b) $\sqrt[4]{80}$; ç) $\sqrt[7]{100}$; d) $\sqrt[10]{1000}$.

297. Kwadratyň tarapy $x = 2{,}4\ m \pm 0{,}05\ m$. Onuň meýdanynyň absolýut we otnositel ýalňyşlyklaryny haýsy çäklerde hasaplamak bolar?

298. Şaryň göwrümini 1%-e çenli takyklykda kesgitlemek üçin onuň radiusyny nähili otnositel ýalňyşlyk bilen ölçemek bolar?

299. Maýatnigiň yrgyldysynyň kömegi bilen agyrlyk güýjüniň tizlenmesini kesgitlemek üçin

$$g = 4\pi^2 l/T^2$$

formula ulanylýar, bu ýerde l – maýatnigiň uzynlygy, T bolsa – maýatnigiň yrgyldysynyň doly periody.

Ölçeglerde goýberilýän otnositel δ ýalňyşlyk g-niň bahasyna nähili täsir eder: a) l uzynlyk ölçenilende; b) T period ölçenilende?

300. $x(x > 0)$ sanyň otnositel ýalňyşlygy δ bolanda ol sanyň onluk logarifminiň absolýut ýalňyşlygyny kesgitlemeli.

301. Şol bir onluk belgili sanlarda tangensiň logarifmik tablisasy boýunça burçlaryň kesgitlenişi sinuslaryň logarifmik tablisasy bilen deňeşdireniňde has takyk kesgitlenýändigini subut etmeli.

§4. Ýokary tertipli önümler we differensiallar

1. ***Ýokary tertipli önümler.*** Eger funksiýanyň $f'(x)$ birinji önüminiň käbir x nokatda önümi bar bolsa, onda bu önüme $y = f(x)$ funksiýanyň nokatdaky ikinji ýa-da ikinji tertipli önümi diýilýär we $f''(x)$ ýa-da $y''(x)$ bilen, ýa-da gysgaça y'' bilen belgilenýär. Umuman, eger $y = f(x)$ funksiýanyň $(n-1)$-nji tertipli $f^{(n-1)}(x)$ önümi kesgitlenen bolsa, onda ol önümiň x nokatdaky birinji önümine $y = f(x)$ funksiýanyň x nokatdaky n-nji önümi ýa-da n tertipli önümi diýilýär:

$$f^{(n)}(x) = [f^{(n-1)}(x)]' = \lim_{\Delta x \to 0} \frac{f^{(n-1)}(x + \Delta x) - f^{(n-1)}(x)}{\Delta x}.$$

Kesgitlemeden görnüşi ýaly, ýokary tertipli önümleri tapmak üçin diňe birinji tertipli önümleri tapmagy başarmaly.

20-nji mysal. $y = \cos x$ funksiýanyň n-nji tertipli önümini tapmaly.

Ç.B. Bu funksiýanyň n-nji tertipli önümi üçin

$$(\cos x)^{(n)} = \cos\left(x + n\frac{\pi}{2}\right) \tag{1}$$

formulanyň dogrudygyny subut edeliň,

$$(\cos x)' = -\sin x = \cos\left(x + \frac{\pi}{2}\right)$$

deňlik (1) formulanyň $n = 1$ bolanda dogrudygyny görkezýär. Goý, ol formula käbir $n = k$ üçin dogry bolsun, onda $n = k + 1$ üçin

$$(\cos x)^{(k+1)} = [(\cos x)^{(k)}]' = \left[\cos\left(x + k\frac{\pi}{2}\right)\right]' =$$
$$= -\sin\left(x + k\frac{\pi}{2}\right) = \cos\left(x + (k+1)\frac{\pi}{2}\right).$$

Bu bolsa (1) formulanyň $n = k + 1$ bolanda hem dogrudygyny görkezýär. Şonuň üçin matematiki induksiýa usuly esasynda (1) formula $\forall n \in N$ üçin dogrudyr. **Ç.S.**

Edil şuňa meňzeş

$$(\sin x)^{(n)} = \sin\left(x + n\frac{\pi}{2}\right), \quad (\ln(1+x))^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

formulalaryň hem dogrudygyny görkezmek bolar.

Eger $u = u(x)$ we $v = v(x)$ funksiýalaryň x nokatda n tertipli önümleri bar bolsa, onda $u \pm v$ we $u \cdot v$ funksiýalaryň x nokatda n tertipli önümleri bardyr we ol önümler üçin

$$(u \pm v)^{(n)} = u^{(n)} \pm v^{(n)}, \tag{2}$$

$$(u \cdot v)^{(n)} = \sum_{m=0}^{n} C_n^m u^{(n-m)} v^{(m)} \tag{3}$$

formulalar dogrudyr. Olaryň ikinjisine Leýbnisiň formulasy diýilýär.

21-nji mysal. $y = x^3\cos x$ funksiýanyň n-nji önümini tapmaly.

Ç.B. Goý, $u = \cos x$ we $v = x^3$ bolsun, onda

$$u^{(n)} = (\cos x)^{(n)} = \cos\left(x + n\frac{\pi}{2}\right),$$

$$v' = 3x^2, \quad v^{(2)} = 6x, \quad v^{(3)} = 6, \quad v^{(4)} = v^{(5)} = \ldots = 0$$

bolýandygy üçin, Leýbnisiň formulasyndan peýdalanyp taparys:

$$(x^3\cos x)^{(n)} = x^3\cos\left(x + n\frac{\pi}{2}\right) + 3nx^2\cos\left[x + (n-1)\frac{\pi}{2}\right] +$$

$$+3n(n-1)x\cos\left[x + (n-2)\frac{\pi}{2}\right] + \ldots + n(n-1)(n-2)\cos\left[x + (n-3)\frac{\pi}{2}\right].$$ **Ç.S.**

Eger $F(x, y) = 0$ deňleme bilen käbir $y = y(x)$ funksiýa anyk däl görnüşde kesgitlenýän bolsa, onda ol deňlemäniň iki bölegini hem differensirläp, $y'(x)$ önümiň nähili tapylýandygy bize ozaldan mälimdir. Şonuň üçin differensirlenip alnan deňligi ýene bir gezek differensirläp we alnan deňlemede birinji önümiň bahasyny goýup, funksiýanyň ikinji önümini tapmak bolar.

22-nji mysal. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ deňleme bilen kesgitlenýän $y = y(x)$ funksiýanyň ikinji tertipli önümini tapmaly.

Ç.B. Aşakdaky funksiýalara çylşyrymly funksiýa hökmünde garap, deňligiň iki bölegini hem differensirläliň we birinji önümi tapalyň:

$$\frac{2x}{a^2} - \frac{2y}{b^2}y' = 0, \quad \frac{x}{a^2} - \frac{y}{b^2}y' = 0, \quad y' = \frac{b^2 x}{a^2 y}.$$

Differensirlenip alnan deňligi ýene bir gezek differensirläp we birinji önümiň bahasyny deňlemede goýup, ikinji önümi tapalyň:

$$\frac{1}{a^2} - \frac{1}{b^2}y'^2 - \frac{y}{b^2}y'' = 0,$$

$$y'' = \frac{1}{y}\left(\frac{b^2}{a^2} - y'^2\right) = \frac{1}{y}\left(\frac{b^2}{a^2} - \frac{b^4}{a^4}\frac{x^2}{y^2}\right) = -\frac{b^4}{a^2y^3}\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = -\frac{b^4}{a^2y^3}. \quad \textbf{Ç.S.}$$

Eger funksiýa $x = \varphi(t)$, $y = \psi(t)$ parametrik görnüşde berlen bolsa, onda $y = y(x)$ funksiýanyň birinji önümi

$$y'(x) = \frac{\psi'(t)}{\varphi'(t)}$$

formula boýunça tapylýar. Bu formuladan hem-de çylşyrymly we ters funksiýalaryň önümleri tapylýan formulalardan peýdalanyp, ikinji $y''(x)$ önümi taparys:

$$y''(x) = \left[\frac{\psi'(t)}{\varphi'(t)}\right]'_x = \frac{\left[\frac{\psi'(t)}{\varphi'(t)}\right]'_t}{\varphi'(t)} = \frac{\psi''(t)\varphi'(t) - \psi'(t)\varphi''(t)}{\varphi'^3(t)}.$$

Bu önümi ulanyp, funksiýanyň üçünji we soňky önümleri tapylýar.

2. *Ýokary tertipli differensiallar.* Mälim bolşy ýaly, eger $y = y(x)$ funksiýa x nokatda differensirlenýän bolsa, onda onuň differensialy

$$dy = f'(x)dx \tag{4}$$

formula boýunça kesgitlenilýär we oňa $y = y(x)$ funksiýanyň x nokatdaky birinji ýa-da birinji tertipli differensialy diýilýär.

$y = f(x)$ funksiýanyň birinji dy differensialynyň x nokatdaky differensialyna $y = f(x)$ funksiýanyň x nokatdaky ikinji differensialy diýilýär we d^2y ýa-da $d^2f(x)$ bilen belgilenilýär.

Şeýlelikde,

$$d^2y = d(dy) \quad \text{ýa-da} \quad d^2f(x) = d(df(x)).$$

Şuňa meňzeşlikde,

$$d^ny = d(d^{n-1}y) \quad \text{we} \quad d^ny = y^{(n)}dx^n.$$

Ýokary tertipli differensiallar üçin

$$d^n(u \pm v) = (u \pm v)^{(n)}dx^n = u^{(n)}dx^n \pm v^{(n)}dx^n = d^nu \pm d^nv,$$

$$d^n(u \cdot v) = (u \cdot v)^{(n)}dx^n = \sum_{m=0}^{n} C_n^m u^{(n-m)} v^{(m)} dx^n =$$

$$= \sum_{m=0}^{n} C_n^m u^{(n-m)} dx^{n-m} v^{(m)} dx^m = \sum_{m=0}^{n} C_n^m d^{n-m} u \cdot d^m v$$

formulalar dogrudyr.

Gönükmeler

Funksiýalaryň ikinji tertipli önümlerini tapmaly:

302. $y = x\sqrt{1+x^2}$.

303. $y = \dfrac{x}{\sqrt{1-x^2}}$.

304. $y = e^{-x^2}$.

305. $y = \mathrm{tg}x$.

306. $y = (1+x^2)\mathrm{arctg}x$.

307. $y = \dfrac{\arcsin x}{\sqrt{1-x^2}}$.

308. $y = x\ln x$.

309. $y = \ln f(x)$.

310. $y = x[\sin(\ln x) + \cos(\ln x)]$.

311. $y = e^{\sin x}\cos(\sin x)$ funksiýanyň $y(0)$, $y'(0)$ we $y''(0)$ bahalaryny tapmaly.

Iki gezek differensirlenýän $u = \varphi(x)$ we $\upsilon = \psi(x)$ funksiýalar üçin y'' önümi tapmaly:

312. $y = u^2$.

313. $y = \ln\dfrac{u}{\upsilon}$.

314. $y = \sqrt{u^2 + \upsilon^2}$.

315. $y = u^{\upsilon}$ $(u > 0)$.

Üç gezek differensirlenýän $f(x)$ funksiýa üçin y'' we y''' önümleri tapmaly:

316. $y = f(x^2)$.

317. $y = f\left(\dfrac{1}{x}\right)$.

318. $y = f(e^x)$.

319. $y = f(\ln x)$.

320. $y = f(\varphi(x))$; bu ýerde $\varphi(x)$ ýeterlik tertipde differensirlenýän funksiýa.

321. $y = e^x$ funksiýanyň iki ýagdaýda-da ikinji differensialyny tapmaly:
a) x baglanyşyksyz üýtgeýän ululyk; b) x aralyk argument.

Baglanyşyksyz üýtgeýän x ululyk üçin d^2y ikinji differensialy tapmaly:

322. $y = \sqrt{1 + x^2}$. **323.** $y = \frac{\ln x}{x}$.

324. $y = x^x$.

Iki gezek differensirlenýän u we ϑ funksiýalar üçin d^2y differensialy tapmaly:

325. $y = u\vartheta$. **326.** $y = \frac{u}{\vartheta}$.

327. $y = u^m\vartheta^n$(bu ýerde m we n hemişelik ululyklar).

328. $y = a^u$; $(a > 0)$. **329.** $y = \ln\sqrt{u^2 + \vartheta^2}$.

330. $y = \operatorname{arctg}\frac{u}{\vartheta}$.

Parametrik görnüşde berlen $y = y(x)$ funksiýanyň y_x', y''_{x^2}, y'''_{x^3} önümlerini tapmaly:

331. $x = 2t - t^2$, $y = 3t - t^3$. **332.** $x = a\cos t$, $y = a\sin t$.

333. $x = a(t - \sin t)$, $y = a(1 - \cos t)$. **334.** $x = e^t\cos t$, $y = e^t\sin t$.

335. $x = f'(t)$, $y = tf'(t) - f(t)$.

336. Goý, $y = f(x)$ ýeterlik tertipde differensirlenýän funksiýa bolsun. $x = f^{-1}(y)$ ters funksiýanyň x', x'', x''', x^{IV} önümlerini tapmaly (ol önümler bar hasap etmeli).

Anyk däl görnüşde berlen $y = y(x)$ funksiýanyň y_x', y''_{x^2} we y'''_{x^3} önümlerini tapmaly:

337. $x^2 + y^2 = 25$. Funksiýanyň y', y'', y''' önümleriniň $M(3, 4)$ nokatdaky bahalary näçä deň?

338. $y = 2px$. **339.** $x^2 - xy + y^2 = 1$.

Anyk däl görnüşde berlen $y = f(x)$ funksiýanyň y_x' we y''_{x^2} önümlerini tapmaly:

340. $y^2 + 2\ln y = x^4$. **341.** $\sqrt{x^2 + y^2} = ae^{\operatorname{arctg}\frac{y}{x}}$ $(a > 0)$.

342. Goý, $f(x)$ funksiýa $x \le x_0$ bolanda kesgitlenen we iki gezek differensirlenýän bolsun. a, b, c koeffisiýentleri nähili saýlanyňda

$$F(x) = \begin{cases} f(x), & \text{eger} \quad x \le x_0; \\ a(x - x_0)^2 + b(x - x_0) + c, & \text{eger} \quad x > x_0 \end{cases}$$

bolsa, funksiýa iki gezek differensirlenýär?

343. Nokat

$$s = 10 + 20t - 5t^2$$

düzgün boýunça göni çyzykly hereket edýär. Hereketiň tizligini we tizlenmesini tapmaly. $t = 2$ pursatda tizlik we tizlenme näçä deň bolar?

344. $M(x, y)$ nokat $x^2 + y^2 = a^2$ töwerek boýunça T *sekuntda* bir aýlaw geçip hereket edýär. $t = 0$ bolanda $M_0(a, 0)$ nokatda ýerleşýän M nokadyň Ox okuna proýeksiýasynyň ϑ tizligini we j tizlenmesini tapmaly.

345. Agyr material $M(x, y)$ nokat dik Oxy tekizliginde gorizontal α burç we başlangyç ϑ_0 tizlik bilen zyňylan. Howanyň garşylygyny hasaba almazdan, hereketiň deňlemesini düzmeli we tizligiň ϑ we tizlenmäniň j ululygyny, şeýle hem hereketiň traýektoriýasyny kesgitlemeli. Nokadyň galan iň ýokarky beýikligi we uçuşyň daşlygy näçä deň?

346. Nokadyň hereket deňlemesi:

$$x = 4\sin \omega t - 3\cos \omega t, \quad y = 3\sin \omega t + 4\cos \omega t$$

bu ýerde ω – hemişelik.

Hereketiň traýektoriýasyny we tizligiň, tizlenmäniň ululygyny kesgitlemeli.

Funksiýalaryň görkezilen tertipdäki önümlerini tapmaly:

347. Eger $y = x(2x - 1)^2(x + 3)^3$ bolsa, onda $y^{(6)}$ we $y^{(7)}$.

348. Eger $y = \dfrac{a}{x^m}$ bolsa, onda y'''.

349. Eger $y = \sqrt{x}$ bolsa, onda $y^{(10)}$.

350. Eger $y = \dfrac{x^2}{1 - x}$ bolsa, onda $y^{(8)}$.

351. Eger $y = \dfrac{1 + x}{\sqrt{1 - x}}$ bolsa, onda $y^{(100)}$.

352. Eger $y = x^2e^{2x}$ bolsa, onda $y^{(20)}$.

353. Eger $y = \dfrac{e^x}{x}$ bolsa, onda $y^{(10)}$.

354. Eger $y = x\ln x$ bolsa, onda $y^{(5)}$.

355. Eger $y = \dfrac{\ln x}{x}$ bolsa, onda $y^{(5)}$.

356. Eger $y = x^2\sin 2x$ bolsa, onda $y^{(50)}$.

357. Eger $y = \dfrac{\cos 3x}{\sqrt[3]{1 - 3x}}$ bolsa, onda y'''.

358. Eger $y = \sin x \sin 2x \sin 3x$ bolsa, onda $y^{(10)}$.

359. Eger $y = x\operatorname{sh}x$ bolsa, onda $y^{(100)}$.

360. Eger $y = e^x \cos x$ bolsa, onda y^{IV}.

361. Eger $y = \sin^2 x \ln x$ bolsa, onda $y^{(6)}$.

Baglanyşyksyz üýtgeýän x ululyk üçin funksiýalaryň görkezilen tertipdäki differensiallaryny tapmaly:

362. Eger $y = x^5$ bolsa, onda d^5y.

363. Eger $y = 1/\sqrt{x}$ bolsa, onda d^3y.

364. Eger $y = x\cos 2x$ bolsa, onda $d^{10}y$.

365. Eger $y = e^x \ln x$ bolsa, onda d^4y.

366. Eger $y = \cos x \cdot \operatorname{ch}x$ bolsa, onda d^6y.

Ýeterlik tertipde differensirlenýän u funksiýa üçin aşakdaky funksiýalaryň görkezilen tertipdäki differensiallaryny tapmaly:

367. Eger $y = u^2$ bolsa, onda $d^{10}y$.

368. Eger $y = e^u$ bolsa, onda d^4y.

369. Eger $y = \ln u$ bolsa, onda d^3y.

370. $y = f(x)$ funksiýanyň x-iň käbir baglanyşyksyz üýtgeýän ululygyň funksiýasy bolan haly üçin $d^2(y)$, $d^3(y)$ we $d^4(y)$ differensiallaryny tapmaly.

371. $y = f(x)$ funksiýanyň y'' we y''' önümlerini x-i baglanyşyksyz üýtgeýän ululyk hasap etmezden, üýtgeýän x we y ululyklaryň yzygiderli differensiallary arkaly aňlatmaly.

372. Hemişelik erkin C_1 we C_2 üçin $y = C_1\cos x + C_2\sin x$ funksiýanyň

$$y'' + y = 0$$

deňlemäni kanagatlandyrýandygyny subut etmeli.

373. Hemişelik erkin C_1 we C_2 üçin $y = C_1\operatorname{ch}x + C_2\operatorname{sh}x$ funksiýanyň

$$y'' - y = 0$$

deňlemäni kanagatlandyrýandygyny subut etmeli.

374. Hemişelik erkin C_1 we C_2, şeýle hem hemişelik λ_1, λ_2 üçin $y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$ funksiýanyň

$$y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y = 0$$

deňlemäni kanagatlandyrýandygyny subut etmeli.

375. Hemişelik erkin C_1, C_2 we n üçin

$$y = x^n[C_1\cos(\ln x) + C_2\sin(\ln x)],$$

funksiýanyň

$$x^2y'' + (1 - 2n)xy' + (1 + n^2)y = 0$$

deňlemäni kanagatlandyrýandygyny subut etmeli.

376. Hemişelik erkin C_1, C_2, C_3 we C_4 üçin

$$y = e^{x/\sqrt{2}}\left(C_1\cos\frac{x}{\sqrt{2}} + C_2\sin\frac{x}{\sqrt{2}}\right) + e^{-x/\sqrt{2}}\left(C_3\cos\frac{x}{\sqrt{2}} + C_4\sin\frac{x}{\sqrt{2}}\right)$$

funksiýanyň

$$y^{\mathrm{IV}} + y = 0$$

deňlemäni kanagatlandyrýandygyny subut etmeli.

377. Eger $f(x)$ funksiýanyň n tertipli önümi bar bolsa, onda

$$[f(ax + b)]^{(n)} = a^n f^{(n)}(ax + b)$$

deňligi subut etmeli.

378. $P(x) = a_0x^n + a_1x^{n-1} + ... + a_n$ funksiýanyň $P^{(n)}(x)$ önümini tapmaly.

Funksiýalaryň $y^{(n)}$ önümini tapmaly:

379. $y = \dfrac{ax + b}{cx + d}$.

380. $y = \dfrac{1}{x(1 - x)}$.

381. $y = \dfrac{1}{x^2 - 3x + 2}$ (*Görkezme*: *Funksiýany ýönekeý droblara dagytmaly*).

382. $y = \dfrac{1}{\sqrt{1 - 2x}}$.

383. $y = \dfrac{x}{\sqrt[3]{1 + x}}$.

384. $y = \sin^2 x$.

385. $y = \cos^2 x$.

386. $y = \sin^3 x$.

387. $y = \cos^3 x$.

388. $y = \sin ax \sin bx$.

389. $y = \cos ax \cos bx$.

390. $y = \sin ax \cos bx$.

391. $y = \sin^2 ax \cos bx$.

392. $y = \sin^4 x + \cos^4 x$.

393. $y = x\cos ax$.

394. $y = x^2\sin ax$.

395. $y = (x^2 + 2x + 2)e^{-x}$.

396. $y = e^x/x$.

397. $y = e^x\cos x$.

398. $y = e^x\sin x$.

399. $y = \ln\dfrac{a + bx}{a - bx}$.

400. $y = e^{ax}P(x)$, bu ýerde $P(x)$ – köpagza.

401. $y = x\operatorname{sh}x$.

Funksiýalaryň $d^n y$ differensialyny tapmaly:

402. $y = x^n e^x$. **403.** $y = \dfrac{\ln x}{x}$.

404. Deňlikleri subut etmeli:
1) $[e^{ax}\sin(bx + c)]^{(n)} = e^{ax}(a^2 + b^2)^{n/2}\sin(bx + c + n\varphi)$ we
2) $[e^{ax}\cos(bx + c)]^{(n)} = e^{ax}(a^2 + b^2)^{n/2}\cos(bx + c + n\varphi)$, bu ýerde

$$\sin\varphi = \frac{b}{\sqrt{a^2 + b^2}} \quad \text{we} \quad \cos\varphi = \frac{a}{\sqrt{a^2 + b^2}}.$$

405. Funksiýalaryň $y^{(n)}$ önümini tapmaly:
a) $y = \text{ch}ax \cos bx$; b) $y = \text{ch}ax \sin bx$.

406. $f(x) = \sin^{2p}x$ (p – natural san) funksiýany

$$f(x) = \sum_{k=0}^{p} A_k \cos 2kx$$

köpagza özgerdip, $f^{(n)}(x)$ önümi tapmaly. (*Görkezme*: $\sin x = \dfrac{1}{2i}(t - \bar{t})$ *almaly, bu ýerde* $t = \cos x + i\sin x$ *we* $\bar{t} = \cos x - i\sin x$ *we Muawryň formulasyndan peýdalanmaly*).

407. Funksiýalaryň $f^{(n)}(x)$ önümini tapmaly;
a) $f(x) = \sin^{2p+1}x$; b) $f(x) = \cos^{2p}x$; ç) $f(x) = \cos^{2p+1}x$.
Bu ýerde p – bitin položitel san (öňündäki mysala seret).
Eger

$$f(x) = f_1(x) + if_2(x)$$

bolsa, onda

$$f'(x) = f_1'(x) + if_2'(x),$$

bu ýerde $i = \sqrt{-1}$ we $f_1(x), f_2(x)$ hakyky x-iň hakyky funksiýalary.

408. $\dfrac{1}{x^2 + 1} = \dfrac{1}{2i}\left(\dfrac{1}{x - i} - \dfrac{1}{x + i}\right)$ toždestwodan peýdalanyp,

$$\left(\frac{1}{x^2 + 1}\right)^{(n)} = \frac{(-1)^n n!}{(1 + x^2)^{(n+1)/2}}\sin[(n + 1)\text{arcctg}x]$$

deňligi subut etmeli. (*Görkezme*: *Muawryň formulasyndan peýdalanmaly*).

409. $f(x) = \text{arctg}x$ funksiýanyň $f^{(n)}(x)$ önümini tapmaly.

Berlen $f(x)$ funksiýalar üçin $f^{(n)}(0)$ tapmaly:

410. a) $f(x)=\dfrac{1}{(1-2x)(1+x)}$; b) $f(x)=\dfrac{x}{\sqrt{1-x}}$.

411. a) $f(x)=x^2e^{ax}$; b) $f(x)=\operatorname{arctg}x$; ç) $f(x)=\arcsin x$.

412. a) $f(x)=\cos(m\arcsin x)$; b) $f(x)=\sin(m\arcsin x)$.

413. a) $f(x)=(\operatorname{arctg}x)^2$; b) $f(x)=(\arcsin x)^2$.

414. $f(x)=(x-a)^n\varphi(x)$ funksiýanyň $f^{(n)}(a)$ önümini tapmaly, bu ýerde $\varphi(x)$ funksiýanyň a nokadyň golaý töwereginde $(n-1)$ tertipli üznüksiz önümi bardyr.

415. $f(x)=\begin{cases} x^{2n}\sin\dfrac{1}{x}, & \text{eger } x\neq 0,\\ 0, & \text{eger } x=0\end{cases}$

funksiýanyň $x=0$ nokatda n-e (n – natural san) çenli tertipli önümleriniň bardygyny we $(n+1)$ tertipli önüminiň ýokdugyny subut etmeli.

416. $f(x)=\begin{cases} e^{-1/x^2}, & \text{eger } x\neq 0,\\ 0, & \text{eger } x=0\end{cases}$

funksiýanyň $x=0$ nokatda tükeniksiz tertipli önüminiň bardygyny subut etmeli.

417. Çebyşewiň

$$T_m(x)=\frac{1}{2^{m-1}}\cos(m\arccos x)\qquad (m=1,2,\ldots)$$

köpagzasynyň

$$(1-x^2)T_m''(x)-xT_m'(x)+m^2T_m(x)=0$$

deňlemäni kanagatlandyrýandygyny subut etmeli.

418. Ležandryň

$$P_m(x)=\frac{1}{2^m m!}[(x^2-1)^m]^{(m)}\qquad (m=0,1,\ldots)$$

köpagzasynyň

$$(1-x^2)P_m''(x)-2xP_m'(x)+m(m+1)P_m(x)=0$$

deňlemäni kanagatlandyrýandygyny subut etmeli. (*Görkezme*: $(x^2-1)u'=2mxu$ *deňligi* $m+1$ *gezek differensirlemeli, bu ýerde* $u=(x^2-1)^m$).

419. Çebyşew-Lagerranyň köpagzalary

$$L_m(x)=e^x(x^me^{-x})^{(m)}\qquad (m=0,1,\ldots)$$

formula bilen kesgitlenýär.

$L_m(x)$ köpagzany anyk görnüşde kesgitlemeli. $L_m(x)$ funksiýanyň

$$xL_m''(x)+(1-x)L_m'(x)+mL(x)=0$$

deňlemäni kanagatlandyrýandygyny subut etmeli. (*Görkezme*: $xu'+(x-m)u=0$ *deňligi ulanmaly, bu ýerde* $u = x^m e^{-x}$).

420. Goý, $y = f(u)$ we $u = \varphi(x)$ bolsun, bu ýerde $f(u)$ we $\varphi(x)$ n gezek differensirlenýän funksiýalar.

$$\frac{d^n y}{dx^n} = \sum_{k=1}^{n} A_k(x) f^{(k)}(u)$$

deňligi subut etmeli, bu ýerde $A_k(x)$ ($k = 0, 1, ..., n$) we $f(u)$ funksiýalar bagly däldirler.

421. Çylşyrymly $y = f(x^2)$ funksiýanyň n-nji tertipli önümi üçin

$$\frac{d^n y}{dx^n} = (2x)^n f^{(n)}(x^2) + \frac{n(n-1)}{1!}(2x)^{n-2} f^{(n-1)}(x^2) +$$
$$+\frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} f^{(n-2)}(x^2) + ...$$

formulanyň dogrudygyny subut etmeli.

422. Çebyşew-Ermitiň köpagzalary

$$H_m(x) = (-1)^m e^{x^2}(e^{-x^2})^{(m)} \quad (m = 0, 1, 2, ...)$$

formula bilen kesgitlenýär. $H_m(x)$ köpagzany anyk görnüşde ýazmaly we

$$H_m''(x) - 2x\, H_m'(x) + 2m\, H_m(x) = 0$$

deňlemäni kanagatlandyrýandygyny subut etmeli. (*Görkezme*: $u'+2xu=0$ *deňligi ulanmaly, bu ýerde* $u = e^{-x^2}$).

423. Deňligi subut etmeli:

$$(x^{n-1} e^{1/x})^{(n)} = \frac{(-1)^n}{x^{n+1}} e^{1/x}$$

(*Görkezme*: *Matematiki induksiýa usulyndan peýdalanmaly*).

424. Formulany subut etmeli:

$$\frac{d^n}{dx^n}(x^n \ln x) = n!\left(\ln x + \sum_{k=1}^{n} \frac{1}{k}\right) \quad (x > 0).$$

425. Formulany subut etmeli:

$$\frac{d^{2n}}{dx^{2n}}\left(\frac{\sin x}{x}\right) = \frac{(2n)!}{x^{2n+1}}[C_n(x)\sin x - S_n(x)\cos x],$$

bu ýerde

$$C_n(x) = 1 - \frac{x^2}{2!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!}$$

we

$$S_n(x) = x - \frac{x^3}{3!} + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}.$$

426. Goý, $\frac{d}{dx} = D$ differensirleme amaly we

$$f(D) = \sum_{k=0}^{n} p_k(x) D^k$$

simwolik differensial köpagzasy bolsun, bu ýerde $p_k(x)$ $(k=0, 1, \ldots, n)$ x-e görä käbir üznüksiz funksiýalar.

$$f(D)\{e^{\lambda x}u(x)\} = e^{\lambda x} f(D+\lambda)u(x)$$

deňligi subut etmeli, bu ýerde λ – hemişelik.

427. Eger

$$\sum_{k=0}^{n} a_k x^k y_x^{(k)} = 0$$

deňlemede $x = e^t$ çalşyrma geçirsek, onda bu deňlemäniň

$$\sum_{k=0}^{n} a_k D(D-1)\ldots(D-k+1)y = 0$$

görnüşi alýandygyny subut etmeli, bu ýerde $D = \frac{d}{dt}$ we t baglanyşyksyz üýtgeýän ululyk.

V. DIFFERENSIRLENÝÄN FUNKSIÝALAR HAKYNDAKY ESASY TEOREMALAR

§1. Funksiýanyň orta bahasy hakyndaky teoremalar

1. ***Önümiň noly hakyndaky teoremalar.*** Goý, f funksiýa c nokadyň käbir $U(c)$ golaý töwereginde kesgitlenen bolsun.

Fermanyň teoremasy. Eger f funksiýa $c \in (a, b)$ nokatda differensirlenýän bolup, şol nokatda iň kiçi ýa-da iň uly bahany alýan bolsa, onda $f'(c) = 0$.

Önümiň noly hakyndaky teorema. Eger f funksiýa $[a, b]$ kesimde differensirlenýän bolsa we $f'_+(a) \cdot f'_-(b) < 0$ şert ýerine ýetse, onda $(a, b) \ni c$ nokat tapylyp, $f'(c) = 0$ bolar.

Darbunyň teoremasy. Eger f funksiýa $[a, b]$ kesimde differensirlenýän bolsa, onda onuň önümi $f'_+(a)$ we $f'_-(b)$ bahalaryň arasyndaky ähli bahalary alýar.

Roluň teoremasy. Eger $[a, b]$ kesimde üznüksiz, (a, b) interwalda differensirlenýän f funksiýa üçin $f(a) = f(b)$ bolsa, onda iň bolmanda bir $c \in (a, b)$ nokat tapylyp, $f'(c) = 0$ bolar.

2. ***Orta baha hakyndaky Koşiniň we Lagranžyň teoremalary***

Koşiniň teoremasy. Eger $[a, b]$ kesimde üznüksiz f we g funksiýalar (a, b) interwalda differensirlenýän bolup, $\forall x \in (a, b)$ üçin $g'(x) \neq 0$ bolsa, onda $(a, b) \ni c$ nokat tapylyp,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \tag{1}$$

deňlik ýerine ýetýär.

Lagranžyň teoremasy. Eger $[a, b]$ kesimde üznüksiz f funksiýa, (a, b) interwalda differensirlenýän bolsa, onda $(a, b) \ni c$ nokat tapylyp, Lagranžyň formulasy diýilýän

$$f(b) - f(a) = f'(c)(b - a) \tag{2}$$

deňlik ýerine ýetýär.

Bu formuladaky c nokat a we b nokatlaryň arasyndaky nokatdyr, ýagny $a < c < b$. Onda $\theta = (c - a)/(b - a)$ üçin $0 < \theta < 1$ we $c = a + \theta(b - a)$ bolar. Şonuň üçin Lagranžyň formulasyny

$$f(b) - f(a) = f(a + \theta(b - a))(b - a) \qquad (0 < \theta < 1)$$

görnüşde ýazyp bolar.

1-nji mysal. Eger f funksiýa $[a, b]$ kesimde üznüksiz bolup, hemme içki nokatlarynda onuň otrisatel däl (položitel) önümi bar bolsa, onda onuň $[a, b]$ kesimde kemelmeýän (artýan) funksiýadygyny subut etmeli. Eger-de ol funksiýanyň položitel däl (otrisatel) önümi bar bolsa, onda onuň $[a, b]$ kesimde artmaýan (kemelmeýän) funksiýadygyny subut etmeli.

Ç.B. Goý, $\forall x_1, x_2 \in [a, b]$ üçin $x_1 < x_2$ bolsun. Onda $[x_1, x_2]$ kesimde Lagranžyň teoremasynyň hemme şertleri ýerine ýetýär we şonuň esasynda (x_1, x_2) interwalda c nokat tapylyp,

$$f(x_2) - f(x_1) = f'(c)(x_1 - x_2) \tag{3}$$

deňlik ýerine ýetýär. Şerte görä, $[a, b]$ kesimiň hemme içki nokatlarynda $f'(x) \geq 0$ ($f'(x) > 0$). Şonuň üçin $f'(c) \geq 0$ ($f'(c) > 0$). Şol sebäpli $x_2 > x_1$ bolanda (3) deňligiň sag bölegi otrisatel däldir (položiteldir), ýagny $f(x_2) - f(x_1) \geq 0$ ($f(x_2) - f(x_1) > 0$) bu bolsa f funksiýanyň $[a, b]$ kesimde kemelmeýändigini (artýandygyny) görkezýär. Eger-de $f'(x) \leq 0$ ($f'(x) < 0$) bolsa, onda f funksiýanyň artmaýandygy (kemelýändigi) şonuň ýaly subut edilýär. **Ç.S.**

2-nji mysal. Eger f funksiýa $[a, b]$ kesimde üznüksiz bolup, onuň hemme içki nokatlaryndaky önümi hemişelik k sana (nola) deň bolsa, onda ol $[a, b]$ kesimde çyzykly (hemişelik) funksiýadyr.

Ç.B. Goý, $x \in (a, b]$ erkin nokat bolsun. Onda $[a, b]$ kesimde f funksiýa Lagranžyň teoremasynyň şertlerini kanagatlandyrýar we şol teorema boýunça $(a, b) \ni c$ nokat tapylyp, $f(x) - f(a) = f'(c)(x - a)$ deňlik ýerine ýetýär. Onda bu deňlikden $f'(c) = k$ bolanda $f(x) = xk + f(a) - kb$ deňligi alarys, ýagny $f(x)$ çyzykly funksiýadyr. Bu ýerden $k = 0$ hususy hal üçin $f(x) = f(a)$ deňligi alarys. Ol bolsa funksiýanyň hemişelikdigini aňladýar. **Ç.S.**

3-nji mysal. Eger $[a, b]$ kesimde üznüksiz φ we g funksiýalaryň şol kesimiň içki nokatlaryndaky önümleri deň bolsalar, onda $[a, b]$ kesimde olaryň tapawudy hemişelikdir.

Ç.B. Şertleriň esasynda $f(x) = \varphi(x) - g(x)$ funksiýa $[a, b]$ kesimde üznüksiz bolar we onuň içki nokatlarynda $f'(x) = 0$. Şonuň üçin subudy 2-nji mysalyň ikinji böleginden gelip çykýar. **Ç.S.**

4-nji mysal. $\ln(1 + x) > x - \frac{x^2}{2}$ $(x > 0)$ deňsizligi subut etmeli.

Ç.B. Goý, $\varphi(x) = \ln(1 + x)$ we $g(x) = x - \frac{x^2}{2}$ bolsun. Onda $\varphi(0) = g(0)$,

$\varphi'(x) - g'(x) = \frac{1}{1+x} - (1 - x) > 0$, çünki $x > 0$ bolanda $\frac{1}{1+x} > 1 - x$. Şonuň

üçin $x > 0$ bolanda $\ln(1 + x) > x - \frac{x^2}{2}$. **Ç.S.**

Edil şuňa meňzeşlikde $x > 0$ bolanda $\ln(1 + x) < x$ deňsizligi görkezmek bolar. Şeýlelikde, $x - \frac{x^2}{2} < \ln(1 + x) < x \ (x > 0)$.

Gönükmeler

1. $f(x) = (x - 1)(x - 2)(x - 3)$ funksiýa üçin Roluň teoremasynyň şertleriniň ýerine ýetýändigini barlamaly.

2. $f(x) = 1 - \sqrt[3]{x^2}$ funksiýa $x_1 = -1$, $x_2 = 1$ nokatlarda nola deň, ýöne oňa garamazdan $-1 \le x \le 1$ bolanda $f'(x) \ne 0$. Bu ýagdaýy düşündirmeli.

3. Goý, $f(x)$ funksiýanyň (a, b) interwalyň tükenikli ýa-da tükeniksiz her bir nokadynda tükenikli $f'(x)$ önümi bar we

$$\lim_{x \to a+0} f(x) = \lim_{x \to b-0} f(x)$$

bolsun. $f'(c) = 0$ deňligi subut etmeli, bu ýerde c (a, b) interwalyň käbir nokadydyr.

4. Goý, 1) $f(x)$ funksiýa $[x_0, x_n]$ kesimde kesgitlenen we onuň üznüksiz $(n - 1)$ tertipli $f^{(n-1)}(x)$ önümi bar bolsun; 2) $f(x)$ funksiýanyň (x_0, x_n) interwalda n tertipli $f^{(n)}(x)$ önümi bolsun we 3) aşakdaky deňlikler ýerine ýetsin:

$$f(x_0) = f(x_1) = \ldots = f(x_n) \qquad (x_0 < x_1 < \ldots < x_n).$$

(x_0, x_n) interwalda iň bolmanda bir sany şeýle c nokat bar bolup,

$$f^{(n)}(c) = 0$$

deňligiň ýerine ýetýändigini subut etmeli.

5. Goý, 1) $f(x)$ funksiýa $[a, b]$ kesimde kesgitlenen we onuň üznüksiz $(p + q)$ tertipli $f^{(p+q)}(x)$ önümi bar bolsun; 2) $f(x)$ funksiýanyň (a, b) interwalda $(p + q + 1)$ tertipli $f^{(p+q+1)}(x)$ önümi bar bolsun; 3) aşakdaky deňlikler ýerine ýetsin:

$$f(a) = f'(a) = \ldots = f^{(p)}(a) = 0$$

we

$$f(b) = f'(b) = \ldots = f^{(q)}(b) = 0.$$

Bu halda $f^{(p+q+1)}(c) = 0$ deňligi subut etmeli, bu ýerde c nokat (a, b) interwalyň käbir nokadydyr.

6. Koeffisiýentleri hakyky sanlar bolan

$$P_n(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n \qquad (a_0 \ne 0)$$

köpagzanyň ähli kökleri hakyky sanlar bolsa, onda onuň yzygider

$$P_n'(x), P_n''(x), \ldots, P_n^{(n-1)}(x)$$

önümleriniň hem kökleri diňe hakyky sanlardyr.

7. Ležandryň

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\{(x^2-1)^n\}$$

köpagzasynyň ähli kökleriniň hakyky sanlar bolup, olaryň (–1, 1) interwalda saklanýandygyny subut etmeli.

8. Çebyşew-Lagerranyň

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$$

köpagzasynyň ähli kökleriniň položiteldigini subut etmeli.

9. Çebyşew-Ermitiň

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$$

köpagzasynyň ähli kökleriniň hakykydygyny subut etmeli.

10. $y = x^3$ egri çyzyga haýsy nokatda geçirilen galtaşma $A(-1, -1)$ we $B(2, 8)$ nokatlary birleşdirýän horda parallel bolar?

11. Eger $ab < 0$ bolsa, onda $f(x) = 1/x$ funksiýa üçin $[a, b]$ kesimde tükenikli artymlar formulasy dogry bolarmy?

12. Berlen funksiýalar üçin

$$f(x+\Delta x) - f(x) = \Delta x f'(x+\theta\Delta x) \qquad (0 < \theta < 1)$$

deňligi kanagatlandyrýan $\theta = \theta(x, \Delta x)$ funksiýany tapmaly:

a) $f(x) = ax^2 + bx + c$ $(a \neq 0)$;　　ç) $f(x) = 1/x$;

b) $f(x) = x^3$;　　d) $f(x) = e^x$.

13. Goý, $f(x) \in C^{(1)}(-\infty, +\infty)$ we islendik x we h üçin

$$f(x+h) - f(x) \equiv hf'(x)$$

toždestwo dogry bolsun. Hemişelik a we b üçin

$$f(x) = ax + b$$

deňligi subut etmeli.

14. Goý, $f(x) \in C^{(2)}(-\infty, +\infty)$ we islendik x we h üçin

$$f(x+h) - f(x) \equiv hf'\left(x + \frac{h}{2}\right)$$

toždestwo dogry bolsun. Hemişelik a, b we c üçin

$$f(x) = ax^2 + bx + c$$

deňligi subut etmeli.

15. $x \geq 0$ üçin

$$\sqrt{x+1} - \sqrt{x} = \frac{1}{2\sqrt{x+\theta(x)}}$$

deňligi subut etmeli, bu ýerde $\frac{1}{4} \leq \theta(x) \leq \frac{1}{2}$, şeýle-de,

$$\lim_{x\to+0}\theta(x) = 1/4, \qquad \lim_{x\to+\infty}\theta(x) = 1/2.$$

16. Goý,

$$f(x) = \begin{cases} \dfrac{3-x^2}{2}, & 0 \leq x \leq 1, \\ \dfrac{1}{x}, & 1 < x < +\infty \end{cases}$$

bolsun. $f(x)$ funksiýanyň [0, 2] kesimdäki tükenikli artymlarynyň formulasyndaky aralyk c-niň bahasyny tapmaly.

17. Goý, $f(x) - f(0) = xf'(c(x))$ bolsun, bu ýerde $0 < c(x) < x$. $f(x)=x\sin(\ln x)$, $x > 0$ we $f(0) = 0$ funksiýa üçin $c = c(x)$ funksiýanyň ýeterlik kiçi $(0, \varepsilon)$ $(\varepsilon > 0)$ interwalda üznüklidigini subut etmeli.

18. Goý, $f(x)$ funksiýanyň (a, b) interwalda üznüksiz $f'(x)$ önümi bar bolsun. (a, b) interwalyň islendik c nokady üçin

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \qquad (x_1 < c < x_2)$$

deňligi kanagatlandyrýan şol interwalyň x_1 we x_2 nokatlarynyň bardygyny subut etmeli. Aşakdaky mysaly hem şu görnüşde işlemeli: $f(x) = x^3$ $(-1 \leq x \leq 1)$, bu ýerde $c = 0$.

19. Deňsizlikleri subut etmeli:

a) $|\sin x - \sin y| \leq |x - y|$;

b) $py^{p-1}(x-y) \leq x^p - y^p \leq px^{p-1}(x-y)$; $\quad 0 < y < x$ we $p > 1$;

ç) $|\mathrm{arctg}\, a - \mathrm{arctg}\, b| \leq |a - b|$;

d) $\dfrac{a-b}{a} < \ln\dfrac{a}{b} < \dfrac{a-b}{b}$, $\quad 0 < b < a$.

20. $f(x) = x^2$ we $g(x) = x^3$ funksiýalar üçin $[-1, 1]$ kesimde Koşiniň formulasynyň näme üçin dogry däldigini düşündirmeli.

21. Goý, $f(x)$ funksiýa $[x_1, x_2]$ kesimde differensirlenýän bolsun, şeýle-de, $x_1 \cdot x_2 > 0$.

$$\frac{1}{x_1 - x_2}\begin{vmatrix} x_1 & x_2 \\ f(x_1) & f(x_2) \end{vmatrix} = f(\xi) - \xi f'(\xi)$$

deňligi subut etmeli, bu ýerde $x_1 < \xi < x_2$.

22. Eger $f(x)$ funksiýa tükenikli (a, b) interwalda differensirlenip, ýöne çäkli bolmasa, onda onuň $f'(x)$ önüminiň hem (a, b) interwalda çäkli däldigini subut etmeli. Ters teorema dogry däldir.

23. Eger $f(x)$ funksiýanyň tükenikli ýa-da tükeniksiz (a, b) interwalda çäkli $f'(x)$ önümi bar bolsa, onda $f(x)$ funksiýanyň (a, b) interwalda deňölçegli üznüksizdigini subut etmeli.

24. Eger $f(x)$ funksiýa tükeniksiz $(x_0, +\infty)$ interwalda differensirlenip,

$$\lim_{x \to +\infty} f'(x) = 0$$

bolsa, onda

$$\lim_{x \to +\infty} \frac{f(x)}{x} = 0$$

deňligi, ýagny $x \to +\infty$ bolanda $f(x) = o(x)$ bolýandygyny subut etmeli.

25. Eger $f(x)$ funksiýa tükeniksiz $(x_0, +\infty)$ interwalda differensirlenip, $x \to +\infty$ bolanda

$$f(x) = o(x)$$

bolsa, onda $\varliminf\limits_{x \to +\infty} |f'(x)| = 0$ bolýandygyny subut etmeli.

Hususan-da, eger $\lim\limits_{x \to +\infty} f'(x) = k$ önüm bar bolsa, onda $k = 0$.

26. Subut etmeli:

a) eger 1) $f(x)$ funksiýa $[x_0, X]$ kesimde kesgitlenen we üznüksiz bolsa; 2) $f(x)$ funksiýanyň (x_0, X) interwalda tükenikli $f'(x)$ önümi bar bolsa; 3) tükenikli ýa-da tükeniksiz

$$\lim_{x \to x_0 + 0} f'(x) = f'(x_0 + 0)$$

önüm bar bolsa, onda degişlilikde tükenikli ýa-da tükeniksiz birtaraplaýyn $f'_+(x_0)$ önüm bardyr we

$$f'_+(x_0) = f'(x_0 + 0).$$

b) $f(x) = \operatorname{arctg}\dfrac{1+x}{1-x}$ $(x \neq 1)$ we $f(1) = 0$ funksiýa üçin tükenikli $\lim\limits_{x \to 1} f'(x)$ önümiň bardygyny, ýöne $f(x)$ funksiýanyň tükenikli birtaraplaýyn $f'_-(1)$ we $f'_+(1)$ önümleriniň ýokdugyny subut etmeli.

Bu ýagdaýa geometrik taýdan düşündiriş bermeli.

27. Eger $a < x < b$ bolanda $f'(x) = 0$ bolsa, onda $a < x < b$ bolanda $f(x) = \text{const}$ bolýandygyny subut etmeli.

28. $f'(x) = k$ bolýan ýeke-täk $f(x)$ $(-\infty < x < +\infty)$ funksiýanyň çyzykly $f(x) = kx + b$ funksiýadygyny subut etmeli.

29. Eger $f^{(n)}(x) = 0$ bolsa, onda $f(x)$ barada näme aýdyp bolar?

30. Goý, $f(x) \in C^{(\infty)}(-\infty, +\infty)$ we her bir x üçin şeýle natural n_x $(n_x \leq n)$ san tapylyp, $f^{(n_x)}(x) = 0$ bolsun. $f(x)$ funksiýanyň köpagzadygyny subut etmeli.

31. $y' = \lambda y$ (λ – hemişelik) deňlemäni kanagatlandyrýan ýeke-täk

$$y = y(x) \quad (-\infty < x < +\infty)$$

funksiýanyň görkezijili

$$y = Ce^{\lambda x}$$

funksiýa bolýandygyny subut etmeli, bu ýerde C – hemişelik.

32. $f(x) = \operatorname{arctg}\dfrac{1+x}{1-x}$ we $g(x) = \operatorname{arctg}x$ funksiýalaryň

1) $x < 1$ we 2) $x > 1$

ýaýlalarda deň önümleriniň bardygyny subut etmeli.

33. Toždestwolary subut etmeli:

a) $2\operatorname{arctg}x + \arcsin\dfrac{2x}{1+x^2} = \pi\operatorname{sgn}x$, $|x| \geq 1$ bolanda;

b) $3\arccos x - \arccos(3x - 4x^3) = \pi$, $|x| \leq \dfrac{1}{2}$ bolanda.

34. Eger 1) $f(x)$ funksiýa $[a, b]$ kesimde üznüksiz bolsa;
2) kesimiň içinde tükenikli $f'(x)$ önümi bar bolsa;
3) çyzykly däl bolsa, onda (a, b) interwalda iň bolmanda bir c nokat tapylyp,

$$|f'(c)| > \left|\frac{f(b) - f(a)}{b - a}\right|$$

deňsizligiň ýerine ýetýändigini subut etmeli.

Oňa geometrik taýdan düşündiriş bermeli.

35. Eger, 1) $f(x)$ funksiýanyň $[a, b]$ kesimde ikinji tertipli $f''(x)$ önümi bar we 2) $f'(a) = f'(b) = 0$ bolsa, onda (a, b) interwalda iň bolmanda bir c nokat tapylyp,

$$|f''(c)| \geq \frac{4}{(b-a)^2}|f(b) - f(a)|$$

deňsizligiň dogrudygyny subut etmeli.

36. Awtomobil başlangyç nokatdan herekete başlap, *t sekunt* ýöräp barmaly ýerine barypdyr we şonda *s metr* ýol geçipdir. Käbir wagt pursadynda awtomobiliň hereketiniň tizlenmesiniň absolýut ulylygynyň $\dfrac{4s}{t^2}\dfrac{m}{s^2}$-dan kiçi däldigini subut etmeli.

§2. Monoton we güberçek funksiýalar. Epin nokatlary

1. ***Monoton funksiýalar.*** Eger $\forall x_1, x_2 \in (a, b)$ üçin $x_1 < x_2$ bolanda $f(x_1) < f(x_2)$ $(f(x_1) > f(x_2))$ bolsa, onda f funksiýa (a, b) artýan (kemelýän) funksiýa diýilýär. Eger-de $f(x_1) \leq f(x_2)$ $(f(x_1) \geq f(x_2))$ bolsa, onda f funksiýa (a, b) interwalda kemelmeýän (artmaýan) funksiýa diýilýär.

2. ***Funksiýanyň monotonlyk nyşanlary.*** (a, b) interwalda differensirlenýän f funksiýanyň şol interwalda kemelmeýän (artmaýan) bolmagy üçin (a, b) aralykda $f'(x) \geq 0$ $(f'(x) \leq 0)$ bolmagy zerur we ýeterlikdir.

(a, b) interwalda differensirlenýän f funksiýanyň şol interwalda artmagy (kemelmegi) üçin $f'(x) > 0$ $(f'(x) < 0)$ bolmagy ýeterlikdir.

3. ***Güberçek funksiýalar.*** Eger (a, b) interwalda kesgitlenen f funksiýa $\forall x_1, x_2 \in (a, b)$ we $\alpha > 0, \beta > 0, \alpha + \beta = 1$ üçin

$$f(\alpha x_1 + \beta x_2) < \alpha f(x_1) + \beta f(x_2),$$

$$(f(\alpha x_1 + \beta x_2) > \alpha f(x_1) + \beta f(x_2))$$

deňsizlikleri kanagatlandyrýan bolsa, onda oňa (a, b) interwalda aşak (ýokaryk) güberçek funksiýa diýilýär.

Differensirlenýän f funksiýanyň (a, b) interwalda aşak (ýokaryk) güberçek bolmagy üçin onuň f' önüminiň (a, b) interwalda kemelmeýän (artmaýan) bolmagy zerur we ýeterlikdir.

(a, b) interwalda ikinji önümi bar bolan f funksiýanyň şol interwalda aşak (ýokaryk) güberçek bolmagy üçin (a, b) interwalda $f''(x) > 0$ $(f''(x) < 0)$ bolmagy zerur we ýeterlikdir.

f funksiýanyň aşak (ýokaryk) güberçekliginiň geometrik manysy şeýledir: funksiýanyň çyzgysynyň, islendik dugasynyň nokatlarynyň şol dugany dartýan hordadan aşakda (ýokarda) ýerleşýändigini aňladýar.

4. ***Epin nokatlary.*** Eger a nokadyň käbir golaý töwereginde kesgitlenen f funksiýanyň şol golaý töwereginde a nokadyň çepinde we sagynda güberçeklik ugurlary dürli bolsa, onda a nokada f funksiýanyň, $(a, f(a))$ nokada bolsa onuň çyzgysynyň epin nokady diýilýär. Eger a epin nokadynda f funksiýanyň ikinji önümi bar bolsa, onda $f''(a)=0$ (zerur şerti).

Eger a nokadyň käbir golaý töwereginde f funksiýanyň ikinji önümi bar bolup, ýa $f''(a)=0$, ýa-da $f''(a)$ önüm ýok bolup, funksiýanyň ikinji f'' önümi a nokatdan geçende alamatyny üýtgedýän bolsa, onda a nokat f funksiýanyň epin nokadydyr.

1-nji mysal. $f(x) = \dfrac{1}{x} + 4x^2$ funksiýanyň epin nokatlaryny, aşak we ýokaryk güberçek interwallaryny tapmaly.

Ç.B. Funksiýanyň ikinji önümini tapalyň:

$$f'(x) = -\frac{1}{x^2} + 8x, \qquad f''(x) = \frac{2}{x^3} + 8 = 8\frac{x^3 + 1/4}{x^3}.$$

Diýmek, funksiýanyň ikinji önümi $x = 0$ nokatda tükeniksizlige deňdir we $x = -1/\sqrt[3]{4}$ nokatda nola deňdir. Şonuň üçin hem funksiýanyň kesgitleniş ýaýlasyny $(-\infty, -1/\sqrt[3]{4})$, $(-1/\sqrt[3]{4}, 0)$, $(0, +\infty)$ interwallara bölüp, olaryň her birinde ikinji önümiň alamatyny kesgitläliň.

1) eger $x \in (-\infty, -1/\sqrt[3]{4})$ bolsa, onda $f''(x) > 0$ we funksiýa aşaklygyna güberçekdir.

2) eger $x \in (-1/\sqrt[3]{4}, 0)$ bolsa, onda $f''(x) < 0$ we funksiýa ýokarlygyna güberçekdir.

3) eger $x \in (0, +\infty)$ bolsa, onda $f''(x) > 0$ we funksiýa aşaklygyna güberçekdir.

Şeýlelikde, ikinji önüm $x = -1/\sqrt[3]{4}$ we $x = 0$ nokatlardan geçende alamatyny üýtgedýär. Şonda $x = -1/\sqrt[3]{4}$ funksiýanyň epin nokadydyr, $x = 0$ bolsa epin nokady däldir, çünki ol nokatda funksiýa kesgitlenmedikdir. **Ç.S.**

Gönükmeler

Funksiýalaryň artýan ýa-da kemelýän interwallaryny kesgitlemeli.

37. $y = 2 + x - x^2$.

38. $y = 3x - x^3$.

39. $y = \dfrac{2x}{1 + x^2}$.

40. $y = \dfrac{\sqrt{x}}{x + 100}$ $(x \geq 0)$.

41. $y = x + \sin x$.

42. $y = x + |\sin 2x|$.

43. $y = \cos\dfrac{\pi}{x}$.

44. $y = \dfrac{x^2}{2^x}$.

45. $y = x^n e^{-x}$ $(n > 0, x \geq 0)$.

46. $y = x^2 - \ln x^2$.

47. $f(x) = x\left(\sqrt{\dfrac{3}{2}} + \sin \ln x\right)$, $x > 0$ bolanda we $f(0) = 0$.

48. n-iň artmagy bilen töweregiň içinden çyzylan dogry n-burçlugyň p_n perimetriniň artýandygyny we daşyndan çyzylan dogry n-burçlugyň P_n perimetriniň kemelýändigini subut etmeli. Şondan peýdalanyp, $n \to \infty$ bolanda p_n we P_n perimetrleriň umumy predelleriniň bardygyny subut etmeli.

49. $\left(1 + \dfrac{1}{x}\right)^x$ funksiýanyň $(-\infty, -1)$ we $(0, +\infty)$ interwallarda artýandygyny subut etmeli.

50. Bitin rasional

$$P(x) = a_0 + a_1x + ... + a_nx^n \qquad (n \geq 1,\ a_n \neq 0)$$

funksiýanyň ýeterlik uly položitel x_0 üçin

$$(-\infty, -x_0) \quad \text{we} \quad (x_0, +\infty)$$

interwallarda berk monotondygyny subut etmeli.

51. Bitin rasional, toždestwolaýyn hemişelik bolmadyk,

$$R(x) = \frac{a_0 + a_1x + ... + a_nx^n}{b_0 + b_1x + ... + b_mx^m} \qquad (a_nb_m \neq 0)$$

funksiýanyň ýeterlik uly položitel x_0 üçin

$$(-\infty, -x_0) \quad \text{we} \quad (x_0, +\infty)$$

interwallarda berk monotondygyny subut etmeli.

52. Monoton funksiýanyň önümi hökman monoton bolýarmy? Aşakdaky monoton funksiýanyň önümi monoton bolarmy: $f(x)=x+\sin x$?

53. Eger $\varphi(x)$ monoton artýan differensirlenýän funksiýa bolsa we $x \geq x_0$ bolanda $|f'(x)| \leq \varphi'(x)$ bolsa, onda $x \geq x_0$ bolanda

$$|f(x) - f(x_0)| \leq \varphi(x) - \varphi(x_0)$$

deňsizligi subut etmeli. Onuň geometrik taýdan düşündirilişini bermeli.

54. Goý, $f(x)$ funksiýa $a \leq x < +\infty$ aralykda üznüksiz we ondan daşgary $x > a$ bolanda $f'(x) > k > 0$ bolsun (k – hemişelik). Onda $f(a)<0$ bolanda $f(x) = 0$ deňlemäniň $\left(a, a - \dfrac{f(a)}{k}\right)$ interwalda ýeke-täk hakyky köküniň bardygyny subut etmeli.

55. Eger x_0 nokadyň $|x-x_0|<\delta$ golaý töwereginde $f(x)$ funksiýanyň $\Delta f(x_0)=f(x)-f(x_0)$ artymynyň alamaty argumentiň $\Delta x_0=x-x_0$ artymynyň alamaty bilen gabat gelýän bolsa, onda $f(x)$ funksiýa x_0 nokatda artýan funksiýa diýilýär.

Eger $f(x)$ $(a < x < b)$ funksiýa käbir tükenikli ýa-da tükeniksiz $(a,\ b)$ interwalyň her bir nokadynda artýan bolsa, onda ol funksiýanyň şol interwalda artýandygyny subut etmeli.

56. $f(x) = x + x^2 \sin\dfrac{2}{x}$, $x \neq 0$ we $f(0)=0$ funksiýanyň $x=0$ nokatda artýandygyny, ýöne şol nokady özünde saklaýan hiç bir $(-\varepsilon, \varepsilon)$ interwalda artmaýandygyny subut etmeli, bu ýerde $\varepsilon > 0$ ýeterlik kiçi sandyr.

Funksiýanyň grafigini gurmaly.

57. Teoremany subut etmeli:

Eger 1) $\varphi(x)$ we $\psi(x)$ funksiýalar n gezek differensirlenýän;

2) $\varphi^{(k)}(x_0) = \psi^{(k)}(x_0)$ $(k = 0, 1, ..., n-1)$;
3) $x > x_0$ bolanda $\varphi^{(n)}(x) > \psi^{(n)}(x)$ bolsa, onda

$$x > x_0 \quad \text{bolanda} \quad \varphi(x) > \psi(x)$$

deňsizlik ýerine ýetýär.

58. Görkezilen şertlerde deňsizlikleri subut etmeli:

a) $e^x > 1 + x, \quad x \neq 0$;

b) $x - \dfrac{x^2}{2} < \ln(1+x) < x, \quad x > 0$;

ç) $x - \dfrac{x^3}{6} < \sin x < x, \quad x > 0$;

d) $\operatorname{tg} x > x + \dfrac{x^3}{3}, \quad 0 < x < \dfrac{\pi}{2}$;

e) $(x^\alpha + y^\alpha)^{1/\alpha} > (x^\beta + y^\beta)^{1/\beta}, \quad x > 0, \quad y > 0$ we $0 < \alpha < \beta$.

59. Deňsizligi subut etmeli:

$$\frac{2}{\pi}x < \sin x < x, \quad 0 < x < \frac{\pi}{2}.$$

60. $x > 0$ bolanda $\left(1 + \dfrac{1}{x}\right)^x < e < \left(1 + \dfrac{1}{x}\right)^{x+1}$ deňsizligiň dogrudygyny subut etmeli.

61. Arifmetik we geometrik progressiýalaryň agzalarynyň sany we gyraky agzalary birmeňzeş we olaryň ähli agzalary položitel. Arifmetik progressiýanyň agzalarynyň jeminiň geometrik progressiýanyň agzalarynyň jeminden uludygyny subut etmeli.

62. $\sum\limits_{k=1}^{n}(a_k x + b_k)^2 \geq 0$ deňsizlikden peýdalanyp, Koşiniň

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \sum_{k=1}^{n} a_k^2 \cdot \sum_{k=1}^{n} b_k^2$$

deňsizliginiň dogrudygyny subut etmeli, bu ýerde x, a_k, b_k $(k = 1, ..., n)$ hakyky sanlar.

63. Položitel sanlaryň orta arifmetik bahasynyň şol sanlaryň orta kwadratik bahalaryndan uly däldigini, ýagny

$$\frac{1}{n}\sum_{k=1}^{n} x_k \leq \sqrt{\frac{1}{n}\sum_{k=1}^{n} x_k^2}$$

deňsizligiň dogrudygyny subut etmeli.

64. Položitel sanlaryň orta geometrik bahasynyň şol sanlaryň orta arifmetik bahasyndan uly däldigini, ýagny

$$\sqrt[n]{x_1 x_2 \ldots x_n} \leq \frac{1}{n}(x_1 + x_2 + \ldots + x_n)$$

deňsizligi subut etmeli. (*Görkezme*: *Matematiki induksiýa usulyndan peýdalanmaly*).

65. Položitel a we b sanlar üçin

$$\Delta_s(a,b) = \left(\frac{a^s + b^s}{2}\right)^{1/s}, \quad s \neq 0 \quad \text{we} \quad \Delta_0(a,b) = \lim_{s \to 0} \Delta_s(a,b)$$

deňlikler boýunça kesgitlenýän funksiýa ol sanlaryň s tertipdäki orta bahasy diýilýär.

Hususan-da, ondan $s = -1$ bolanda orta garmonik, $s = 0$ bolanda orta geometrik, $s = 1$ bolanda orta arifmetik, $s = 2$ bolanda orta kwadratik sanlar alynýar.

Subut etmeli:

1) $\min(a, b) \leq \Delta_s(a, b) \leq \max(a, b)$;

2) $\Delta_s(a, b)$ $(a \neq b)$ funksiýanyň üýtgeýän s ululyga görä artýandygyny;

3) $\lim\limits_{x \to -\infty} \Delta_s(a,b) = \min(a,b), \quad \lim\limits_{x \to +\infty} \Delta_s(a,b) = \max(a,b).$

(*Görkezme*: $\frac{d}{ds}[\ln \Delta_s(a,b)]$ *peýdalanmaly*).

66. Görkezilen şertlerde deňsizlikleri subut etmeli:

a) $x^\alpha - 1 > \alpha(x - 1), \quad \alpha \geq 2, \quad x > 1$;

b) $\sqrt[n]{x} - \sqrt[n]{a} < \sqrt[n]{x - a}, \quad n > 1, \quad x > a > 0$;

ç) $1 + 2\ln x \leq x^2, \quad x > 0$.

67. $y = 1 + \sqrt[3]{x}$ çyzygyň $A(-1, 0)$, $B(1, 2)$ we $C(0, 0)$ nokatlardaky güberçeklik ugurlaryny derňemeli.

Funksiýalaryň (aşak, ýokaryk) güberçeklik interwallaryny we epin nokatlaryny tapmaly:

68. $y = 3x^2 - x^3$.

69. $y = \dfrac{a^3}{a^2 + x^2} \quad (a > 0)$.

70. $y = x + x^{5/3}$.

71. $y = \sqrt{1 + x^2}$.

72. $y = x + \sin x$.

73. $y = e^{-x^2}$.

74. $y = \ln(1 + x^2)$.

75. $y = x\sin(\ln x) \; (x > 0)$.

76. $y = x^x \quad (x > 0)$.

77. $y = \dfrac{x+1}{x^2+1}$ çyzygyň bir göni çyzykda ýatýan üç sany epin nokatlarynyň bardygyny subut etmeli.

78. h parametri nähili saýlanyňda $y = \dfrac{h}{\sqrt{\pi}} e^{-h^2x^2}$ $(h > 0)$ «ähtimallyk egri çyzygynyň» $x = \pm\,\sigma$ nokatlar epin nokatlary bolar?

79. $x = a(t - \sin t), y = a(1 - \cos t)$ $(a > 0)$ çyzygyň güberçeklik ugruny derňemeli.

80. Goý, $f(x)$ funksiýa $a \leq x < +\infty$ aralykda iki gezek differensirlenýän bolsun hem-de: 1) $f(a) = A > 0$; 2) $f'(a) < 0$; 3) $x > a$ bolanda $f''(x) \leq 0$ şertler ýerine ýetsin.

$f(x) = 0$ deňlemäniň $(a, +\infty)$ interwalda bir we diňe bir köküniň bardygyny subut etmeli.

81. Eger $\forall x_1, x_2 \in (a, b)$ we islendik $\alpha > 0, \beta > 0, \alpha + \beta = 1$ üçin

$$f(\alpha x_1 + \beta x_2) < \alpha f(x_1) + \beta f(x_2),$$
$$(f(\alpha x_1 + \beta x_2) > \alpha f(x_1) + \beta f(x_2))$$

deňsizlik ýerine ýetse, onda $f(x)$ funksiýa (a, b) interwalda aşaklygyna (ýokarlygyna) güberçek funksiýa diýilýär.

Eger $a < x < b$ bolanda $f''(x) > 0$ $(f''(x) < 0)$ bolsa, onda $f(x)$ funksiýanyň (a, b) interwalda aşaklygyna (ýokarlygyna) güberçekdigini subut etmeli.

82. x^n $(n > 1)$, e^x, $x\ln x$ funksiýalaryň $(0, +\infty)$ interwalda aşaklygyna güberçekdigini, x^n $(0 < n < 1)$, $\ln x$ funksiýalaryň $(0, +\infty)$ interwalda ýokarlygyna güberçekdigini subut etmeli.

83. Deňsizlikleri subut etmeli we olaryň geometrik manylaryny anyklamaly:

a) $\dfrac{1}{2}(x^n + y^n) > \left(\dfrac{x+y}{2}\right)^n \quad (x > 0, \ \ y > 0, \ \ x \neq y, \ \ n > 1);$

b) $\dfrac{e^x + e^y}{2} > e^{\frac{x+y}{2}} \quad (x \neq y);$

ç) $x\ln x + y\ln y > (x+y)\ln\dfrac{x+y}{2} \quad (x > 0, \ \ y > 0).$

84. Goý, $a \leq x \leq b$ bolanda $f''(x) \geq 0$ bolsun, onda islendik $\forall x_1, x_2 \in [a, b]$ üçin

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}[f(x_1) + f(x_2)]$$

deňsizligiň dogrudygyny subut etmeli.

85. Çäkli güberçek funksiýanyň hemme ýerde üznüksizdigini we birtaraplaýyn çep we sag önümleriniň bardygyny subut etmeli.

86. Goý, $f(x)$ funksiýa (a, b) interwalda iki gezek differensirlenýän we $f''(c) \neq 0$ bolsun, bu ýerde $a < c < b$. (a, b) interwalda şeýle x_1 we x_2 nokatlar tapylyp,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

deňligiň ýerine ýetýändigini subut etmeli.

87. Eger $f(x)$ funksiýa tükeniksiz $(x_0, +\infty)$ interwalda iki gezek differensirlenýän we

$$\lim_{x \to x_0 + 0} f(x) = 0, \qquad \lim_{x \to +\infty} f(x) = 0$$

bolsa, onda $(x_0, +\infty)$ interwalda iň bolmanda bir c nokat tapylyp, $f''(c) = 0$ bolar.

§3. Lopitalyň kesgitsizlikleri açmak düzgünleri

Lopitalyň 1-nji düzgüni (0/0 kesgitsizlik üçin). Goý, $f(x)$ we $g(x)$ funksiýalar a nokadyň käbir $U(a)$ $(x \neq a)$ golaý töwereginde kesgitlenen bolup:

1) $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$ bolsun;

2) a nokadyň, mümkin bolan a nokadyň özünden başga, $U(a)$ golaý töwereginde $f'(x)$ we $g'(x)$ önümler bar bolup, $\forall x \in U(a)$ $(x \neq a)$ üçin $f'^2(x) + g'^2(x) \neq 0$ bolsun;

3) tükenikli ýa-da tükeniksiz $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)} = K$ predel bar bolsun, onda

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = K$$

predel bardyr.

Lopitalyň 2-nji düzgüni (∞/∞ kesgitsizlik üçin). Goý,

1) $f(x)$ we $g(x)$ funksiýalar $x \to a$ bolanda tükeniksizlige ymtylýan bolsun;

2) a nokadyň, mümkin bolan a nokadyň özünden başga, $U(a)$ golaý töwereginde $f'(x)$ we $g'(x)$ önümler bar bolup, $\forall x \in U(a)$ $(x \neq a)$ üçin $f'^2(x) + g'^2(x) \neq 0$ bolsun;

3) tükenikli ýa-da tükeniksiz $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)} = K$ predel bar bolsun, onda

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = K$$

predel bardyr.

Kesgitsizlikleriň $0 \cdot \infty$, $\infty - \infty$, 1^∞, 0^0, ∞^0 we ş.m. görnüşleri hem algebraik özgertmeleri ulanmak we logarifmlemek bilen $\dfrac{0}{0}$ we $\dfrac{\infty}{\infty}$ görnüşdäki kesgitsizliklere getirilýär.

Bellik. Käbir hallarda $\frac{0}{0}$ we $\frac{\infty}{\infty}$ görnüşdäki kesgitsizlikleri açmaklyk $f(x)$ we $g(x)$ funksiýalar üçin ulanylýan Lopitalyň düzgünlerini ol funksiýalaryň birinji ýa-da ondan hem ýokary tertipdäki önümleri ulanylyp ýerine ýetirilýär.

Mysal. $\lim\limits_{x\to 0}\frac{x-\sin x}{x^3}$ predeli tapmaly.

Ç.B. $f(x) = x - \sin x$ we $g(x) = x^3$ funksiýalaryň birinji $f'(x)=1-\cos x$ we $g'(x)=3x^2$ we önümleriniň, şeýle hem ikinji $f''(x)=\sin x$ we $g''(x)=6x$ önümleriniň $x\to 0$ bolanda nola ymtylýandyklary üçin Lopitalyň 1-nji düzgünini berlen $f(x)$ we $g(x)$ funksiýalaryň ikinji tertipli önümlerini ulanyp, aşakdaky predeli taparys:

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\lim_{x\to 0}\frac{\sin x}{6x}=\frac{1}{6}\lim_{x\to 0}\frac{\sin x}{x}=\frac{1}{6}\lim_{x\to 0}\frac{\cos x}{1}=\frac{1}{6}.$$ **Ç.S.**

Gönükmeler

Predelleri tapmaly:

88. $\lim\limits_{x\to 0}\frac{\sin ax}{\sin bx}$.

89. $\lim\limits_{x\to 0}\frac{\operatorname{ch}x-\cos x}{x^2}$.

90. $\lim\limits_{x\to 0}\frac{\operatorname{tg}x-x}{x-\sin x}$.

91. $\lim\limits_{x\to 0}\frac{3\operatorname{tg}4x-12\operatorname{tg}x}{3\sin 4x-12\sin x}$.

92. $\lim\limits_{x\to \frac{\pi}{2}}\frac{\operatorname{tg}3x}{\operatorname{tg}x}$.

93. $\lim\limits_{x\to 0}\frac{x\operatorname{ctg}x-1}{x^2}$.

94. $\lim\limits_{x\to \frac{\pi}{4}}\frac{\sqrt[3]{\operatorname{tg}x}-1}{2\sin^2 x-1}$.

95. $\lim\limits_{x\to 0}\frac{x(e^x+1)-2(e^x-1)}{x^3}$.

96. $\lim\limits_{x\to 0}\frac{1-\cos x^2}{x^2\sin x^2}$.

97. $\lim\limits_{x\to 0}\frac{\arcsin 2x-2\arcsin x}{x^3}$.

98. $\lim\limits_{x\to 0}\frac{1}{x\sqrt{x}}\left(\sqrt{a}\operatorname{arctg}\sqrt{\frac{x}{a}}-\sqrt{b}\operatorname{arctg}\sqrt{\frac{x}{b}}\right)$.

99. $\lim\limits_{x\to 0}\frac{a^x-a^{\sin x}}{x^3}\quad (a>0)$.

100. $\lim\limits_{x\to 1}\left(\frac{x^x-x}{\ln x-x+1}\right)$.

101. $\lim\limits_{x\to 0}\frac{\ln(\sin ax)}{\ln(\sin bx)}$.

102. $\lim\limits_{x\to 0}\frac{\ln(\cos ax)}{\ln(\cos bx)}$.

103. $\lim\limits_{x\to 0}\frac{\cos(\sin x)-\cos x}{x^4}$.

104. $\lim\limits_{x\to 0}\frac{1}{x}\left(\frac{1}{\operatorname{th}x}-\frac{1}{\operatorname{tg}x}\right)$.

105. $\lim\limits_{x\to 0}\dfrac{\mathrm{Arsh}(\mathrm{sh}x)-\mathrm{Arsh}(\sin x)}{\mathrm{sh}x-\sin x}$, bu ýerde $\mathrm{Arsh}x=\ln(x+\sqrt{1+x^2})$.

106. $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x^{\varepsilon}}\quad(\varepsilon>0)$.

107. $\lim\limits_{x\to+\infty}\dfrac{x^{n}}{e^{ax}}\quad(a>0,n>0)$.

108. $\lim\limits_{x\to 0}\dfrac{e^{-1/x^2}}{x^{100}}$.

109. $\lim\limits_{x\to+\infty}x^{2}e^{-0{,}01x}$.

110. $\lim\limits_{x\to 1-0}\ln x\cdot\ln(1-x)$.

111. $\lim\limits_{x\to+0}x^{\varepsilon}\ln x\quad(\varepsilon>0)$.

112. $\lim\limits_{x\to+0}x^{x}$.

113. $\lim\limits_{x\to 0}x^{x^{x}-1}$.

114. $\lim\limits_{x\to 0}(x^{x^{x}}-1)$.

115. $\lim\limits_{x\to+0}x^{k/(1+\ln x)}$.

116. $\lim\limits_{x\to 1}x^{1/(1-x)}$.

117. $\lim\limits_{x\to 1}(2-x)^{\mathrm{tg}\frac{\pi x}{2}}$.

118. $\lim\limits_{x\to\frac{\pi}{4}}(\mathrm{tg}x)^{\mathrm{tg}2x}$.

119. $\lim\limits_{x\to 0}(\mathrm{ctg}x)^{\sin x}$.

120. $\lim\limits_{x\to+0}\left(\ln\dfrac{1}{x}\right)^{x}$.

121. $\lim\limits_{x\to\infty}\left(\mathrm{tg}\dfrac{\pi x}{2x+1}\right)^{\frac{1}{x}}$.

122. $\lim\limits_{x\to a}\left(\dfrac{\mathrm{tg}x}{\mathrm{tg}a}\right)^{\mathrm{ctg}(x-a)}$.

123. $\lim\limits_{x\to 0}\left(\dfrac{a^{x}-x\ln a}{b^{x}-x\ln b}\right)^{\frac{1}{x^2}}$.

124. $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{e^{x}-1}\right)$.

125. $\lim\limits_{x\to 1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$.

126. $\lim\limits_{x\to 0}\left(\mathrm{ctg}x-\dfrac{1}{x}\right)$.

127. $\lim\limits_{x\to 0}\left[\dfrac{1}{\ln(x+\sqrt{1+x^2})}-\dfrac{1}{\ln(1+x)}\right]$.

128. $\lim\limits_{x\to a}\dfrac{a^{x}-x^{a}}{x-a}\quad(a>0)$.

129. $\lim\limits_{x\to 0}\dfrac{(1+x)^{1/x}-e}{x}$.

130. $\lim\limits_{x\to 0}\dfrac{(a+x)^{x}-a^{x}}{x^{2}}\quad(a>0)$.

131. $\lim\limits_{x\to+\infty}\left(\dfrac{2}{\pi}\mathrm{arctg}x\right)^{x}$.

132. $\lim\limits_{x\to+\infty}(\mathrm{th}x)^{x}$.

133. $\lim\limits_{x\to 0}\left(\dfrac{\arcsin x}{x}\right)^{1/x^2}$.

134. $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2}$.

135. $\lim\limits_{x\to 0}\left(\dfrac{\mathrm{tg}x}{x}\right)^{1/x^2}$.

136. $\lim\limits_{x\to 0}\left(\dfrac{\operatorname{arctg}x}{x}\right)^{1/x^2}$.

137. $\lim\limits_{x\to 0}\left(\dfrac{\operatorname{Arsh}x}{x}\right)^{1/x^2}$, bu ýerde $\operatorname{Arsh}x = \ln\left(x + \sqrt{1+x^2}\right)$.

138. $\lim\limits_{x\to 0}\left[\dfrac{(1+x)^{1/x}}{e}\right]^{1/x}$.

139. $\lim\limits_{x\to 0}\left(\dfrac{2}{\pi}\arccos x\right)^{1/x}$.

140. $\lim\limits_{x\to 0}\left(\dfrac{\cos x}{\operatorname{ch}x}\right)^{1/x^2}$.

141. $\lim\limits_{x\to 0}\dfrac{\ln \operatorname{ch}x}{\sqrt[m]{\operatorname{ch}x} - \sqrt[n]{\operatorname{ch}x}}$.

142. $\lim\limits_{x\to 0}\left(\dfrac{1+e^x}{2}\right)^{\operatorname{cth}x}$.

143. $\lim\limits_{x\to +\infty}\dfrac{x^{\ln x}}{(\ln x)^x}$.

144. $\lim\limits_{x\to +\infty}\left[\sqrt[3]{x^3+x^2+x+1} - \sqrt{x^2+x+1}\cdot\dfrac{\ln(e^x+x)}{x}\right]$.

145. $\lim\limits_{x\to +\infty}\left[(x+a)^{1+1/x} - x^{1+1/(x+a)}\right]$.

146. Eger $y = f(x)$ egri çyzyk $x \to 0$ bolanda $\left(\lim\limits_{x\to 0} f(x) = f(0) = 0\right)$ (0, 0) koordinatalar başlangyjyna α burç bilen girýän bolsa, onda $\lim\limits_{x\to 0}\dfrac{y}{x}$ predeli tapmaly.

147. Eger üznüksiz $y = f(x)$ egri çyzyk $x \to +0$ bolanda $\left(\lim\limits_{x\to +0} f(x) = 0\right)$ (0, 0) koordinatalar başlangyjyna girýän bolsa we $0 < x < \varepsilon$ bolanda tutuşlygyna $y = -kx$ we $y = kx$ ($k \neq \infty$) çyzyklar bilen emele gelýän ýiti burçuň içinde ýerleşýän bolsa, onda

$$\lim_{x\to +0} x^{f(x)} = 1$$

deňligi subut etmeli.

148. Eger $y = f(x)$ funksiýanyň ikinji tertipli $f''(x)$ önümi bar bolsa, onda deňligi subut etmeli:

$$f''(x) = \lim_{h\to 0}\frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$

149. $f(x) = \begin{cases} \dfrac{1}{x} - \dfrac{1}{e^x - 1}, & \text{eger } \ x \neq 0; \\ 1/2, & \text{eger } \ x = 0 \end{cases}$ bolsa, funksiýanyň $x = 0$ nokatda differensirlenýändigini derňemeli.

150. $y = \dfrac{x^{1+x}}{(1+x)^x}$ $(x > 0)$ çyzygyň asimptotasyny tapmaly.

151. Aşakdaky mysallarda Lopitalyň düzgünini ulanyp, derňemeli:

a) $\lim\limits_{x\to 0}\dfrac{x^2\sin\dfrac{1}{x}}{\sin x}$; ç) $\lim\limits_{x\to +\infty}\dfrac{e^{-2x}(\cos x+2\sin x)+e^{-x^2}\sin^2 x}{e^{-x}(\cos x+\sin x)}$;

b) $\lim\limits_{x\to \infty}\dfrac{x-\sin x}{x+\sin x}$; d) $\lim\limits_{x\to \infty}\dfrac{1+x+\sin x\cos x}{(x+\sin x\cos x)e^{\sin x}}$.

152. Radiusy üýtgemeýän R ulylyga deň, hordasy b we peýkamy h bolan tegelek segmentiň meýdanynyň şol segmentiň içinden çyzylan deňýanly üçburçlugyň meýdanyna bolan gatnaşygynyň segmentiň dugasynyň nola ymtylandaky predelini tapmaly. Alnan netijäni ulanyp, segmentiň meýdany üçin takmyny formulany getirip çykarmaly:

$$S\approx\frac{2}{3}bh.$$

§4. Teýloryň formulasy

1. ***Teýloryň lokal formulasy.*** Eger a nokadyň käbir golaý töwereginde kesgitlenen f funksiýanyň a nokatda n tertipli önümi bar bolsa, onda Teýloryň

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+o((x-a)^n),\qquad x\to a$$

formulasy dogrudyr, bu ýerdäki $r_n(x)=o((x-a)^n)$, $x\to a$ funksiýa Teýloryň formulasynyň Peano görnüşindäki galyndy agzasy diýilýär.

2. ***Teýloryň umumy formulasy.*** Eger a nokadyň käbir golaý töwereginde kesgitlenen f funksiýanyň şol golaý töwerekde $n+1$ tertipli önümi bar bolsa, onda şol golaý töwerege degişli islendik x üçin a we x nokatlaryň arasynda şeýle c nokat tapylyp, Teýloryň

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+r_n(x)$$

formulasy dogrudyr, bu ýerdäki $c=a+\theta(x-a)$ $(0<\theta<1)$ bolanda alynýan

$$r_n(x)=\frac{f^{(n+1)}(a+\theta(x-a))}{(n+1)!}(x-a)^{n+1}\qquad(0<\theta<1),$$

$$r_n(x)=\frac{f^{(n+1)}(a+\theta(x-a))}{n!}(x-a)^{n+1}(1-\theta)^n\qquad(0<\theta<1)$$

formulalara degişlilikde Lagranž we Koşi görnüşindäki galyndy agzalary diýilýär.

Teýloryň formulasyndan $a=0$ bolanda alynýan

$$f(x)=\sum_{k=0}^{n}\frac{f^{(n)}(0)}{n!}x^{n}+r_{n}(x)$$

formula Maklореniň formulasy diýilýär. Onuň galyndy agzalary aşakdakylardan ybaratdyr:

1. Peano görnüşinde $r_n(x) = o(x^n)$,

2. Lagranž görnüşinde $r_n(x)=\dfrac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$ $(0<\theta<1)$,

3. Koşi görnüşinde $r_n(x)=\dfrac{f^{(n+1)}(\theta x)}{n!}(1-\theta)^{n}x^{n+1}$ $(0<\theta<1)$.

Bäş sany elementar funksiýanyň dagydylyş formulalary:

I. $e^{x}=1+x+\dfrac{x^{2}}{2!}+\ldots+\dfrac{x^{n}}{n!}+o(x^{n})$.

II. $\sin x=x-\dfrac{x^{3}}{3!}+\ldots+(-1)^{n-1}\dfrac{x^{2n-1}}{(2n-1)!}+o(x^{2n})$.

III. $\cos x=1-\dfrac{x^{2}}{2!}+\ldots+(-1)^{n}\dfrac{x^{2n}}{(2n)!}+o(x^{2n+1})$.

IV. $(1+x)^{m}=1+mx+\dfrac{m(m-1)}{2!}x^{2}+$

$$+\ldots+\frac{m(m-1)\ldots(m-n+1)}{n!}x^{n}+o(x^{n}).$$

V. $\ln(1+x)=x-\dfrac{x^{2}}{2}+\ldots+(-1)^{n-1}\dfrac{x^{n}}{n}+o(x^{n})$.

Mysal. $f(x) = (1 + x)/(1 + x^2)$ funksiýany üýtgeýän x ululygyň bitin položitel derejesi boýunça x^4-e çenli dagytmaly we $f^{(4)}(0)$ bahany hasaplamaly.

Ç.B. Berlen funksiýany $f(x) = 1 + (x - x^2)\cdot(1 + x^2)^{-1}$ görnüşde ýazyp, IV formulany ulanarys:

$$f(x) = 1 + (x - x^2)\cdot(1 - x^2 + x^4 + o(x^4)) = 1 + x - x^2 - x^3 + x^4 + o(x^4).$$

Bu aňlatmany Maklореniň formulasy bilen deňeşdirip, $\dfrac{f^{(4)}(0)}{4!}=1$ deňligi alarys. Bu ýerden $f^{(4)}(0) = 24$. **Ç.S.**

Gönükmeler

153. $P(x) = 1 + 3x + 5x^2 - 2x^3$ köpagzany $x + 1$ ikiagzanyň bitin položitel derejeleri boýunça dagytmaly.

Berlen funksiýalary x-iň bitin položitel derejeleri boýunça aşakda görkezilen agzalarynyň tertibine çenli dagytmaly (olary hem girizip):

154. $f(x) = \dfrac{1 + x + x^2}{1 - x + x^2}$, x^4-e çenli. ($f^{(4)}(0)$ näçä deň?).

155. $\dfrac{(1 + x)^{100}}{(1 - 2x)^{40}(1 + 2x)^{60}}$, x^2-a çenli.

156. $\sqrt[m]{a^m + x}$ $(a > 0)$, x^2-a çenli.

157. $\sqrt{1 - 2x + x^3} - \sqrt[3]{1 - 3x + x^2}$, x^3-e çenli.

158. $e^{2x - x^2}$, x^5-e çenli.

159. $\dfrac{x}{e^x - 1}$, x^4-e çenli.

160. $\sqrt[3]{\sin x^3}$, x^{13}-e çenli.

161. $\ln\cos x$, x^6-e çenli.

162. $\sin(\sin x)$, x^3-e çenli.

163. $\mathrm{tg}x$, x^5-e çenli.

164. $\ln\dfrac{\sin x}{x}$, x^6-a çenli.

165. $f(x) = \sqrt{x}$ funksiýanyň $(x - 1)$ tapawudyň bitin položitel derejeleri boýunça dagydylmasynyň üç agzasyny tapmaly.

166. $f(x) = x^x - 1$ funksiýany $(x - 1)$ tapawudyň bitin položitel derejeleri boýunça $(x - 1)^3$-a çenli dagytmaly.

167. $y = a\,\mathrm{ch}\dfrac{x}{a}$ $(a > 0)$ funksiýany $x = 0$ nokadyň golaý töwereginde ikinji tertipli parabola bilen takmynan çalşyrmaly.

168. $f(x) = \sqrt{1 + x^2} - x$ $(x > 0)$ funksiýany $1/x$ drobuň bitin položitel derejeleri boýunça $1/x^2$-a çenli dagytmaly.

169. $f(h) = \ln(x + h)$ $(x > 0)$ funksiýany h artymyň bitin položitel derejeleri boýunça h^n-e çenli dagytmaly (n – natural san).

170. Goý, $f(x + h) = f(x) + hf'(x) + \dots + \dfrac{h^n}{n!}f^{(n)}(x + \theta h)$ $(0 < \theta < 1)$ bolsun, şeýle-de, $f^{(n+1)}(x) \neq 0$. Deňligi subut etmeli:

$$\lim_{h \to 0} \theta = \frac{1}{n + 1}.$$

171. Goý, $x \to 0$ bolanda $f(x) = 1 + kx + o(x)$ bolsun. Deňligi subut etmeli: $\lim\limits_{x\to 0}[f(x)]^{1/x} = e^k$.

172. Goý, $f(x) \in C^{(2)}[0, 1]$, $f(0) = f(1) = 0$ bolsun, şeýle-de, $x \in (0, 1)$ bolanda $|f''(x)| \le A$. $0 \le x \le 1$ bolanda $|f'(x)| \le \frac{A}{2}$ deňsizligi subut etmeli.

173. Goý, $f(x)$ $(-\infty < x < +\infty)$ iki gezek differensirlenýän funksiýa we $M_k = \sup\limits_{-\infty<x<+\infty} |f^{(k)}(x)| < +\infty$ $(k=0,1,2)$ bolsun. Deňsizligi subut etmeli: $M_1^2 \le 2M_0M_2$.

174. Takmyny formulalaryň absolýut ýalňyşlyklaryny bahalandyrmaly:

a) $e^x \approx 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}$, $0 \le x \le 1$;

b) $\sin x \approx x - \frac{x^3}{6}$, $|x| \le \frac{1}{2}$;

ç) $\operatorname{tg} x \approx x + \frac{x^3}{3}$, $|x| \le 0{,}1$;

d) $\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$, $0 \le x \le 1$.

175. x-iň haýsy bahalarynda 0,0001 takyklykda $\cos x = 1 - \frac{x^2}{2}$ takmyny formula dogry?

176. $\sqrt[n]{a^n + x} = a + \frac{x}{na^{n-1}} - r$ $(n \ge 2,\ a > 0,\ x > 0)$ formulany subut etmeli, bu ýerde $0 < r < \frac{n-1}{2n^2} \cdot \frac{x^2}{a^{2n-1}}$.

177. Teýloryň formulasyny ulanyp, takmyny bahalary tapmaly:

a) $\sqrt[3]{30}$;
b) $\sqrt[5]{250}$;
ç) $\sqrt[12]{4000}$;
d) $\sqrt{e}$;
e) $\sin 18^0$;
ä) $\ln 1{,}2$;
f) arctg 0,8;
g) arcsin 0,45;
h) $(1{,}1)^{1{,}2}$.

178. Hasaplamaly:

a) e sany 10^{-9} takyklykda;
b) $\sin 1^\circ$ sany 10^{-8} takyklykda;
ç) $\cos 9^\circ$ sany 10^{-5} takyklykda;
d) $\sqrt{5}$ sany 10^{-4} takyklykda;
e) lg11 sany 10^{-5} takyklykda.

I-V dagytmalardan peýdalanyp, predelleri tapmaly:

179. $\lim\limits_{x\to 0}\dfrac{\cos x - e^{-x^2/2}}{x^4}$. **180.** $\lim\limits_{x\to 0}\dfrac{e^x\sin x - x(1+x)}{x^3}$.

181. $\lim\limits_{x\to +\infty} x^{3/2}(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x})$.

182. $\lim\limits_{x\to +\infty}(\sqrt[6]{x^6+x^5}-\sqrt[6]{x^6-x^5})$.

183. $\lim\limits_{x\to +\infty}\left[\left(x^3-x^2+\dfrac{x}{2}\right)e^{1/x}-\sqrt{x^6+1}\right]$.

184. $\lim\limits_{x\to 0}\dfrac{a^x+a^{-x}-2}{x^2}\quad (a>0)$. **185.** $\lim\limits_{x\to \infty}\left[x-x^2\ln\left(1+\dfrac{1}{x}\right)\right]$.

186. $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right)$. **187.** $\lim\limits_{x\to 0}\dfrac{1}{x}\left(\dfrac{1}{x}-\operatorname{ctg}x\right)$.

188. $\lim\limits_{x\to 0}\dfrac{\sin(\sin x)-x\sqrt[3]{1-x^2}}{x^5}$. **189.** $\lim\limits_{x\to 0}\dfrac{1-(\cos x)^{\sin x}}{x^3}$.

190. $\lim\limits_{x\to 0}\dfrac{\operatorname{sh}(\operatorname{tg}x)-x}{x^3}$.

$x\to 0$ bolanda tükeniksiz kiçi y ululyklaryň Cx^n görnüşdäki baş agzasyny kesgitlemeli (C – hemişelik san):

191. $y=\operatorname{tg}(\sin x)-\sin(\operatorname{tg}x)$. **192.** $y=(1+x)^x-1$.

193. $y=1-\dfrac{(1+x)^{1/x}}{e}$.

194. a we b koeffisiýentler nähili saýlanyp alnanda

$$x-(a+b\cos x)\sin x$$

ululyk x-a görä 5-nji tertipli tükeniksiz kiçi bolar?

195. A we B koeffisiýentleri $x\to 0$ bolanda

$$\operatorname{ctg}x=\frac{1+Ax^2}{x+Bx^3}+O(x^5)$$

asimptotik deňlik ýerine ýeter ýaly saýlap almaly.

196. A, B, C we D koeffisiýentleriň haýsy bahalarynda $x\to 0$ bolanda

$$e^x=\frac{1+Ax+Bx^2}{1+Cx+Dx^2}+O(x^5)$$

asimptotik formula dogry bolar?

197. $|x|$ ululygy tükeniksiz kiçi ululyk hasap edip, berlen aňlatmalar üçin ýönekeý ýakynlaşma formulalary getirip çykarmaly:

a) $\dfrac{1}{R^2} - \dfrac{1}{(R+x)^2} \quad (R > 0);$

b) $\sqrt[3]{\dfrac{1+x}{1-x}} - \sqrt[3]{\dfrac{1-x}{1+x}};$

ç) $\dfrac{A}{x}\left[1 - \left(1 + \dfrac{x}{100}\right)^{-n}\right];$

d) $\dfrac{\ln 2}{\ln\left(1 + \dfrac{x}{100}\right)}.$

198. Absolýut ululygy boýunça tükeniksiz kiçi x üçin

$$x = \alpha \sin x + \beta \operatorname{tg} x$$

görnüşdäki ýakynlaşma formulany x^5 agza çenli takyklykda getirip çykarmaly.

Bu formulany kiçi burç ululykly dugalaryň ýakynlaşma göneltmeleri üçin ulanmaly.

199. Çebyşewiň aşakdaky düzgüniniň otnositel ýalňyşlygyny bahalandyrmaly: tegelek duga beýikligi onuň ugrunyň $\sqrt{4/3}$ bölegine deň bolan we şol duganyň hordasy esasynda gurlan deňýanly üçburçlugyň gapdal taraplarynyň jemine takmynan deňdir.

§5. Funksiýanyň ekstremumy. Funksiýanyň iň uly we iň kiçi bahalary

Eger a nokadyň $U(a, \delta)$ golaý töwereginde kesgitlenen f funksiýa $\forall x \in U(a, \delta)$ üçin $f(x) \leq f(a)$ ($f(x) \geq f(a)$) deňsizligi kanagatlandyrýan bolsa, onda a nokada f funksiýanyň maksimum (minimum) nokady diýilýär. Funksiýanyň maksimum we minimum nokatlaryna onuň ekstremum nokatlary diýilýär.

Eger a ekstremum nokadynda f funksiýanyň önümi bar bolsa, onda $f'(a) = 0$ (ekstremumyň zerur şerti).

Funksiýanyň ekstremumyny tapmak üçin şeýle düzgünlerden peýdalanmak bolar:

Birinji düzgün (birinji ýeterlik şert). Goý, f funksiýa a nokadyň, mümkin a nokadyň özünden başga, käbir golaý töwereginde differensirlenýän bolup, a nokatda üznüksiz bolsun. Eger a nokatdan geçende funksiýanyň f' önümi alamatyny üýtgedýän bolsa, onda a nokatda f funksiýanyň ekstremumy bardyr.

Şonda, eger

1. $a - \delta < x < a$ bolanda $f'(x) > 0$ we $a < x < a + \delta$ bolanda $f'(x) < 0$ bolsa, onda a nokat funksiýanyň maksimum nokadydyr.

2. $a - \delta < x < a$ bolanda $f'(x) < 0$ we $a < x < a + \delta$ bolanda $f'(x) > 0$ bolsa, onda a nokat funksiýanyň minimum nokadydyr.

Eger-de funksiýanyň önümi nokatdan geçende alamatyny üýtgetmese, onda ol nokatda funksiýanyň ekstremumy ýokdur.

Bu düzgün boýunça funksiýanyň ekstremumyny derňemek üçin onuň kesgitlenen ýaýlasyny funksiýanyň differensirlenmeýän we önüminiň nol nokatlary arkaly interwallara bölmeli. Olaryň her birinde önümiň alamatyny kesgitlemeli (onuň üçin bolsa interwalyň bir nokadynda önümiň alamatyny bilmek ýeterlikdir).

1-nji mysal. $f(x) = (x + 2)^2(x - 1)^3$ funksiýanyň ekstremumyny tapmaly.

Ç.B. San okunyň ähli nokatlarynda aşakdaky funksiýalaryň önümi bardyr:

$$f'(x) = 2(x + 2)(x - 1)^3 + 3(x + 2)^2(x - 1)^2 = (x + 2)(x - 1)^2(5x + 4).$$

Funksiýanyň önüminiň nollary $x_1 = -2$, $x_2 = -0{,}8$, $x_3 = 1$. Şol nokatlar arkaly san okuny $(-\infty, -2)$, $(-2, -0{,}8)$, $(-0{,}8, 1)$, $(1, +\infty)$ interwallara böleliň we şol interwallara degişli bolan -3, -1, 0, 2 nokatlarda funksiýanyň önüminiň alamatlaryny kesgitläp, onuň degişli interwallardaky alamatlaryny anyklarys:

1) $(-\infty, -2)$ interwalda $f'(x) > 0$,
2) $(-2, -0{,}8)$ interwalda $f'(x) < 0$,
3) $(-0{,}8, 1)$ interwalda $f'(x) > 0$,
4) $(1, +\infty)$ interwalda $f'(x) > 0$.

Şeýlelikde, birinji düzgün boýunça $x = -2$ nokat funksiýanyň maksimum, $x=-0{,}8$ nokat minimum nokadydyr, $x = 1$ nokatda bolsa onuň ekstremumy ýokdur. Şunlukda, funksiýanyň maksimum bahasy $f(-2) = 0$, minimum bahasy bolsa $f(-0{,}8) = -8{,}4$. **Ç.S.**

Ikinji düzgün (ikinji ýeterlik şert). Goý, f funksiýanyň a nokatda ikinji önümi bar bolup, $f'(a) = 0$ bolsun. Onda $f''(a) < 0$ bolanda a nokat funksiýanyň maksimum, $f''(a) > 0$ bolanda bolsa minimum nokadydyr.

Üçünji düzgün (üçünji ýeterlik şert). Goý, f funksiýanyň a nokatda n tertipli önümi bar bolup,

$$f^{(i)}(a) = 0 \quad (i = 1, ..., n - 1), \quad f^{(n)}(a) \neq 0$$

şertler ýerine ýetsin. Onda n jübüt bolup, $f^{(n)}(a) < 0$ bolanda a nokat f funksiýanyň maksimum, $f^{(n)}(a) > 0$ bolanda bolsa minimum nokadydyr. Eger n täk bolsa, onda a nokatda f funksiýanyň ekstremumy ýokdur. Bu halda $f^{(n)}(a)<0$ bolanda nokat funksiýanyň kemelýän, $f^{(n)}(a) > 0$ bolanda artýan nokadydyr.

2-nji mysal. $f(x) = e^x + e^{-x} + 2\cos x$ funksiýa üçin $x = 0$ önümiň nol nokadydyr. Şonda

$$f''(x) = e^x + e^{-x} - 2\cos x, \quad f''(0) = 0,$$

$$f'''(x) = e^x - e^{-x} + 2\sin x, \quad f'''(0) = 0,$$

$$f^{\mathrm{IV}}(x) = e^x + e^{-x} + 2\cos x, \quad f^{\mathrm{IV}}(0) = 4 > 0.$$

Diýmek, üçünji düzgün boýunça $x = 0$ nokat funksiýanyň minimum nokadydyr.

Gönükmeler

Funksiýalaryň ekstremumyny derňemeli:

200. $y = 2 + x - x^2$.

201. $y = (x - 1)^3$.

202. $y = (x - 1)^4$.

203. $y = x^m(1 - x)^n$, (n we m bitin položitel sanlar).

204. $y = \cos x + \operatorname{ch} x$.

205. $y = (x + 1)^{10}e^{-x}$.

206. $y = \left(1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}\right)e^{-x}$ (n – natural san).

207. $y = |x|$.

208. $y = x^{1/3}(1 - x)^{2/3}$.

209. $f(x) = (x - x_0)^n\varphi(x)$ funksiýanyň $x = x_0$ nokatdaky ekstremumyny derňemeli, bu ýerde $\varphi(x)$ funksiýa $x = x_0$ nokatda üznüksiz we $\varphi(x_0) \neq 0$ (n – natural san).

210. Goý, $f(x) = \frac{P(x)}{Q(x)}$, $f'(x) = \frac{P_1(x)}{Q^2(x)}$, we x_0 nokat $f(x)$ funksiýanyň önüminiň noly, ýagny $P_1(x_0)= 0$, $Q(x_0) \neq 0$ bolsun.

$$\operatorname{sgn} f''(x_0) = \operatorname{sgn} P_1'(x_0)$$

deňligi subut etmeli.

211. Eger x_0 nokat $f(x)$ funksiýanyň maksimum nokady bolsa, onda şol nokadyň ýeterlik kiçi golaý töwereginde x_0 nokatdan çepde $f(x)$ funksiýa artýar, sagda bolsa kemelýär diýmek bolarmy? Aşakdaky mysaly şol soraglar boýunça derňemeli: $f(x) = 2 - x^2\left(2 + \sin\frac{1}{x}\right)$, eger $x \neq 0$ we $f(0) = 2$.

212. Goý, $f(x) = e^{-1/x^2}$, eger $x \neq 0$ we $f(0) = 0$ hem-de $g(x) = xe^{-1/x^2}$, eger $x \neq 0$ we $g(0) = 0$ funksiýalar üçin

$$f^{(n)}(0) = 0, \quad g^{(n)}(0) = 0 \quad (n = 1, 2, ...)$$

deňlikler ýerine ýetýän bolsun.

$x = 0$ nokadyň $f(x)$ funksiýanyň minimum nokady bolýandygyny we şol nokatda $g(x)$ funksiýanyň ekstremumynyň ýokdugyny subut etmeli.

213. Funksiýalaryň ekstremumyny derňemeli:

a) $f(x) = e^{-1/|x|}\left(\sqrt{2} + \sin\frac{1}{x}\right)$, $x \neq 0$, $f(0) = 0$;

b) $f(x) = e^{-1/|x|}\left(\sqrt{2} + \cos\frac{1}{x}\right)$, $x \neq 0$, $f(0) = 0$.

Bu funksiýalaryň grafiklerini gurmaly.

214. $f(x)=|x|\left(2+\cos\frac{1}{x}\right)$, eger $x\neq 0$ we $f(0)=0$ funksiýanyň $x=0$ nokatda ekstremumyny derňemeli. Bu funksiýanyň grafigini gurmaly.

Berlen funksiýalaryň ekstremumlaryny tapmaly:

215. $y=x^3-6x^2+9x-4$.

216. $y=2x^2-x^4$.

217. $y=x(x-1)^2(x-2)^3$.

218. $y=x+\frac{1}{x}$.

219. $y=\frac{2x}{1+x^2}$.

220. $y=\frac{x^2-3x+2}{x^2+2x+1}$.

221. $y=\sqrt{2x-x^2}$.

222. $y=x\sqrt[3]{x-1}$.

223. $y=xe^{-x}$.

224. $y=\sqrt{x}\ln x$.

225. $y=\frac{\ln^2 x}{x}$.

226. $y=\cos x+\frac{1}{2}\cos 2x$.

227. $y=\frac{10}{1+\sin^2 x}$.

228. $y=\operatorname{arctg}x-\frac{1}{2}\ln(1+x^2)$.

229. $y=e^x\sin x$.

230. $y=|x|e^{-|x-1|}$.

Funksiýalaryň görkezilen kesimlerdäki iň uly we iň kiçi bahalaryny tapmaly:

231. $f(x)=2^x$, $[-1, 5]$.

232. $f(x)=x^2-4x+6$, $[-3, 10]$.

233. $f(x)=|x^2-3x+2|$, $[-10, 10]$.

234. $f(x)=x+\frac{1}{x}$, $[0{,}01, 100]$.

235. $f(x)=\sqrt{5-4x}$, $[-1, 1]$.

Funksiýalaryň berlen interwallardaky aşaky (inf) we ýokarky (sup) takyk çäklerini tapmaly:

236. $f(x)=xe^{-0{,}01x}$, $(0, +\infty)$.

237. $f(x)=\left(1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}\right)e^{-x}$, $(0, +\infty)$.

238. $f(x)=\frac{1+x^2}{1+x^4}$, $(0, +\infty)$.

239. $f(x)=e^{-x^2}\cos x^2$, $(-\infty, +\infty)$.

240. $f(\xi)=\frac{1+\xi}{3+\xi^2}$ funksiýanyň $x<\xi<+\infty$ interwaldaky takyk aşaky we takyk ýokarky çäklerini kesgitlemeli.

$$M(x) = \sup_{x<\xi<+\infty} f(\xi) \quad \text{we} \quad m(x) = \inf_{x<\xi<+\infty} f(\xi)$$

funksiýalaryň grafiklerini çyzmaly.

241. Goý, $M_k = \sup_x |f^{(k)}(x)|$, $k = 0, 1, 2,...$ bolsun. $f(x) = e^{-x^2}$ funksiýa üçin M_0, M_1 we M_2 sanlary tapmaly.

242. Yzygiderligiň iň uly agzasyny tapmaly:

a) $\dfrac{n^{10}}{2^n}$ $(n = 1, 2, ...)$;

b) $\dfrac{\sqrt{n}}{n + 10000}$ $(n = 1, 2, ...)$;

ç) $\sqrt[n]{n}$ $(n = 1, 2, ...)$.

243. Deňsizlikleri subut etmeli:

a) $|3x - x^3| \leq 2$, eger $|x| \leq 2$;

b) $\dfrac{1}{2^{p-1}} \leq x^p + (1 - x)^p \leq 1$, eger $0 \leq x \leq 1$ we $p > 1$;

ç) $x^m (a - x)^n \leq \dfrac{m^n n^n}{(m + n)^{m+n}} a^{m+n}$, eger $m > 0$, $n > 0$ we $0 \leq x \leq a$;

d) $\dfrac{x + a}{2^{(n-1)/n}} \leq \sqrt[n]{x^n + a^n} \leq x + a$ $(x > 0, a > 0, n > 1)$;

e) $|a \sin x + b \cos x| \leq \sqrt{a^2 + b^2}$.

244. $-\infty < x < +\infty$ bolanda deňsizligi subut etmeli:

$$\frac{2}{3} \leq \frac{x^2 + 1}{x^2 + x + 1} \leq 2.$$

245. $P(x) = x(x - 1)^2(x + 2)$ köpagzanyň $[-2, 1]$ kesimde «noldan gyşarmasyny», ýagny $E_P = \sup_{-2 \leq x \leq 1} |P(x)|$ sany kesgitlemeli.

246. $P(x) = x^2 + q$ köpagzanyň q koeffisiýentini nähili kesgitläniňde ol köpagzanyň $[-1, 1]$ kesimde noldan gyşarmasy iň az bolar, ýagny $E_P = \sup_{-1 \leq x \leq 1} |P(x)| = \min$?

247. $f(x)$ we $g(x)$ funksiýalaryň $[a, b]$ kesimdäki absolýut gyşarmasy diýip $\Delta = \sup_{a \leq x \leq b} |f(x) - g(x)|$ sana aýdylýar.

$f(x) = x^2$ we $g(x) = x^3$ funksiýalaryň $[0, 1]$ kesimdäki absolýut gyşarmasyny kesgitlemeli.

248. $f(x)$ we $g(x)$ funksiýalaryň $[x_1, x_2]$ kesimde absolýut gyşarmalary iň kiçi bolar ýaly, $f(x) = x^2$ funksiýany $[x_1, x_2]$ kesimde çyzykly $g(x) = (x_1 + x_2)x + b$ funksiýa bilen takmyny çalşyrmaly we absolýut gyşarmasyny kesgitlemeli.

249. $f(x) = \max\{2|x|, |1 + x|\}$ funksiýanyň minimumyny kesgitlemeli.

Berlen deňlemeleriň hakyky kökleriniň sanyny kesgitlemeli we olary anyklamaly:

250. $x^3 - 6x^2 + 9x - 10 = 0.$

251. $x^3 - 3x^2 - 9x + h = 0.$

252. $3x^4 - 4x^3 - 6x^2 + 12x - 20 = 0.$

253. $x^5 - 5x = a.$

254. $\ln x = kx.$

255. $e^x = ax^2.$

256. $\sin^3 x \cdot \cos x = a, \quad 0 \le x \le \pi.$

257. $\operatorname{ch} x = kx.$

258. Haýsy şertlerde $x^3 + px + q = 0$ deňlemäniň:
a) bir hakyky köki bar;
b) üç hakyky köki bar?
(p, q) tekizliginde degişli ýaýlalary şekillendirmeli.

§6. Häsiýetlendiriji nokatlary boýunça funksiýalaryň grafiklerini gurmak

$y = f(x)$ funksiýasynyň grafigini gurmak üçin aşakdakylar gerekdir: 1) bu funksiýanyň barlyk ýaýlasyny kesgitlemeli we ondaky çäk nokatlarda funksiýanyň häsiýetlerini barlamaly; 2) grafigiň simmetrikdigini we periodikdigini anyklamaly; 3) funksiýanyň üzülme nokatlaryny we üznüksiz interwallaryny tapmaly; 4) funksiýanyň nollaryny we alamatlarynyň hemişelik ýaýlalaryny kesgitlemeli; 5) ekstremum nokatlaryny tapmaly we funksiýanyň artýan we kemelýän interwallaryny anyklamaly; 6) epin nokatlaryny kesgitlemeli we funksiýanyň grafiginiň güberçeklik interwallaryny tapmaly; 7) asimptotalary bar bolan ýagdaýynda, olary tapmaly; 8) grafigiň ol ýa-da beýleki aýratynlyklaryny görkezmeli.

Gönükmeler

Funksiýalaryň grafiklerini gurmaly:

259. $y = 3x - x^3.$

260. $y = 1 + x^2 - \dfrac{x^4}{2}.$

261. $y = (x + 1)(x - 2)^2.$

262. $y = \dfrac{2 - x^2}{1 + x^4}.$

263. $y = \dfrac{x^2-1}{x^2-5x+6}$.

264. $y = \dfrac{x}{(1+x)(1-x)^2}$.

265. $y = \dfrac{x^4}{(1+x)^3}$.

266. $y = \left(\dfrac{1+x}{1-x}\right)^4$.

267. $y = \dfrac{x^2(x-1)}{(x+1)^2}$.

268. $y = \dfrac{x}{(1-x^2)^2}$.

269. $y = \dfrac{(x+1)^3}{(x-1)^2}$.

270. $y = \dfrac{x^4+8}{x^3+1}$.

271. $y = \dfrac{1}{1+x} - \dfrac{10}{3x^2} + \dfrac{1}{1-x}$.

272. $y = (x-3)\sqrt{x}$.

273. $y = \pm\sqrt{8x^2 - x^4}$.

274. $y = \dfrac{x-2}{\sqrt{x^2+1}}$.

275. $y = \pm\sqrt{(x-1)(x-2)(x-3)}$.

276. $y = \sqrt[3]{x^3 - x^2 - x + 1}$.

277. $y = \sqrt[3]{x^2} - \sqrt[3]{x^2+1}$.

278. $y = (x+2)^{2/3} - (x-2)^{2/3}$.

279. $y = (x+1)^{2/3} + (x-1)^{2/3}$.

280. $y = \dfrac{x}{\sqrt[3]{x^2-1}}$.

281. $y = \dfrac{x^2\sqrt{x^2-1}}{2x^2-1}$.

282. $y = \dfrac{|1+x|^{3/2}}{\sqrt{x}}$.

283. $y = 1 - x + \sqrt{\dfrac{x^3}{3+x}}$.

284. $y = \sqrt[3]{\dfrac{x^2}{x+1}}$.

285. $y = \sqrt{(x^4+3)/(x^2+1)}$.

286. $y = \sin x + \cos^2 x$.

287. $y = (7 + 2\cos x)\sin x$.

288. $y = \sin x + \dfrac{1}{3}\sin 3x$.

289. $y = \cos x - \dfrac{1}{2}\cos 2x$.

290. $y = \sin^4 x + \cos^4 x$.

291. $y = \sin x \cdot \sin 3x$.

292. $y = \dfrac{\sin x}{\sin(x+\pi/4)}$.

293. $y = \dfrac{\cos x}{\cos 2x}$.

294. $y = \dfrac{\sin x}{2+\cos x}$.

295. $y = 2x - \operatorname{tg} x$.

296. $y = e^{2x-x^2}$.

297. $y = (1+x^2)e^{-x^2}$.

298. $y = x + e^{-x}$.

299. $y = x^{2/3}e^{-x}$.

300. $y = e^{-2x}\sin^2 x$.

301. $y = \dfrac{e^x}{1 + x}$.

302. $y = \sqrt{1 - e^{-x^2}}$.

303. $y = \dfrac{\ln x}{\sqrt{x}}$.

304. $y = \ln\left(x + \sqrt{x^2 + 1}\right)$.

305. $y = \sqrt{x^2 + 1} \cdot \ln\left(x + \sqrt{x^2 + 1}\right)$.

306. $y = \dfrac{\arcsin x}{\sqrt{1 - x^2}}$.

307. $y = x + \mathrm{arctg}x$.

308. $y = \dfrac{x}{2} + \mathrm{arcctg}x$.

309. $y = x\,\mathrm{arctg}x$.

310. $y = \arcsin\dfrac{2x}{1 + x^2}$.

311. $y = \arccos\dfrac{1 - x^2}{1 + x^2}$.

312. $y = (x + 2)e^{1/x}$.

313. $y = 2^{\sqrt{x^2+1} - \sqrt{x^2-1}}$.

314. $y = \ln\dfrac{x^2 - 3x + 2}{x^2 + 1}$.

315. $y = a\arcsin\dfrac{x}{a} - \sqrt{a^2 - x^2} \quad (a > 0)$.

316. $y = \arccos\dfrac{1 - x}{1 - 2x}$.

317. $y = x^x$.

318. $y = x^{1/x}$.

319. $y = (1 + x)^{1/x}$.

320. $y = x\left(1 + \dfrac{1}{x}\right)^x \quad (x > 0)$.

321. $y = \dfrac{e^{1(1-x^2)}}{1 + x^2}$ (oýuklygyny barlamazdan).

Parametrik görnüşde berlen egri çyzyklary gurmaly:

322. $x = \dfrac{(t + 1)^2}{4}, \quad y = \dfrac{(t - 1)^2}{4}$.

323. $x = 2t - t^2, \quad y = 3t - t^3$.

324. $x = \dfrac{t^2}{t - 1}, \quad y = \dfrac{t}{t^2 - 1}$.

325. $x = \dfrac{t^2}{1 - t^2}, \quad y = \dfrac{1}{1 + t^2}$.

326. $x = t + e^{-t}, \quad y = 2t + e^{-2t}$.

327. $x = a\cos 2t, \quad y = a\cos 3t, \quad (a > 0)$.

328. $x = \cos^4 t, \quad y = \sin^4 t$.

329. $x = t\ln t, \quad y = \ln t/t$.

330. $x = \dfrac{a}{\cos^3 t}, \quad y = a\,\mathrm{tg}^3 t, \quad (a > 0)$.

331. $x = a(\mathrm{sh}t - t), \quad y = a(\mathrm{ch}t - 1) \quad (a > 0)$.

Çyzyklaryň berlen deňlemelerini parametrik görnüşde aňladyp, olaryň grafiklerini gurmaly:

332. $x^3 + y^3 - 3axy = 0 \quad (a > 0)$. (*Görkezme*: $y = tx$ *girizmeli*).

333. $x^2 + y^2 = x^4 + y^4$.

334. $x^2y^2 = x^3 - y^3$.

335. $x^y = y^x \quad (x > 0, y > 0)$.

336. $\mathrm{ch}^2x - \mathrm{ch}^2y = 1$ çyzygyň grafigini gurmaly.

(φ, r) $(r \geq 0)$ polýar koordinatalar sistemasynda berlen funksiýalaryň grafiklerini gurmaly:

337. $r = a + b\cos\varphi \ (0 < a \leq b)$.

338. $r = a\sin3\varphi \ (a > 0)$.

339. $r = \dfrac{a}{\sqrt{\cos 3\varphi}} \quad (a > 0)$.

340. $r = a\dfrac{\mathrm{th}\varphi}{\varphi - 1}$, $(a > 0)$ bu ýerde $\varphi > 1$.

341. $\varphi = \arccos\dfrac{r - 1}{r^2}$.

Funksiýalaryň grafiklerini gurmaly (a – üýtgeýän parametr):

342. $y = x^2 - 2x + a$.

343. $y = x + \dfrac{a^2}{x}$.

344. $y = x \pm \sqrt{a(1 - x^2)}$.

345. $y = \dfrac{x}{2} + e^{-ax}$.

346. $y = xe^{-x/a}$.

§7. Funksiýalaryň maksimumlaryny we minimumlaryny tapmaklyga degişli meseleler

Gönükmeler

347. Otrisatel däl $f(x)$ funksiýa üçin

$$F(x) = Cf^2(x) \qquad (C > 0)$$

funksiýanyň hem edil $f(x)$ funksiýanyňky bilen birmeňzeş ekstremum nokatlarynyň bardygyny subut etmeli.

348. Eger $\varphi(x)$ funksiýa $-\infty < x < +\infty$ bolanda artýan bolsa, onda $f(x)$ we $\varphi(f(x))$ funksiýalaryň birmeňzeş ekstremum nokatlarynyň bolýandygyny subut etmeli.

349. Jemleri hemişelik we a sana deň bolan iki položitel sanyň položitel m we n derejeleriniň köpeltmek hasylynyň iň uly bahasyny tapmaly.

350. Köpeltmek hasyly hemişelik we a sana deň bolan iki položitel sanyň položitel m we n derejeleriniň jemleriniň iň kiçi bahasyny tapmaly.

351. Logarifmleriň haýsy sistemasynda özüniň logarifmine deň bolan sanlar bar?

352. Berlen meýdanlary S bolan ähli gönüburçluklardan perimetri iň kiçi bolýanyny kesgitlemeli.

353. Kateti bilen gipotenuzasynyň jemi hemişelik bolan iň uly meýdanly göni burçly üçburçlugy tapmaly.

354. Berlen V göwrümli ýapyk silindr şekilli bankanyň çyzyk ölçegleri nähili bolanda onuň doly üsti iň kiçi baha eýe bolar?

355. Ýarym tegelekden uly bolmadyk berlen tegelek segmentiň içinden iň uly meýdanly gönüburçluk çyzmaly.

356. Berlen $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipsiň içinden taraplary ellipsiň oklaryna parallel bolan iň uly meýdanly gönüburçluk çyzmaly.

357. Esasy b we beýikligi h bolan üçburçlugyň içinden iň uly perimetrli gönüburçluk çyzmaly.

358. Diametri d bolan tegelek agaç böleginden kese-kesiginde esasy b we beýikligi h bolan gönüburçluk alynýan pürsi kesmeli. Berkligi bh^2-a proporsional bolan agaç böleginiň ölçegleri nähili bolanda onuň berkligi iň uly bolar?

359. Radiusy R bolan ýarym şaryň içinden esasy kwadrat bolan iň uly göwrümli göni burçly parallelepiped çyzmaly.

360. Radiusy R bolan şaryň içinden iň uly göwrümli silindr çyzmaly.

361. Radiusy R bolan şaryň içinden doly üsti iň uly bolan silindr çyzmaly.

362. Berlen şaryň daşyndan iň kiçi göwrümli konus çyzmaly.

363. Emele getirijisi l bolan iň uly göwrümli konusy tapmaly.

364. Esasynyň radiusy R we ok kesiginde 2α burçly göni tegelek konusyň içinden doly üsti iň uly bolan silindr çyzmaly.

365. $M(p, p)$ nokatdan $y^2 = 2px$ parabola çenli iň ýakyn uzaklygy tapmaly.

366. $x^2 + y^2 = 1$ töwerekden $A(2, 0)$ nokada çenli in ýakyn we iň daş uzaklyklary tapmaly.

367. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ $(0 < b < a)$ ellipsiň $B(0, -b)$ depesinden geçýän iň uly hordasyny tapmaly.

368. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ellipsiň $M(x, y)$ nokady arkaly koordinata oklary bilen iň kiçi meýdanly üçburçlugy emele getirýän galtaşýan çyzyk geçirmeli.

369. Jisim ýokarsy ýarym şar bolan göni tegelek silindri aňladýar. Göwrümi V bolan ol jisimiň çyzyk ölçegleri nähili bolanda onuň doly üsti iň kiçi bolar?

370. Açyk kanalyň kese-kesigi deňýanly trapesiýa görnüşindedir. Suwuň kanaldaky «kese-kesiginiň» meýdany S we suwuň derejesi h bolan kesigiň «öl perimetri» gapdallarynyň nähili φ gyşarmasynda iň kiçi bolar?

371. S meýdany çäklendirýän ýapyk konturyň «egrelmesi» diýlip konturyň perimetriniň şol S meýdanly tegelegi çäklendirýän töweregiň uzynlygyna bolan gatnaşygyna aýdylýar.

AD esasy $2a$ we ýiti BAD burçy α bolan iň kiçi egrelmesi bolan deňýanly $ABCD$ $(AD \parallel BC)$ trapesiýa nähili görnüşdedir?

372. Radiusy R bolan tegelekden nähili sektor kesilip alnanda, onuň galan böleginden iň uly göwrümli guýguç ýasamak bolar?

373. A zawod günortadan demirgazyga tarap gidýän we B şäher arkaly geçýän demir ýoldan iň ýakyny a *km* uzaklykda ýerleşýär. Eger 1 *tonna* ýüki gara ýol boýunça daşamagyň bahasy p *man*, demir ýol boýunça daşamagyň bahasy q *man* $(p > q)$ we B şäher A zawoddan b *km* demirgazykda ýerleşýän bolsa, onda zawoddan demir ýola haýsy φ burç boýunça gara ýol çekilende A zawoddan B şähere ýük daşamak has amatly bolar?

374. Iki gämi hemişelik u we ϑ tizlik bilen özaralarynda θ burçy emele getirýän göni çyzyklar boýunça ýüzýärler. Eger käbir pursatda olaryň kesişme nokatlardan uzaklyklary degişlilikde a we b deň bolsa, onda gämileriň arasyndaky iň gysga uzaklyk näçä deň bolar?

375. A we B nokatlarda güýçleri degişlilikde S_1 we S_2 deň bolan ýagtylyk çeşmeleri ýerleşdirilen. $AB = a$ kesimde iň kiçi ýagtylykly K nokady tapmaly.

376. Ýagtylandyryjy nokat radiuslary R we r $(R > r)$ bolan kesişmeýän şarlaryň merkezlerini birleşdirýän göni çyzykda we ol şarlaryň daşynda ýerleşýär. Nokat nirede ýerleşdirilende şarlaryň ýagtylanýan bölekleriniň üstleriniň jemi iň uly bolar?

377. Radiusy a deň bolan tegelek stoluň merkeziniň ýokarsynda nähili beýiklikde elektrik çyrasy ýerleşdirilende stoluň gyralarynyň ýagtylanyşy iň uly bolar? (*Görkezme*: *Ýagtylygyň ýagtylandyryjysy* $I = k\sin\varphi/r^2$ *formula boýunça tapylýar*). Bu ýerde: φ – şöhleleriň gyşarma burçy, r – ýagtylandyryjy şöhle bilen ýagtylanýan meýdanyň arasyndaky uzaklyk, k – ýagtylyk şöhläniň güýji.

378. Ini a *m* bolan deryadan göni burç boýunça ini b *metr* bolan kanal gurlan. Şol kanala iň uly uzynlykdaky nähili gämi girip biler?

379. Gämi suwda ýüzende bir gije-gündizde harç edilýän ýangyç iki bölekden ybarat: a manada deň hemişelik bölek we tizligiň kubuna proporsional bolan üýtgeýän bölek. Gämi haýsy ϑ tizlikde ýüzende iň amatly bolar?

380. P agramly ýük büdür-südür tekizlikde ýatyr. Ony güýç bilen ýerinden süýşürmek talap edilýän bolsun. Eger ýüküň sürtülme koeffisiýenti k bolsa, onda güýjüň gorizonta haýsy gyşarmasynda onuň ululygy iň kiçi bolar?

381. Radiusy a deň bolan ýarym şar görnüşli gaba uzynlygy $l > 2a$ bolan steržen goýberilen. Sterženiň deňagramly ýagdaýyny tapmaly.

§8. Egri çyzyklaryň galtaşmasy. Egriligiň tegelegi. Ewolýuta

1. ***n* tertipli galtaşýan.** Eger $y = \varphi(x)$ we $y = \psi(x)$ çyzyklar üçin

$$\varphi^{(k)}(x_0) = \psi^{(k)}(x_0) \quad (k = 0, 1, ..., n) \qquad \text{we} \qquad \varphi^{(n+1)}(x_0) \neq \psi^{(n+1)}(x_0)$$

bolsa, onda olaryň x_0 nokatda n tertipli galtaşýany bar diýilýär. Şonda, $x \to x_0$ bolanda

$$\varphi(x) - \psi(x) = O^*[x - x_0]^{n+1}.$$

2. ***Egriligiň tegelegi.*** Berlen $y = f(x)$ egri çyzyk bilen 2-nji tertipden pes bolmadyk galtaşýany bolan $(x - \xi)^2 + (y - \eta)^2 = R^2$ töwerege *egriligiň tegelegi* diýilýär. Bu tegelegiň

$$R = \frac{(1 + y'^2)^{3/2}}{|y''|}$$

radiusyna egriligiň radiusy, $k = 1/R$ ululyga bolsa egrilik diýilýär.

3. ***Ewolýuta.*** Egrilik tegelekleriniň

$$\xi = x - \frac{y'(1 + y'^2)}{y''}, \qquad \eta = y + \frac{1 + y'^2}{y''}$$

merkezleriniň (*egrilik merkezleriniň*) (ξ, η) geometrik ornuna berlen $y = f(x)$ çyzygyň ewolýutasy diýilýär.

Gönükmeler

382. $y = kx + b$ göni çyzykdaky k we b parametrleri şol göni çyzygyň $y = x^3 - 3x^2 + 2$ çyzyk bilen galtaşýanynyň tertibi birden ýokary bolar ýaly saýlap almaly.

383. a, b we c koeffisiýentleri nähili saýlanylanda $y = ax^2 + bx + c$ parabolanyň $x = x_0$ nokatda $y = e^x$ çyzyk bilen 2-nji tertipli galtaşýany bolar?

384. Berlen çyzyklaryň $x = 0$ nokatda Ox oky bilen galtaşma tertibi nähili bolar:
a) $y = 1 - \cos x$; b) $y = \mathrm{tg}x - \sin x$; ç) $y = e^x - (1 + x + x^2/2)$?

385. $y = e^{-1/x^2}$, $x \neq 0$ bolanda we $y = 0$ çyzygyň $x = 0$ nokatda Ox oky bilen tükeniksiz uly tertipdäki galtaşmasynyň bardygyny subut etmeli.

386. $xy = 1$ giperbolanyň: a) $M(1, 1)$; b) $N(100; 0,01)$ nokatlardaky egrilik radiusyny we merkezini tapmaly.

Çyzyklaryň egrilik radiusyny kesgitlemeli:

387. $y^2 = 2px$ parabolanyň.

388. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $(a \geq b > 0)$ ellipsiň.

389. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ giperbolanyň.

390. $x^{2/3} + y^{2/3} = a^{2/3}$ astroidanyň.

391. $x = a\cos t$, $y = b\sin t$ ellipsiň.

392. $x = a(t - \sin t)$, $y = a(1 - \cos t)$ çyzygyň.

393. $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$ tegelegiň ewolwentasynyň.

394. Ikinji tertipli $y^2 = 2px - qx^2$ çyzygyň egrilik radiusynyň normalyň kesiminiň kubuna proporsionaldygyny subut etmeli.

395. Polýar koordinatalarynda berlen çyzygyň egrilik radiusynyň formulasyny ýazmaly.

Polýar koordinatalarynda berlen çyzyklaryň egrilik radiuslaryny kesgitlemeli (parametrler položitel):

396. $r = a\varphi$ Arhimediň spiralynyň.

397. $r = ae^{m\varphi}$ logarifmik spiralyň.

398. $r = a(1 + \cos\varphi)$ kardioidanyň.

399. $r^2 = a^2\cos 2\varphi$ lemniskatanyň.

400. $y = \ln x$ çyzykda egriligi iň uly bolýan nokady tapmaly.

401. $y = \frac{kx^3}{6}$ $(0 \le x < +\infty,\ k > 0)$ kubiki parabolanyň maksimal egrelmesi 1/1000-e deň. Şol maksimal bahany alýan x nokady tapmaly.

Berlen çyzyklaryň ewolýutalarynyň deňlemesini düzmeli:

402. $y^2 = 2px$ parabolanyň. **403.** $x^2/a^2 + y^2/b^2 = 1$ ellipsiň.

404. $x^{2/3} + y^{2/3} = a^{2/3}$ astroidanyň.

405. $x = a\ln\frac{a+\sqrt{a^2-y^2}}{y} - \sqrt{a^2-y^2}$ traktrisanyň.

406. $r = ae^{m\varphi}$ logarifmik spiralyň.

407. $x = a(t - \sin t)$, $y = a(1 - \cos t)$ egri çyzygyň ewolýutasynyň ýene-de şol çyzyk bolýandygyny, ýöne onuň berlenden diňe ýerleşişi boýunça tapawutlanýandygyny subut etmeli.

§9. Deňlemeleriň takmyny çözüwi

1. ***Proporsional bölekler düzgüni (hordalar usuly).*** Eger $f(x)$ funksiýa $[a, b]$ kesimde üznüksiz bolsa we

$$f(a)f(b) < 0,$$

şeýle-de, $a < x < b$ bolanda $f'(x) \neq 0$ bolsa, onda

$$f(x) = 0 \tag{1}$$

deňlemäniň (a, b) interwalda ýeke-täk hakyky ξ köki bardyr. Bu kököň birinji ýakynlaşmasy hökmünde $x_1 = a + \delta_1$ ululygy almak bolar, bu ýerde $\delta_1 = -\frac{f(a)}{f(b)-f(a)}(b-a)$.

Mundan beýläk bu usuly uçlarynda $f(x)$ funksiýanyň bahalarynyň alamatlary dürli bolan (a, x_1) ýa-da (x_1, b) interwallaryň birinde ulanyp, ξ kököň ikinji x_2 ýakynlaşmasyny alarys.

n-nji x_n ýakynlaşmany bahalandyrmak üçin

$$|x_n - \xi| \le \frac{|f(x_n)|}{m} \tag{2}$$

formula dogrudyr, bu ýerde $m = \inf\limits_{a<x<b}|f'(x)|$, şeýle-de, $\lim\limits_{n\to\infty} x_n = \xi$.

2. ***Nýutonyň galtaşýanlar usuly.*** Eger $[a, b]$ kesimde $f''(x) \neq 0$ we $f(a)f''(a) > 0$ bolsa, onda (1) deňlemäniň ξ kökünüň birinji ýakynlaşmasy hökmünde aşakdaky ξ_1 sany almak bolar:

$$\xi_1 = a - \frac{f(a)}{f'(a)}.$$

Bu usuly gaýtalap, ξ köke çalt ýygnanýan ξ_n (n = 1, 2, ...) yzygiderli ýakynlaşmalary alarys, olaryň takyklygy, meselem, (2) formula boýunça bahalandyrylýar.

Takyk däl çemeleşme üçin $y = f(x)$ funksiýasynyň grafiginiň garalamasyny çyzmak peýdaly bolar.

Gönükmeler

Proporsional bölekler usulyndan peýdalanyp, 0,001-e çenli takyklykda deňlemeleriň köklerini tapmaly:

408. $x^3 - 6x + 2 = 0$.

409. $x^4 - x - 1 = 0$.

410. $x - 0{,}1\sin x = 2$.

411. $\cos x = x^2$.

Nýutonyň usulyndan peýdalanyp, görkezilen takyklykda berlen deňlemeleriň köklerini tapmaly:

412. $x^2 + \frac{1}{x^2} = 10x$ (10^{-3}).

413. $x \lg x = 1$ (10^{-4}).

414. $\cos x \cdot \operatorname{ch} x = 1$ (10^{-3}).

415. $x + e^x = 0$ (10^{-5}).

416. $x \operatorname{th} x = 1$ (10^{-6}).

417. $\operatorname{tg} x = x$ deňlemäniň 0,001 çenli takyklykda ilkinji üç položitel köküni tapmaly.

418. $\operatorname{ctg} x = \frac{1}{x} - \frac{x}{2}$ deňlemäniň 10^{-3}-e çenli takyklykda iki položitel köküni tapmaly.

VI. KESGITSIZ INTEGRAL

§1. Kesgitsiz integral we integrirlemek usullary

1. *Kesgitsiz integral düşünjesi.* Eger F funksiýa (a, b) interwalda differensirlenýän bolup, $F'(x) = f(x)$ deňlik ýerine ýetse, onda F funksiýa (a, b) interwalda f funksiýanyň asyl funksiýasy diýilýär. f funksiýanyň interwaldaky asyl funksiýalarynyň köplügine ol funksiýanyň kesgitsiz integraly diýilýär we ol $\int f(x)dx$ görnüşde belgilenýär.

Şeýlelikde, kesgitleme boýunça, eger F funksiýa (a, b) interwalda f funksiýanyň asyl funksiýalarynyň biri bolsa, onda

$$\int f(x)dx = F(x) + C \qquad (1)$$

bu ýerde C – hemişelik sandyr.

2. *Kesgitsiz integralyň häsiýetleri*

1) $\left(\int f(x)\,dx\right)' = f(x)$.

2) $d\left(\int f(x)\,dx\right)' = f(x)dx$.

3) $\int df(x) = f(x) + C$.

4) $\int kf(x)dx = k\int f(x)dx$.

5) $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$.

6) $\int f(x)dx = F(x) + C \Rightarrow \int f(ax + b)dx = \frac{1}{a}F(ax + b) + C \ (a \neq 0)$.

3. *Kesgitsiz integralyň tablisasy*

1. $\int dx = x + C$.

2. $\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha + 1} + C \quad (a \neq -1)$.

3. $\int a^x dx = \frac{a^x}{\ln a} + C \quad (0 < a \neq 1); \qquad \int e^x dx = e^x + C$.

4. $\int \frac{dx}{x} = \ln|x| + C \quad (x \neq 0)$.

5. $\int \cos x dx = \sin x + C$.

6. $\int \sin x dx = -\cos x + C.$

7. $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C \quad \left(x \neq \frac{\pi}{2} + k\pi,\ k \in Z\right).$

8. $\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C \quad (x \neq k\pi,\ k \in Z).$

9. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C = -\arccos \frac{x}{a} + C \quad (a \neq 0).$

10. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C = -\frac{1}{a} \operatorname{arcctg} \frac{x}{a} + C \quad (a \neq 0).$

11. $\int \operatorname{ch} x dx = \operatorname{sh} x + C.$

12. $\int \operatorname{sh} x dx = \operatorname{ch} x + C.$

13. $\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x + C.$

14. $\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x + C \quad (x \neq 0).$

15. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C \quad (a \neq 0).$

16. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{a + x}{a - x}\right| + C \quad (a \neq 0).$

Integraly hasaplamaklyk integral astyndaky funksiýany özgerdip, ony integral tablisasyna getirmekden ybaratdyr.

4. ***Kesgitsiz integraly hasaplamagyň usullary***

a) ***täze üýtgeýän ululygy girizme usuly.*** Bu usul üznüksiz differensirlenýän $u = \varphi(x)$ funksiýa üçin (1) formuladan gelip çykýan

$$\int f(u)du = F(u) + C$$

formula esaslanýar. Ol formula integral tablisasynyň üýtgeýän x ululygyň ýerine $u = \varphi(x)$ funksiýa ýazylanda hem dogrudygyny aňladýar.

1-nji mysal. $\int \frac{dx}{\sin x}$ integraly hasaplamaly.

Ç.B. $$\int \frac{dx}{\sin x} = \int \frac{d\left(\frac{x}{2}\right)}{\sin \frac{x}{2} \cos \frac{x}{2}} = \int \frac{du}{\sin u \cos u} = \int \frac{\frac{1}{\cos^2 u} du}{\frac{\sin u \cos u}{\cos^2 u}} =$$

$$= \int \frac{d\operatorname{tg} u}{\operatorname{tg} u} = \ln|\operatorname{tg} u| + C = \ln\left|\operatorname{tg} \frac{x}{2}\right| + C.$$ **Ç.S.**

2-nji mysal. $\int \frac{1}{\sqrt{\operatorname{arctg} x}} \frac{dx}{1+x^2}$ integraly hasaplamaly.

Ç.B. $$\int \frac{1}{\sqrt{\operatorname{arctg} x}} \frac{dx}{1+x^2} = \int \frac{d(\operatorname{arctg} x)}{\sqrt{\operatorname{arctg} x}} = \int u^{-\frac{1}{2}} du =$$
$$= 2\sqrt{u} + C = 2\sqrt{\operatorname{arctg} x} + C. \quad \textbf{Ç.S.}$$

b) *dagytma usuly.* Bu usul integral astyndaky funksiýany asyl funksiýalary aňsat tapylýan funksiýalar arkaly $f(x) = \sum_{i=1}^{n} k_i f_i(x)$ görnüşde ýazyp, berlen integraly integrallaryň jemi görnüşinde tapmaklygy aňladýar.

3-nji mysal. $\int x(1-2x)^{43} dx$ integraly hasaplamaly.

Ç.B. $$\int x(1-2x)^{43} dx = \int \left[-\frac{1}{2}(1-2x) + \frac{1}{2}\right](1-2x)^{43} dx =$$
$$= \frac{1}{2}\int (1-2x)^{44} dx + \frac{1}{2}\int (1-2x)^{43} dx =$$
$$= \left(-\frac{1}{2}\right)^2 \int (1-2x)^{44} d(1-2x) + \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\int (1-2x)^{43} d(1-2x) =$$
$$= \frac{1}{180}(1-2x)^{45} - \frac{1}{176}(1-2x)^{44} + C. \quad \textbf{Ç.S.}$$

4-nji mysal. $\int \frac{dx}{\sqrt{x+1} + \sqrt{x-1}}$ integraly hasaplamaly.

Ç.B. $$\int \frac{dx}{\sqrt{x+1} + \sqrt{x-1}} = \int \frac{\sqrt{x+1} - \sqrt{x-1}}{2} dx =$$
$$= \frac{1}{2}\int \sqrt{x+1}\, dx - \frac{1}{2}\int \sqrt{x+1}\, dx = \frac{1}{3}(x+1)^{3/2} - \frac{1}{3}(x-1)^{3/2} + C. \quad \textbf{Ç.S.}$$

ç) *üýtgeýän ululygy çalşyrma usuly.* Bu usul $\int f(x)dx$ integraly hasaplamaklygy $x = \varphi(t)$ çalşyrmany girizip, ol integraly integrirlemek üçin amatly bolan görnüşe getirip, ýagny
$$\int f(x)dx = \int f[\varphi(t)]\varphi'(t)dt$$
formulany ulanyp, integraly hasaplamagy aňladýar. Bu formuladan görnüşi ýaly, eger $G(t)$ funksiýa $f[\varphi(t)]\varphi'(t)$ funksiýanyň asyl funksiýasy bolsa, onda $f(x)$ funksiýanyň asyl funksiýasy $F(x) = G(\varphi^{-1}(x))$ bolar. Çalşyrmadaky $\varphi(t)$ funksiýa integral astyndaky aňlatmanyň anyk görnüşi boýunça kesgitlenýär.

5-nji mysal. $\int \frac{dx}{x^2\sqrt{1+x^2}}$ integraly hasaplamaly.

Ç.B. Integraly hasaplamak üçin $x = \frac{1}{t}\ (x = \varphi(t))$ çalşyrmany ulanarys, şonda $dx = -\frac{dt}{t^2}$ bolar we integral aňsat hasaplanylýan görnüşi alar:

$$\int \frac{dx}{x^2\sqrt{1+x^2}} = -\int \frac{t^2 dt}{t^2\sqrt{1+\frac{1}{t^2}}} = -\int \frac{tdt}{\sqrt{t^2+1}} =$$

$$= -\int d\sqrt{t^2+1} = -\sqrt{t^2+1} + C = -\sqrt{\frac{1}{x^2}+1} + C \qquad \textbf{Ç.S.}$$

d) *bölekleýin integrirleme usuly.* Bu usul boýunça $\int udv$ görnüşdäki integraly hasaplamaklyk $\int udv = uv - \int vdu$ formulany ulanmak bilen $\int vdu$ görnüşdäki integraly hasaplamaklyga getirilýär. Bu usul, köplenç, integral astynda dürli «jynsdaky» funksiýalaryň köpeltmek hasyly, mysal üçin, e^{ax} we x^b, e^{ax} we $\cos bx$, x we $\ln x$, x we $\mathrm{arctg}x$ we ş.m. bolanda ulanylýar. Käbir hallarda integraly hasaplamak üçin bu usuly birnäçe gezek ulanmaly bolýar, şonda gözlenýän integral çyzykly deňlemäni çözüp tapylýar.

6-njy mysal. $I = \int e^{ax}\sin bxdx$ (a, b – hemişelik ululyk) integraly hasaplamaly.

Ç.B. Eger $u = e^{ax}$, $d\upsilon = \sin bxdx$ bolsa, onda $du = ae^{ax}dx$, $\upsilon = -\frac{1}{b}\cos bx$ bolar. Şonuň üçin (8) formulany ulanyp alarys:

$$I = -\frac{1}{b}\cos bx \cdot e^{ax} + \frac{a}{b}\int e^{ax}\cos bxdx.$$

Eger $u = e^{ax}$, $d\upsilon = \cos bxdx$ bolsa, onda $du = ae^{ax}dx$, $\upsilon = \frac{1}{b}\sin bx$ bolar. Şonuň üçin integrala ýene-de (8) formulany ulanyp, I integrala görä çyzykly

$$I = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}I$$

deňlemäni alarys. Ol deňlemäni

$$I\left(\frac{a^2+b^2}{b^2}\right) = \frac{e^{ax}(a\sin bx - b\cos bx)}{b^2}$$

görnüşde ýazyp we ony I integrala görä çozüp, integraly taparys:

$$\int e^{ax}\sin bxdx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C. \qquad \textbf{Ç.S.}$$

Gönükmeler

Integral tablisasyny ulanmak bilen integrallary tapmaly:

1. $\int (3-x^2)^3 dx.$

2. $\int x^2(5-x)^4 dx.$

3. $\int (1-x)(1-2x)(1-3x)dx.$

4. $\int \left(\frac{1-x}{x}\right)^2 dx.$

5. $\int \left(\frac{a}{x}+\frac{a^2}{x^2}+\frac{a^3}{x^3}\right)dx.$

6. $\int \frac{x+1}{\sqrt{x}}dx.$

7. $\int \frac{\sqrt{x}-2\sqrt[3]{x^2}+1}{\sqrt[4]{x}}dx.$

8. $\int \frac{(1-x)^3}{x\sqrt[3]{x}}dx.$

9. $\int \left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,dx.$

10. $\int \frac{(\sqrt{2x}-\sqrt[3]{3x})^2}{x}dx.$

11. $\int \frac{\sqrt{x^4+x^{-4}+2}}{x^3}dx.$

12. $\int \frac{x^2 dx}{1+x^2}.$

13. $\int \frac{x^2 dx}{1-x^2}.$

14. $\int \frac{x^2+3}{x^2-1}dx.$

15. $\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}dx.$

16. $\int \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^4-1}}dx.$

17. $\int (2^x+3^x)^2 dx.$

18. $\int \frac{2^{x+1}-5^{x-1}}{10^x}dx.$

19. $\int \frac{e^{3x}+1}{e^x+1}dx.$

20. $\int (1+\sin x+\cos x)dx.$

21. $\int \sqrt{1-\sin 2x}\,dx.\ (0 \le x \le \pi)$

22. $\int \mathrm{ctg}^2 x dx.$

23. $\int \mathrm{tg}^2 x dx.$

24. $\int (a\,\mathrm{sh}x+b\,\mathrm{ch}x)dx.$

25. $\int \mathrm{th}^2 x dx.$

26. $\int \mathrm{cth}^2 x dx.$

27. Eger $\int f(x)dx = F(x)+C$ bolsa, onda

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b)+C \quad (a \ne 0)$$

deňligi subut etmeli.

Integrallary tapmaly:

28. $\int \frac{dx}{x+a}.$

29. $\int (2x-3)^{10} dx.$

30. $\int \sqrt[3]{1-3x}\,dx.$

31. $\int \dfrac{dx}{\sqrt{2-5x}}.$

32. $\int \dfrac{dx}{(5x-2)^{5/2}}.$

33. $\int \dfrac{\sqrt[5]{1-2x+x^2}}{1-x}dx.$

34. $\int \dfrac{dx}{2+3x^2}.$

35. $\int \dfrac{dx}{2-3x^2}.$

36. $\int \dfrac{dx}{\sqrt{2-3x^2}}.$

37. $\int \dfrac{dx}{\sqrt{3x^2-2}}.$

38. $\int (e^{-x}+e^{-2x})dx.$

39. $\int (\sin 5x - \sin 5\alpha)dx.$

40. $\int \dfrac{dx}{\sin^2\left(2x+\frac{\pi}{4}\right)}.$

41. $\int \dfrac{dx}{1+\cos x}.$

42. $\int \dfrac{dx}{1-\cos x}.$

43. $\int \dfrac{dx}{1+\sin x}.$

44. $\int [\operatorname{sh}(2x+1)+\operatorname{ch}(2x-1)]dx.$

45. $\int \dfrac{dx}{\operatorname{ch}^2\frac{x}{2}}.$

46. $\int \dfrac{dx}{\operatorname{sh}^2\frac{x}{2}}.$

Integrallaryň astyndaky aňlatmalary özgerdip, integrallary tapmaly:

47. $\int \dfrac{xdx}{\sqrt{1-x^2}}.$

48. $\int x^2\sqrt[3]{1+x^3}\,dx.$

49. $\int \dfrac{xdx}{3-2x^2}.$

50. $\int \dfrac{xdx}{(1+x^2)^2}.$

51. $\int \dfrac{xdx}{4+x^4}.$

52. $\int \dfrac{x^3dx}{x^8-2}.$

53. $\int \dfrac{dx}{(1+x)\sqrt{x}}$ (*Görkezme*: $\dfrac{dx}{\sqrt{x}} = 2d(\sqrt{x})$).

54. $\int \sin\dfrac{1}{x}\cdot\dfrac{dx}{x^2}.$

55. $\int \dfrac{dx}{x\sqrt{x^2+1}}.$

56. $\int \dfrac{dx}{x\sqrt{x^2-1}}.$

57. $\int \dfrac{dx}{(x^2+1)^{\frac{3}{2}}}.$

58. $\int \frac{xdx}{(x^2-1)^{\frac{3}{2}}}$.

59. $\int \frac{x^2 dx}{(8x^3+27)^{\frac{2}{3}}}$.

60. $\int \frac{dx}{\sqrt{x(1+x)}}$.

61. $\int \frac{dx}{\sqrt{x(1-x)}}$.

62. $\int xe^{-x^2}dx$.

63. $\int \frac{e^x dx}{2+e^x}$.

64. $\int \frac{dx}{e^x+e^{-x}}$.

65. $\int \frac{dx}{\sqrt{1+e^{2x}}}$.

66. $\int \frac{\ln^2 x}{x}dx$.

67. $\int \frac{dx}{x\ln x\ln(\ln x)}$.

68. $\int \sin^5 x\cos x dx$.

69. $\int \frac{\sin x}{\sqrt{\cos^3 x}}dx$.

70. $\int \mathrm{tg} x dx$.

71. $\int \mathrm{ctg} x dx$.

72. $\int \frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}dx$.

73. $\int \frac{\sin x\cos x}{\sqrt{a^2\sin^2 x+b^2\cos^2 x}}dx$.

74. $\int \frac{\sin x}{\sqrt{\cos 2x}}dx$.

75. $\int \frac{\cos x}{\sqrt{\cos 2x}}dx$.

76. $\int \frac{\mathrm{sh} x}{\sqrt{\mathrm{ch} 2x}}dx$.

77. $\int \frac{dx}{\sin^2 x\sqrt[4]{\mathrm{ctg} x}}$.

78. $\int \frac{dx}{\sin^2 x+2\cos^2 x}$.

79. $\int \frac{dx}{\sin x}$.

80. $\int \frac{dx}{\cos x}$.

81. $\int \frac{dx}{\mathrm{sh} x}$.

82. $\int \frac{dx}{\mathrm{ch} x}$.

83. $\int \frac{\mathrm{sh} x\,\mathrm{ch} x}{\sqrt{\mathrm{sh}^4 x+\mathrm{ch}^4 x}}dx$.

84. $\int \frac{dx}{\mathrm{ch}^2 x\sqrt[3]{\mathrm{th}^2 x}}$.

85. $\int \frac{\mathrm{arctg} x}{1+x^2}dx$.

86. $\int \frac{dx}{(\arcsin x)^2\sqrt{1-x^2}}$.

87. $\int \sqrt{\frac{\ln\left(x+\sqrt{1+x^2}\right)}{1+x^2}}\,dx$.

88. $\int \frac{x^2+1}{x^4+1}dx$. (*Görkezme*: $\left(1+\frac{1}{x^2}\right)dx = d\left(x-\frac{1}{x}\right)$).

89. $\int \frac{x^2-1}{x^4+1}dx.$

90. $\int \frac{x^4\,dx}{(x^5+1)^4}.$

91. $\int \frac{x^{n/2}\,dx}{\sqrt{1+x^{n+2}}}.$

92. $\int \frac{1}{1-x^2}\ln\frac{1+x}{1-x}dx.$

93. $\int \frac{\cos x\,dx}{\sqrt{2+\cos 2x}}.$

94. $\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx.$

95. $\int \frac{2^x\cdot 3^x}{9^x-4^x}dx.$

96. $\int \frac{x\,dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}.$

Dagytma usulyny ulanyp, integrallary hasaplamaly:

97. $\int x^2(2-3x^2)^2dx.$

98. $\int x(1-x)^{10}dx.$

99. $\int \frac{1+x}{1-x}dx.$

100. $\int \frac{x^2}{1+x}dx.$

101. $\int \frac{x^3}{3+x}dx.$

102. $\int \frac{(1+x)^2}{1+x^2}dx.$

103. $\int \frac{(2-x)^2}{2-x^2}dx.$

104. $\int \frac{x^2}{(1-x)^{100}}dx.$

105. $\int \frac{x^5}{x+1}dx.$

106. $\int \frac{dx}{\sqrt{x+1}+\sqrt{x-1}}.$

107. $\int x\sqrt{2-5x}\,dx.$ (*Görkezme*: $x\equiv -\frac{1}{5}(2-5x)+\frac{2}{5}$).

108. $\int \frac{x\,dx}{\sqrt[3]{1-3x}}.$

109. $\int x^3\sqrt[3]{1+x^2}\,dx.$

110. $\int \frac{dx}{(x-1)(x+3)}.$ (*Görkezme*: $1\equiv\frac{1}{4}[(x+3)-(x-1)]$).

111. $\int \frac{dx}{x^2+x-2}.$

112. $\int \frac{dx}{(x^2+1)(x^2+2)}.$

113. $\int \frac{dx}{(x^2-2)(x^2+3)}.$

114. $\int \frac{x\,dx}{(x+2)(x+3)}.$

115. $\int \frac{x\,dx}{x^4+3x^2+2}.$

116. $\int \frac{dx}{(x+a)^2(x+b)^2}\quad (a\neq b).$

117. $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}\quad (a^2\neq b^2).$

118. $\int \sin^2 x dx.$

119. $\int \cos^2 x dx.$

120. $\int \sin x \sin(x + \alpha) dx.$

121. $\int \sin 3x \cdot \sin 5x dx.$

122. $\int \cos\frac{x}{2} \cdot \cos\frac{x}{3} dx.$

123. $\int \sin\left(2x - \frac{\pi}{6}\right)\cos\left(3x + \frac{\pi}{4}\right)dx.$

124. $\int \sin^3 x dx.$

125. $\int \cos^3 x dx.$

126. $\int \sin^4 x dx.$

127. $\int \cos^4 x dx.$

128. $\int \mathrm{ctg}^2 x dx.$

129. $\int \mathrm{tg}^3 x dx.$

130. $\int \sin^2 3x \sin^3 2x dx.$

131. $\int \frac{dx}{\sin^2 x \cos^2 x}$. (*Görkezme*: $1 \equiv \sin^2 x + \cos^2 x$).

132. $\int \frac{dx}{\sin^2 x \cdot \cos x}.$

133. $\int \frac{dx}{\sin x \cos^3 x}.$

134. $\int \frac{\cos^3 x}{\sin x} dx.$

135. $\int \frac{dx}{\cos^4 x}.$

136. $\int \frac{dx}{1 + e^x}.$

137. $\int \frac{(1 + e^x)^2}{1 + e^{2x}} dx.$

138. $\int \mathrm{sh}^2 x dx.$

139. $\int \mathrm{ch}^2 x dx.$

140. $\int \mathrm{sh} x \, \mathrm{sh} 2x dx.$

141. $\int \mathrm{ch} x \cdot \mathrm{ch} 3x dx.$

142. $\int \frac{dx}{\mathrm{sh}^2 x \mathrm{ch}^2 x}.$

Amatly orun çalşyrmalary ulanyp, integrallary tapmaly:

143. $\int x^2 \sqrt[3]{1 - x}\, dx.$

144. $\int x^3(1 - 5x^2)^{10} dx.$

145. $\int \frac{x^2}{\sqrt{2 - x}} dx.$

146. $\int \frac{x^5}{\sqrt{1 - x^2}} dx.$

147. $\int x^5(2 - 5x^3)^{2/3} dx.$

148. $\int \cos^5 x \cdot \sqrt{\sin x}\, dx.$

149. $\int \frac{\sin x \cos^3 x}{1 + \cos^2 x} dx.$

150. $\int \frac{\sin^2 x}{\cos^6 x} dx.$

151. $\int \frac{\ln x dx}{x\sqrt{1+\ln x}}$.

152. $\int \frac{dx}{e^{x/2}+e^{x}}$.

153. $\int \frac{dx}{\sqrt{1+e^{x}}}$.

154. $\int \frac{\operatorname{arctg}\sqrt{x}}{\sqrt{x}} \cdot \frac{dx}{1+x}$.

Trigonometrik $x = a\sin t$, $x = a\operatorname{tg} t$, $x = a\sin^2 t$ we ş.m. orun çalşyrmalary ulanyp, integrallary tapmaly (parametrleri položitel):

155. $\int \frac{dx}{(1-x^2)^{3/2}}$.

156. $\int \frac{x^2 dx}{\sqrt{x^2-2}}$.

157. $\int \sqrt{1-x^2}\, dx$.

158. $\int \frac{dx}{(x^2+a^2)^{3/2}}$.

159. $\int \sqrt{\frac{a+x}{a-x}}\, dx$.

160. $\int x\sqrt{\frac{x}{2a-x}}\, dx$.

161. $\int \frac{dx}{\sqrt{(x-a)(b-x)}}$. (*Görkezme*: $x-a=(b-a)\sin^2 t$ *çalşyrmany ulanmaly*).

162. $\int \sqrt{(x-a)(b-x)}\, dx$.

Giperbolik $x = a\operatorname{sh} t$, $x = a\operatorname{ch} t$ we ş.m. orun çalşyrmalary ulanyp, integrallary tapmaly (parametrleri položitel):

163. $\int \sqrt{a^2+x^2}\, dx$.

164. $\int \frac{x^2}{\sqrt{a^2+x^2}} dx$.

165. $\int \sqrt{\frac{x-a}{x+a}}\, dx$.

166. $\int \frac{dx}{\sqrt{(x+a)(x+b)}}$.

167. $\int \sqrt{(x+a)(x+b)}\, dx$. (*Görkezme*: $x+a=(b-a)\operatorname{sh}^2 t$ *almaly*).

Bölekleýin integrirleme usulyny ulanyp, integrallary tapmaly:

168. $\int \ln x dx$.

169. $\int x^n \ln x dx \quad (n \neq -1)$.

170. $\int \left(\frac{\ln x}{x}\right)^2 dx$.

171. $\int \sqrt{x}\ln^2 x dx$.

172. $\int x e^{-x} dx$.

173. $\int x^2 e^{-2x} dx$.

174. $\int x^3 e^{-x^2} dx$.

175. $\int x\cos x dx$.

176. $\int x^2 \sin 2x dx$.

177. $\int x\operatorname{sh} x dx$.

178. $\int x^3 \mathrm{ch} 3x dx$.

179. $\int \mathrm{arctg}\, x dx$.

180. $\int \arcsin x dx$.

181. $\int x\, \mathrm{arctg}\, x dx$.

182. $\int x^2 \arccos x dx$.

183. $\int \frac{\arcsin x}{x^2} dx$.

184. $\int \ln\left(x + \sqrt{1 + x^2}\right) dx$.

185. $\int x \ln \frac{1 + x}{1 - x} dx$.

186. $\int \mathrm{arctg} \sqrt{x}\, dx$.

187. $\int \sin x \cdot \ln(\mathrm{tg} x) dx$.

Integrallary tapmaly:

188. $\int x^5 e^{x^3} dx$.

189. $\int (\arcsin x)^2 dx$.

190. $\int x(\mathrm{arctg} x)^2 dx$.

191. $\int x^2 \ln \frac{1 - x}{1 + x} dx$.

192. $\int \frac{x \ln\left(x + \sqrt{1 + x^2}\right)}{\sqrt{1 + x^2}} dx$.

193. $\int \frac{x^2}{(1 + x^2)^2} dx$.

194. $\int \frac{dx}{(a^2 + x^2)^2}$.

195. $\int \sqrt{a^2 - x^2}\, dx$.

196. $\int \sqrt{x^2 + a}\, dx$.

197. $\int x^2 \sqrt{a^2 + x^2}\, dx$.

198. $\int x \sin^2 x dx$.

199. $\int e^{\sqrt{x}} dx$.

200. $\int x \sin \sqrt{x}\, dx$.

201. $\int \frac{x e^{\mathrm{arctg} x}}{(1 + x^2)^{3/2}} dx$.

202. $\int \frac{e^{\mathrm{arctg} x}}{(1 + x^2)^{3/2}} dx$.

203. $\int \sin(\ln x) dx$.

204. $\int \cos(\ln x) dx$.

205. $\int e^{ax} \cos bx dx$.

206. $\int e^{ax} \sin bx dx$.

207. $\int e^{2x} \sin^2 x dx$.

208. $\int (e^x - \cos x)^2 dx$.

209. $\int \frac{\mathrm{arcctg}\, e^x}{e^x} dx$.

210. $\int \frac{\ln(\sin x)}{\sin^2 x} dx$.

211. $\int \frac{x dx}{\cos^2 x}$.

212. $\int \frac{x e^x}{(x + 1)^2} dx$.

Aşakdaky integrallary tapmaklyk kwadrat üçagzany ýönekeý görnüşe getirmeklige we şol formulalary ulanmaklyga esaslanan:

I. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C \quad (a \neq 0).$

II. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right| + C \quad (a \neq 0).$

III. $\int \frac{xdx}{a^2 \pm x^2} = \pm\frac{1}{2}\ln|a^2 \pm x^2| + C.$

IV. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C \quad (a > 0).$

V. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 + a^2}| + C \quad (a > 0).$

VI. $\int \frac{xdx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2} + C \quad (a > 0)$

VII. $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C \quad (a > 0).$

VIII. $\int \sqrt{x^2 \pm a^2}\,dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln|x + \sqrt{x^2 \pm a^2}| + C \quad (a > 0).$

Integrallary tapmaly:

213. $\int \frac{dx}{a + bx^2} \quad (ab \neq 0).$

214. $\int \frac{dx}{x^2 - x + 2}.$

215. $\int \frac{dx}{3x^2 - 2x - 1}.$

216. $\int \frac{xdx}{x^4 - 2x^2 - 1}.$

217. $\int \frac{(x + 1)}{x^2 + x + 1}dx.$

218. $\int \frac{xdx}{x^2 - 2x\cos\alpha + 1}.$

219. $\int \frac{x^3 dx}{x^4 - x^2 + 2}.$

220. $\int \frac{x^5 dx}{x^6 - x^3 - 2}.$

221. $\int \frac{dx}{3\sin^2 x - 8\sin x\cos x + 5\cos^2 x}.$

222. $\int \frac{dx}{\sin x + 2\cos x + 3}.$

223. $\int \frac{dx}{\sqrt{a + bx^2}} \quad (b \neq 0).$

224. $\int \frac{dx}{\sqrt{1 - 2x - x^2}}.$

225. $\int \frac{dx}{\sqrt{x + x^2}}.$

226. $\int \frac{dx}{\sqrt{2x^2 - x + 2}}$.

227. Eger $y = ax^2 + bx + c \quad (a \neq 0)$ bolsa, onda

$$\int \frac{dx}{\sqrt{y}} = \frac{1}{\sqrt{a}} \ln\left| \frac{y'}{2} + \sqrt{ay} \right| + C \qquad (a > 0)$$

we

$$\int \frac{dx}{\sqrt{y}} = \frac{1}{\sqrt{-a}} \arcsin \frac{-y'}{\sqrt{b^2 - 4ac}} + C \qquad (a < 0)$$

deňlikleri subut etmeli.

Integrallary tapmaly:

228. $\int \frac{xdx}{\sqrt{5 + x - x^2}}$.

229. $\int \frac{x + 1}{\sqrt{x^2 + x + 1}} dx$.

230. $\int \frac{xdx}{\sqrt{1 - 3x^2 - 2x^4}}$.

231. $\int \frac{\cos x dx}{\sqrt{1 + \sin x + \cos^2 x}}$.

232. $\int \frac{x^3 dx}{\sqrt{x^4 - 2x^2 - 1}}$.

233. $\int \frac{x + x^3}{\sqrt{1 + x^2 - x^4}} dx$.

234. $\int \frac{dx}{x\sqrt{x^2 + x + 1}}$.

235. $\int \frac{dx}{x^2 \sqrt{x^2 + x - 1}}$.

236. $\int \frac{dx}{(x + 1)\sqrt{x^2 + 1}}$.

237. $\int \frac{dx}{(x - 1)\sqrt{x^2 - 2}}$.

238. $\int \frac{dx}{(x + 2)^2 \sqrt{x^2 + 2x - 5}}$.

239. $\int \sqrt{2 + x - x^2}\, dx$.

240. $\int \sqrt{2 + x + x^2}\, dx$.

241. $\int \sqrt{x^4 + 2x^2 - 1}\, xdx$.

242. $\int \frac{1 - x + x^2}{x\sqrt{1 + x - x^2}} dx$.

243. $\int \frac{x^2 + 1}{x\sqrt{x^4 + 1}} dx$.

§2. Rasional funksiýalaryň integrirlenişi

1. ***Ýönekeý rasional droblaryň integrirlenişi.*** Her bir $\frac{P(x)}{Q(x)}$ görnüşdäki rasional funksiýa $P(x)$ köpagza bilen $Q(x)$ köpagzanyň köklerine baglylykda

I. $\frac{A}{x - a}$, II. $\frac{A}{(x - a)^k}$, III. $\frac{Mx + N}{x^2 + px + q}$, IV. $\frac{Mx + N}{(x^2 + px + q)^k}$ $(p^2 - 4q < 0)$

ýönekeý rasional droblaryň jemi görnüşinde aňladylýar. Şonuň üçin hem rasional funksiýalary integrirlemek olary ýönekeý rasional droblara dagytmaklyga we ýönekeý rasional droblary we köpagzalary integrirlemeklige getirilýär.

Ýönekeý rasional droblary integrirlemek aşakdaky ýaly ýerine ýetirilýär:

I. $\int \frac{A}{x-a}dx = A\int \frac{d(x-a)}{x-a} = A\ln|x-a| + C.$

II. $\int \frac{A}{(x-a)^k}dx = A\int (x-a)^{-k}d(x-a) = \frac{A}{(1-k)(x-a)^{k-1}} + C.$

III. $\int \frac{Mx+N}{x^2+px+q}dx = \int \frac{M(x+p/2)+(N-Mp/2)}{(x+p/2)^2+(q-p^2/4)}d(x+p/2) =$

$$= M\int \frac{tdt}{t^2+a^2} + (N-Mp/2)\int \frac{dt}{t^2+a^2} =$$

$$= \frac{M}{2}\int \frac{d(t^2+a^2)}{t^2+a^2} + \left(N - \frac{Mp}{2}\right)\int \frac{dt}{t^2+a^2} =$$

$$= \frac{M}{2}\ln|t^2+a^2| + \left(N - \frac{Mp}{2}\right)\frac{1}{a}\operatorname{arctg}\frac{t}{a} + C.$$

IV. $\int \frac{Mx+N}{(x^2+px+q)^m}dx = \frac{M}{2}\int \frac{d(t^2+a^2)}{(t^2+a^2)} + \left(N - \frac{MP}{2}\right)\int \frac{dt}{(t^2+a^2)^m} =$

$$= \frac{Mx+N}{2(1-m)(t^2+a^2)^{m-1}} + \left(N - \frac{MP}{2}\right)\int \frac{dt}{(t^2+a^2)^m},$$

bu ýerde $a^2 = q - p^2/4$, $t = x + p/2$.

7-nji mysal. $I_m = \int \frac{dx}{(x^2+a^2)^m}$ integral üçin rekurrent formulany getirip çykarmaly.

Ç.B. Özgertmeler geçirip, ilki ony

$$I_m = \frac{1}{a^2}\int \frac{x^2+a^2-x^2}{(x^2+a^2)^m}dx = \frac{1}{a^2}I_{m-1} - \frac{1}{2a^2}\int x\frac{d(a^2+x^2)}{(x^2+a^2)}$$

görnüşe getireris we soňky integrala bölekleýin integrirleme usulyny ulanarys. Goý, $u = x$, $d\upsilon = \frac{d(a^2+x^2)}{(x^2+a^2)^k}$ bolsun, onda $du = dx$, $\upsilon = -\frac{1}{(k-1)(x^2+a^2)^{k-1}}$.

Şonuň üçin

$$I_m = \frac{1}{a^2}I_{m-1} + \frac{x}{2a^2(m-1)(x^2+a^2)^{m-1}} - \frac{1}{2a^2(m-1)}I_{m-1}.$$

Bu ýerden bolsa I_m integral üçin rekurrent formula alynýar:

$$I_m = \frac{x}{2a^2(m-1)(x^2+a^2)^{m-1}} + \frac{2m-3}{2a^2(m-1)} B_{m-1}. \quad \textbf{Ç.S.}$$

Alnan formulanyň kömegi bilen $\forall m = 2, 3, ...$ üçin I_m integraly hasaplap bolar. Hakykatdan-da, mälim bolan

$$I_m = \int \frac{dx}{x^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C$$

integraly ulanyp, I_2 integraly hasaplarys. Şonuň ýaly-da I_2 integraly ulanyp, I_3 integraly taparys. Şonuň ýaly dowam etdirip, $\forall k \in N$ üçin integraly I_m hasaplap bileris.

Bellik. $\frac{P(x)}{Q(x)}$ görnüşdäki rasional droby ýönekeý rasional droblaryň jemine dagytma näbelli koeffisiýentler usulyny ulanmak arkaly amala aşyrylýar. Rasional drobuň maýdalawjysynyň n kratny a köküne degişli bolan

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-a)^n Q_1(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + ... + \frac{A_n}{(x-a)^n} + \frac{P(x)}{Q_1(x)}$$

dagytmasyndaky näbelli koeffisiýentleri tapmak üçin

$$A_{n-k} = \frac{1}{k!}\left(\frac{P(x)}{Q_1(x)}\right)^{(k)}\Bigg|_{x=a}, \qquad k = \overline{0, n-1} \tag{2}$$

formuladan peýdalanmak bolar. Maýdalawjynyň beýleki hakyky köklerine degişli näbelli koeffisiýentleri hem şolar ýaly tapylýar.

8-nji mysal. $\int \frac{xdx}{(x+1)(x-2)^2}$ integraly hasaplamaly.

Ç.B. Ilki bilen integral astyndaky funksiýany ýönekeý rasional droblaryň jemi görnüşinde aňladalyň:

$$\frac{x}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B_1}{x-2} + \frac{B_2}{(x-2)^2}.$$

(2) formulany ulanyp, näbelli koeffisiýentleri taparys:

$$A = \frac{x}{(x-2)^2}\Bigg|_{x=-1} = -\frac{1}{9}, \quad B_2 = \frac{x}{x+1}\Bigg|_{x=2} = \frac{2}{3},$$

$$B_1 = \left(\frac{x}{x+1}\right)'\Bigg|_{x=2} = \frac{1}{(x+1)^2}\Bigg|_{x=2} = \frac{1}{9}.$$

Şeýlelikde,

$$\int \frac{xdx}{(x+1)(x-2)^2} = -\frac{1}{9}\int\frac{dx}{x+1} + \frac{1}{9}\int\frac{dx}{x-2} + \frac{2}{3}\int\frac{dx}{(x-2)^2} =$$

$$= \frac{1}{9}\ln\left|\frac{x-2}{x+1}\right| - \frac{2}{3}\frac{1}{x-2} + C. \quad \textbf{Ç.S.}$$

Näbelli koeffisiýentleri tapmak, köplenç, köp hasaplamalary talap edýär. Şonuň üçin rasional funksiýalaryň käbirleriniň integrallaryny hasaplamagy, özgertmeleri geçirip, ýerine ýetirmek amatly bolýar.

9-njy mysal. $\int \frac{dx}{(x+2)^2(x-3)^2}$ integraly hasaplamaly.

Ç.B. $$\int \frac{dx}{(x+2)^2(x-3)^2} = \int\left[\frac{1}{5}\left(\frac{1}{x-3} - \frac{1}{x+2}\right)\right]^2 dx =$$

$$= \frac{1}{25}\int\frac{dx}{(x-3)^2} + \frac{1}{25}\int\frac{dx}{(x+2)^2} - \frac{2}{25}\int\frac{dx}{(x+2)(x-3)} =$$

$$= \frac{1}{25}\cdot\frac{1}{x-3} - \frac{1}{25}\cdot\frac{1}{x+2} - \frac{2}{125}\int\left[\frac{1}{x-3} - \frac{1}{x+2}\right]dx =$$

$$= -\frac{1}{25}\cdot\frac{1}{x-3} - \frac{1}{25}\cdot\frac{1}{x+2} - \frac{2}{125}\ln|x-3| + \frac{2}{125}\ln|x+2| + C. \quad \textbf{Ç.S.}$$

Gönükmeler

Näbelli koeffisiýentler usulyny ulanyp, integrallary tapmaly:

244. $\int \frac{2x+3}{(x-2)(x+5)}dx.$

245. $\int \frac{xdx}{(x+1)(x+2)(x+3)}.$

246. $\int \frac{x^{10}dx}{x^2+x-2}.$

247. $\int \frac{x^3+1}{x^3-5x^2+6x}dx.$

248. $\int \frac{x^4}{x^4+5x^2+4}dx.$

249. $\int \frac{xdx}{x^3-3x+2}.$

250. $\int \frac{x^2+1}{(x+1)^2(x-1)}dx.$

251. $\int \left(\frac{x}{x^2-3x+2}\right)^2 dx.$

252. $\int \frac{dx}{(x+1)(x+2)^2(x+3)^3}.$

253. $\int \frac{dx}{x^5+x^4-2x^3-2x^2+x+1}.$

254. $\int \frac{x^2+5x+4}{x^4+5x^2+4}dx.$

255. $\int \frac{dx}{(x+1)(x^2+1)}.$

256. $\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)}.$

257. $\int \frac{xdx}{(x-1)^2(x^2+2x+2)}.$

258. $\int \frac{dx}{x(1+x)(1+x+x^2)}$.

259. $\int \frac{dx}{x^3+1}$.

260. $\int \frac{xdx}{x^3-1}$.

261. $\int \frac{dx}{x^4-1}$.

262. $\int \frac{dx}{x^4+1}$.

263. $\int \frac{dx}{x^4+x^2+1}$.

264. $\int \frac{dx}{x^6+1}$.

265. $\int \frac{dx}{(1+x)(1+x^2)(1+x^3)}$.

266. $\int \frac{dx}{x^5-x^4+x^3-x^2+x-1}$.

267. $\int \frac{x^2dx}{x^4+3x^3+\frac{9}{2}x^2+3x+1}$.

268. Haýsy şertlerde integral $\int \frac{ax^2+bx+c}{x^3(x-1)^2}dx$ rasional funksiýany aňladýar?

2. ***Ostrogradskiniň usuly.*** Eger $\frac{P(x)}{Q(x)}$ dogry rasional drobuň maýdalawjysynyň kratny kökleri, aýratyn-da, kompleks kökleri bar bolsa, onda ol rasional droby integrirlemek üçin Ostrogradskiniň

$$\int \frac{P(x)}{Q(x)}dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)}dx$$

formulasyndan peýdalanmak amatly bolýar, bu ýerde $Q_2(x)$ köpagza bolup, onuň hem kökleri $Q(x)$ köpagzanyňky ýalydyr, ýöne onuň ähli kökleri ýönekeýdir (birgatdyr), $Q_1(x) = Q(x)/Q_2(x)$. $P_1(x)$ we $P_2(x)$ bolsa köpagzalar bolup, olaryň derejeleri degişlilikde $Q_1(x)$ we $Q_2(x)$ köpagzalaryň derejelerinden kiçidir.

Ostrogradskiniň usulyny ulanyp, integrallary tapmaly:

269. $\int \frac{xdx}{(x-1)^2(x+1)^3}$.

270. $\int \frac{dx}{(x^3+1)^2}$.

271. $\int \frac{dx}{(x^2+1)^3}$.

272. $\int \frac{x^2dx}{(x^2+2x+2)^2}$.

273. $\int \frac{dx}{(x^4+1)^2}$.

274. $\int \frac{x^2+3x-2}{(x-1)(x^2+x+1)^2}dx$.

275. $\int \frac{dx}{(x^4-1)^3}$.

Aşakdaky integrallaryň algebraik bölegini bölüp çykarmaly:

276. $\int \frac{x^2+1}{(x^4+x^2+1)^2}dx.$

277. $\int \frac{dx}{(x^3+x+1)^3}.$

278. $\int \frac{4x^5-1}{(x^5+x+1)^2}dx.$

279. $\int \frac{dx}{x^4+2x^3+3x^2+2x+1}$ integraly tapmaly.

280. Haýsy şertlerde $\int \frac{\alpha x^2+2\beta x+\gamma}{(ax^2+2bx+c)^2}dx$ integral rasional funksiýany aňladýar?

Dürli usullary ulanyp, aşakdaky integrallary tapmaly:

281. $\int \frac{x^3}{(x-1)^{100}}dx.$

282. $\int \frac{xdx}{x^8-1}.$

283. $\int \frac{x^3dx}{x^8+3}.$

284. $\int \frac{x^2+x}{x^6+1}dx.$

285. $\int \frac{x^4-3}{x(x^8+3x^4+2)}dx.$

286. $\int \frac{x^4dx}{(x^{10}-10)^2}.$

287. $\int \frac{x^{11}dx}{x^8+3x^4+2}.$

288. $\int \frac{x^9dx}{(x^{10}+2x^5+2)^2}.$

289. $\int \frac{x^{2n-1}}{x^n+1}dx.$

290. $\int \frac{x^{3n-1}}{(x^{2n}+1)^2}dx.$

291. $\int \frac{dx}{x(x^{10}+2)}.$

292. $\int \frac{dx}{x(x^{10}+1)^2}.$

293. $\int \frac{1-x^7}{x(1+x^7)}dx.$

294. $\int \frac{x^4-1}{x(x^4-5)(x^5-5x+1)}dx.$

295. $\int \frac{x^2+1}{x^4+x^2+1}dx.$

296. $\int \frac{x^2-1}{x^4+x^3+x^2+x+1}dx.$

297. $\int \frac{x^5-x}{x^8+1}dx.$

298. $\int \frac{x^4+1}{x^6+1}dx.$

299. $I_n = \int \frac{dx}{(ax^2 + bx + c)^n}$ $(a \neq 0)$ integraly hasaplamak üçin rekurrent formulany getirip çykarmaly. Bu formulany ulanyp, $I_3 = \int \frac{dx}{(x^2 + x + 1)^3}$ integraly hasaplamaly. (*Görkezme*: *Toždestwony ulanmaly:* $4a(ax^2+bx+c)=(2ax+b)^2+(4ac-b^2)$).

300. $I = \int \frac{dx}{(x + a)^m (x + b)^n}$ integraly hasaplamak üçin $t = \frac{x + a}{x + b}$ orun çalşyrmany ulanmaly (m we n – natural sanlar).

Bu orun çalşyrmany ulanyp, $\int \frac{dx}{(x - 2)^2 (x + 3)^3}$ integraly tapmaly.

301. $\int \frac{P_n(x)}{(x - a)^{n+1}} dx$ integraly hasaplamaly, bu ýerde $P_n(x)$ funksiýa x-a görä n-derejeli köpagza. (*Görkezme*: *Teýloryň formulasyny ulanmaly*).

302. Goý, $R(x) = \overline{R}(x^2)$ bolsun, bu ýerde $\overline{R}$ rasional funksiýa. $R(x)$ funksiýany rasional droblara dagytmagyň nähili aýratynlyklary bar?

303. $\int \frac{dx}{1 + x^{2n}}$ integraly hasaplamaly, bu ýerde n bitin položitel san.

§3. Irrasional funksiýalaryň integrirlenişi

1. ***Rasionallaşdyrmak usuly.*** Bu usul amatly orun çalşyrmalary girizip, irrasional funksiýalaryň intergallaryny rasional funksiýalaryň integrallaryna getirmekligi aňladýar. Bu ýerde garalýan $R(x, u_1, ..., u_k)$ görnüşdäki irrasional funksiýalar x, $u_1, ..., u_k$ argumentleriň her biri boýunça rasional funksiýalardyr.

Mysal üçin, irrasional

$$\frac{x^3 + \sqrt{x}}{1 + \sqrt{1 + x^2}} = R(x, u_1, ..., u_k)$$

funksiýa x, $u_1 = \sqrt{x}$, $u_2 = \sqrt{1 + x^2}$ argumentlere görä rasional funksiýadyr.

2. $\int R\left[x, \left(\frac{ax + b}{cx + d}\right)^{r_1}, ..., \left(\frac{ax + b}{cx + d}\right)^{r_k}\right] dx$ görnüşdäki integral. Bu ýerde $r_1, ..., r_k$ – rasional sanlar, a, b, c we d – hemişelik sanlar we $ad - bc \neq 0$. Ol integraly tapmak üçin $t^m = \frac{ax + b}{cx + d}$ çalşyrma ulanylýar, bu çalşyrmada m san $r_1, ..., r_k$ rasional sanlaryň iň kiçi umumy maýdalawjysydyr.

1-nji mysal. $\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx$ integraly hasaplamaly.

Ç.B. Bu integraldaky funksiýa x, $u_1=\sqrt[3]{x}$, $u_2=\sqrt[6]{x}$ argumentlere görä rasional funksiýadyr, $r_1 = 1/3$, $r_2 = 1/6$. Şonuň üçin olaryň umumy maýdalawjysy $m = 6$ bolýar. Diýmek, $x = t^6$, $dx = 6t^5dt$ çalşyrmany girizmek bolar. Şonuň esasynda

$$\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx = 6\int \frac{t^6+t^4+t}{t^6(1+t^2)}t^5\,dt = 6\int \frac{t^5+t^3+1}{1+t^2}dt =$$

$$= 6\int t^3\,dt + 6\int \frac{dt}{1+t^2} = \frac{3}{2}t^4 + 6\operatorname{arctg}t + C = \frac{3}{2}\sqrt[3]{x^2} + 6\operatorname{arctg}\sqrt[6]{x} + C. \quad \textbf{Ç.S.}$$

Bellik. Beýleki görnüşdäki käbir integrallar hem ýönekeý özgertmeleriň kömegi bilen 2-nji görnüşdäki integrala getirilýär.

2-nji mysal. $\int \frac{dx}{\sqrt[3]{(2+x)(2-x)^5}}$ integraly hasaplamaly.

Ç.B. Integraly

$$\int \frac{dx}{\sqrt[3]{(2+x)(2-x)^5}} = \int \sqrt[3]{\frac{2-x}{2+x}}\frac{dx}{(2-x)^2}$$

görnüşde ýazyp, $\frac{2-x}{2+x}=t^3$ çalşyrmany girizeliň. Şonda

$$x = 2\frac{1-t^3}{1+t^3}, \qquad dx = -12\frac{t^2\,dt}{(1+t^3)^2}, \qquad \frac{1}{2-x} = \frac{1+t^3}{4t^3}$$

bolar. Şeýlelikde,

$$\int \frac{dx}{\sqrt[3]{(2+x)(2-x)^5}} = \int \sqrt[3]{\frac{2-x}{2+x}}\frac{dx}{(2-x)^2} = -12\int \frac{(t^3+1)^2t^3\,dt}{16t^6(t^3+1)^2} =$$

$$= -\frac{3}{4}\int \frac{dt}{t^3} = \frac{3}{8}\frac{1}{t^2} + C = \frac{3}{8}\sqrt[3]{\left(\frac{2+x}{2-x}\right)^2} + C. \quad \textbf{Ç.S.}$$

3. $\int R(x, \sqrt{ax^2+bx+c})dx$ görnüşdäki integral. Ilki bilen bu integralyň hususy görnüşi bolan integrala aýratyn garalyň.

3.1. $\int \frac{R_1(x)}{\sqrt{ax^2+bx+c}}dx$ görnüşdäki integral, bu ýerde $R_1(x)$ rasional funksiýa. Ony $R_1(x) = P_n(x) + \frac{F(x)}{Q(x)}$ görnüşde aňladyp we $\frac{F(x)}{Q(x)}$ droby ýönekeý droblaryň jemi görnüşinde ýazyp, integraly aşakdaky integrallaryň birine getirmek bolar:

A. $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx,$ $\quad P_n(x)$ – köpagza;

B. $\int \frac{M}{(x-\alpha)^k\sqrt{ax^2+bx+c}}dx,$ $\quad M$ – hemişelik;

C. $\int \frac{Mx+N}{(x^2+px+q)^m\sqrt{ax^2+bx+c}}dx,$ $\quad M, N$ – hemişelik

we x^2+px+q üçagzanyň hakyky kökleri ýokdur.

Bu integrallaryň hasaplanyş usullaryny görkezeliň.

A. Bu görnüşdäki integraly hasaplamak üçin

$$\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}} = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \alpha\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

formula ulanylýar, bu ýerde $Q_{n-1}(x)$ derejesi $n-1$ bolan köpagzadyr, α bolsa hemişelik sandyr. $Q_{n-1}(x)$ köpagzanyň näbelli koeffisiýentleri we α san ol deňligi differensirläp tapylýar.

3-nji mysal. $\int \frac{x^3-2}{\sqrt{x^2+x+1}}dx$ integraly hasaplamaly.

Ç.B. Sanawjyda üçünji derejeli köpagza bolany üçin formula şeýle görnüşi alar:

$$\int \frac{(x^3-2)dx}{\sqrt{x^2+x+1}} = (b_2x^2+b_1x+b_0)\sqrt{x^2+x+1} + \alpha\int\frac{dx}{\sqrt{x^2+x+1}}.$$

Bu deňligi differensirläp we soňra alnan deňligi $\sqrt{x^2+x+1}$ aňlatma köpeldip alarys:

$$2(x^3-2) = (4b_2x+2b_1)(x^2+x+1) + (b_2x^2+b_1x+b_o)(2x+1) + 2\alpha.$$

Bu ýerden deňlemäniň çep we sag bölegindäki üýtgeýän x ululygyň deň derejeleriniň koeffisiýentlerini deňläp, näbellileri tapmak üçin aşakdaky deňlemeler sistemasyny alarys:

$$\left.\begin{aligned} 4b_2 + 2b_2 &= 2,\\ 4b_2 + 2b_1 + b_2 + 2b_1 &= 0,\\ 4b_2 + 2b_1 + b_1 + 2b_o &= 0,\\ 2b_1 + b_o + 2\alpha &= -4. \end{aligned}\right\} \Rightarrow \begin{cases} b_2 = \frac{1}{3}, & b_1 = -\frac{5}{12},\\ b_o = -\frac{1}{24}, & \alpha = -\frac{25}{16}. \end{cases}$$

Şeýlelikde,

$$\int \frac{x^3-2}{\sqrt{x^2+x+1}}dx =$$

$$= \left(\frac{1}{3}x^2 - \frac{5}{12}x - \frac{1}{24}\right)\sqrt{x^2+x+1} - \frac{25}{16}\int \frac{d\left(x+\frac{1}{2}\right)}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}} =$$

$$= \left(\frac{1}{3}x^2 - \frac{5}{12}x - \frac{1}{24}\right)\sqrt{x^2+x+1} - \frac{25}{16}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right| + C. \quad \textbf{Ç.S.}$$

B. Bu görnüşdäki integral $x - \alpha = \frac{1}{t}$ çalşyrmany ulanylyp, seredilen görnüşdäki integrala getirilýär.

4-nji mysal. $\int \frac{dx}{x^3\sqrt{x^2+1}}$ integraly hasaplamaly.

Ç.B. $x = \frac{1}{t}$, $dx = -\frac{dt}{t^2}$ çalşyrmany ulanyp alarys:

$$\int \frac{dx}{x^3\sqrt{x^2+1}} = -\int \frac{t^3\,dt}{t^2\sqrt{\frac{1}{t^2}+1}} = -\int \frac{t^2\,dt}{\sqrt{1+t^2}} =$$

$$= -\int \frac{(1+t^2-1)dt}{\sqrt{1+t^2}} = -\int \sqrt{t^2+1}\,dt + \int \frac{dt}{\sqrt{1+t^2}} =$$

$$= -\frac{t\sqrt{1+t^2}}{2} - \frac{1}{2}\ln\left|t+\sqrt{1+t^2}\right| + \ln\left|t+\sqrt{1+t^2}\right| + C =$$

$$= -\frac{t\sqrt{1+t^2}}{2} + \frac{1}{2}\ln\left|t+\sqrt{1+t^2}\right| + C, \quad t = \frac{1}{x}. \quad \textbf{Ç.S.}$$

Ç. Bu görnüşdäki integral üçin ilki

$$ax^2 + bx + c = a(x^2 + px + q)$$

bolýan hala aýratyn garalyň. Bu halda integraly

$$\int \frac{Mx+N}{(x^2+px+q)^m\sqrt{ax^2+bx+c}}dx = \int \frac{M_1x+N_1}{(x^2+px+q)^{m+1/2}}dx$$

görnüşde ýazyp, $M_1x + N_1 = \frac{M_1}{2}(2x+p) + N_1 - \frac{M_1p}{2}$ deňligi ulansak, onda integraly şeýle görnüşde ýazmak bolar:

$$\int \frac{M_1x+N_1}{(x^2+px+q)^{m+1/2}}dx = K_1\int \frac{d(x^2+px+q)}{(x^2+px+q)^{m+1/2}} + K_2\int \frac{dx}{(x^2+px+q)^{m+1/2}}.$$

Olaryň birinjisi tablisanyň integraly bolup, ikinjisini integrirlemek üçin Abeliň $t=\left(\sqrt{x^2+px+q}\right)'$ çalşyrmasy ulanylýar.

Umumy ýagdaýda, ýagny ax^2+bx+c we x^2+px+q üçagzalaryň gatnaşygy hemişelik bolmadyk halda üçagzalarda birinji derejeli agzalar ýok bolar ýaly orun çalşyrma ulanylýar. Mysal üçin, $p \neq b/a$ bolanda $x=\dfrac{\alpha t+\beta}{t+1}$ we $p=b/a$ bolanda $x=t-\dfrac{p}{2}$ çalşyrmany ulanmak bolar. Netijede, $\displaystyle\int \frac{Kt+L}{(t^2+\lambda)^m\sqrt{\delta t^2+r}}dt$ görnüşdäki integral alnar. Hasaplamak üçin ol integral

$$\int \frac{Kt+L}{(t^2+\lambda)^m\sqrt{\delta t^2+r}}dt = L\int \frac{dt}{(t^2+\lambda)^m\sqrt{\delta t^2+r}}$$

görnüşde ýazylyp, birinjisine $u=\sqrt{\delta t^2+r}$ çalşyrma, ikinjisine bolsa $v=\left(\sqrt{\delta t^2+r}\right)'$ çalşyrma ulanylýar.

5-nji mysal. $\displaystyle\int \frac{dx}{\sqrt{(x^2+x+2)^5}}$ integraly hasaplamaly.

Ç.B. Abeliň

$$t=\left(\sqrt{x^2+x+2}\right)'=\frac{2x+1}{2\sqrt{x^2+x+2}} \tag{1}$$

çalşyrmasyny ulanalyň. Bu deňligiň iki bölegini hem kwadrata göterip we soňra $4(x^2+x+2)$ aňlatma köpeldip,

$$4t^2(x^2+x+2)=4x^2+4x+1=4(x^2+x+2)-7$$

deňligi alarys we ondan x^2+x+2 aňlatmany taparys:

$$x^2+x+2=\frac{-7}{4t^2-1}. \tag{2}$$

(1) deňlikden alynýan $t\sqrt{x^2+x+2}=x+\dfrac{1}{2}$ deňligi differensirläp,

$$dt\sqrt{x^2+x+2}+\frac{(2x+1)tdx}{2\sqrt{x^2+x+2}}=dx$$

deňligi we (1) deňlik esasynda ondan $dt\sqrt{x^2+x+2}+t^2dx=dx$ deňligi alarys. Bu deňlikden bolsa

$$\frac{dx}{\sqrt{x^2+x+2}}=\frac{dt}{1-t^2} \tag{3}$$

deňlik gelip çykýar. (2) we (3) deňlikleri ulanyp, integraly taparys:

$$\int \frac{dx}{\sqrt{(x^2+x+2)^{5/2}}} = \int \frac{dx}{\sqrt{x^2+x+2}} \cdot \frac{1}{(x^2+x+2)} =$$

$$= \int \frac{dt}{1-t^2} \cdot \frac{(4t^2-4)^2}{49} = \frac{16}{49}\int (1-t^2)dt = \frac{16}{49}\left(t - \frac{t^3}{3}\right) + C =$$

$$= \frac{16}{49}\left[\frac{2x+1}{2\sqrt{x^2+x+2}} - \frac{1}{24}\left(\frac{2x+1}{\sqrt{x^2+x+2}}\right)^3\right] + C \qquad \textbf{Ç.S.}$$

6-njy mysal. $\int \frac{(x+2)dx}{(x^2+1)\sqrt{x^2+2}}$ integraly hasaplamaly.

Ç.B. Integraly

$$\int \frac{(x+2)dx}{(x^2+1)\sqrt{x^2+2}} = \int \frac{xdx}{(x^2+1)\sqrt{x^2+2}} + \int \frac{2dx}{(x^2+1)\sqrt{x^2+2}}$$

görnüşde ýazyp, olaryň integrirlenişini subut etmeli.

$$\int \frac{xdx}{(x^2+1)\sqrt{x^2+2}} = \frac{1}{2}\int \frac{du}{(u+1)\sqrt{u+2}} = \frac{1}{2}\int \frac{2zdz}{(z^2-1)z} =$$

$$= \int \frac{dz}{(z^2-1)} = \frac{1}{2}\ln\left|\frac{z-1}{z+1}\right| + C = \frac{1}{2}\ln\left|\frac{\sqrt{x^2+2}-1}{\sqrt{x^2+2}+1}\right| + C.$$

Integrallaryň ikinjisi üçin Abeliň orun çalşyrmasyny ulanarys:

$$t = (\sqrt{x^2+2})' = \frac{x}{\sqrt{x^2+2}}, \tag{4}$$

onda $t^2 = \frac{x^2}{x^2+2}$ we $x^2 = \frac{2t^2}{1-t^2}$ bolar. Şonuň üçin

$$x^2 + 1 = \frac{2t^2}{1-t^2} + 1 = \frac{t^2+1}{1-t^2}. \tag{5}$$

$t\sqrt{x^2+2} = x$ deňligi differensirläp we (4) deňligi ulanyp alarys:

$$dt\sqrt{x^2+2} + \frac{xtdx}{\sqrt{x^2+2}} = dx, \quad dt\sqrt{x^2+2} + t^2dx = dx.$$

Bu ýerden

$$\frac{dx}{\sqrt{x^2+2}} = \frac{dt}{1-t^2} \tag{6}$$

deňlik gelip çykýar. (5) we (6) deňlikleri ulanyp, integraly taparys:

$$= \int \frac{2dx}{(x^2+1)\sqrt{x^2+2}} = 2\int \frac{dx}{\sqrt{x^2+2}} \cdot \frac{1}{x^2+1} = 2\int \frac{dt}{1-t^2} \cdot \frac{1}{\frac{t^2+1}{1-t^2}}$$

$$= 2\int \frac{dt}{1+t^2} = 2\mathrm{arctg}t + C = 2\mathrm{arctg}\frac{x}{\sqrt{x^2+2}} + C.$$

Şeýlelikde,

$$\int \frac{(x+2)dx}{(x^2+1)\sqrt{x^2+2}} = \frac{1}{2}\ln\left|\frac{\sqrt{x^2+2}-1}{\sqrt{x^2+2}+1}\right| + 2\mathrm{arctg}\frac{x}{\sqrt{x^2+2}} + C. \text{ **Ç.S.**}$$

7-nji mysal. $\int \frac{(11x-13)dx}{(x^2-x+1)\sqrt{x^2+1}}$ integraly hasaplamaly.

Ç.B. Bu mysalda x^2-x+1 we x^2+1 üçagzalaryň gatnaşyklary hemişelik bolmandygy üçin ilki $x = \frac{\alpha t+\beta}{t+1}$ çalşyrmany girizeliň. Şonda

$$x^2-x+1 = \frac{\alpha^2t^2+2\alpha\beta t+\beta^2-(\alpha t+\beta)(t+1)+t^2+2t+1}{(t+1)^2},$$

$$x^2+1 = \frac{\alpha^2t^2+2\alpha\beta t+\beta^2+t^2+2t+1}{(t+1)^2}$$

deňlikleri alarys. Olardaky t-niň koeffisiýentlerini nola deňläp,

$$\begin{cases} 2\alpha\beta-\alpha-\beta+2=0, \\ 2\alpha\beta+2=0 \end{cases}$$

deňlemeler sistemasyny alarys we ony çözüp, näbelli koeffisiýentleri taparys: $-\alpha=\beta-1$. Şonuň esasynda integralda $x=\frac{t-1}{t+1}$ çalşyrmany ulanmaly. Bu çalşyrmada $dx = \frac{2dt}{(t+1)^2}$ we

$$x^2-x+1=\frac{t^2+3}{(t+1)^2}, \quad x^2+1=\frac{2t^2+2}{(t+1)^2}, \quad 11x-3=\frac{-2t-24}{t+1}$$

bolar. Şonuň üçin integral şeýle görnüşi alar:

$$\int \frac{(11x-13)dx}{(x^2-x+3)\sqrt{x^2+1}} = -2\sqrt{2}\int \frac{(t+12)dt}{(t^2+3)\sqrt{t^2+1}}.$$

Bu integraly integrallaryň jemi görnüşinde ýazyp taparys:

$$\int \frac{tdt}{(t^2+3)\sqrt{t^2+1}} = \int \frac{d\sqrt{t^2+1}}{t^2+3} = \int \frac{du}{u^2+2} =$$

$$= \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{u}{\sqrt{2}} + C = \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\sqrt{t^2+1}}{\sqrt{2}} + C.$$

Ikinji $\int \frac{dt}{(t^2+3)\sqrt{t^2+1}}$ integraly hasaplamak üçin Abeliň $z = \left(\sqrt{t^2+1}\right)'$ çalşyrmasyny ulanarys. Şonda *(11-nji we 12-nji mysallardaka meňzeşlikde)*

$$\frac{dz}{1-z^2} = \frac{dt}{\sqrt{t^2+1}}, \qquad t^2+3 = \frac{3-2z^2}{1-z^2}$$

bolýandygyndan peýdalanyp, integraly taparys:

$$\int \frac{dt}{(t^2+3)\sqrt{t^2+1}} = \int \frac{dt}{\sqrt{t^2+1}} \cdot \frac{1}{(t^2+3)} = \int \frac{dz}{3-2z^2} =$$

$$= \frac{1}{2\sqrt{6}} \ln\left|\frac{\sqrt{3}+z\sqrt{2}}{\sqrt{3}-z\sqrt{2}}\right| + C = \frac{1}{2\sqrt{6}} \ln\left|\frac{\sqrt{3t^2+3}+\sqrt{2}\,t}{\sqrt{3t^2+3}-\sqrt{2}\,t}\right| + C.$$

Şeýlelikde,

$$\int \frac{(11x-13)dx}{(x^2-x+1)\sqrt{x^2+1}} = -\,2\operatorname{arctg}\frac{\sqrt{t^2+1}}{\sqrt{2}} - 4\sqrt{3}\ln\left|\frac{\sqrt{3t^2+3}+\sqrt{2}\,t}{\sqrt{3t^2+3}-\sqrt{2}\,t}\right| + C,$$

$$t = \frac{x+1}{1-x}. \quad \textbf{Ç.S.}$$

Üýtgeýän ululyklaryna görä rasional bolan $R(x, u)$ funksiýa üçin $\int R\left(x, \sqrt{ax^2+bx+c}\right)dx$ görnüşdäki integraly umumy halda rasionallaşdyrmak üçin aşakdaky üç görnüşdäki Eýleriň orun çalşyrma usullarynyň biri ulanylýar:

1. Eger $a > 0$ bolsa, onda $\sqrt{ax^2+bx+c} = \pm x\sqrt{a} + t$.

2. Eger $c > 0$ bolsa, onda $\sqrt{ax^2+bx+c} = \pm\sqrt{c} + tx$.

3. Eger $ax^2 + bx + c$ kwadrat üçagzanyň hakyky sanlar bolan dürli x_1 we x_2 kökleri bar bolsa, onda $\sqrt{ax^2+bx+c} = t(x - x_1)$.

8-nji mysal. $\int \frac{dx}{x+\sqrt{x^2+2x+2}}$ integraly hasaplamaly.

Ç.B. Integraly başga görnüşe getireliň:

$$\int \frac{dx}{x+\sqrt{x^2+2x+2}} = \int \frac{d(x+1)}{(x+1)-1+\sqrt{(x+1)^2+1}}.$$

Bu integraly hasaplamak üçin ilki $x + 1 = t$ we soňra $a = 1 > 0$ bolany üçin, $\sqrt{t^2+1} = u - t$ çalşyrma girizeliň. Ony kwadrata göterip alarys:

$$1 = u^2 - 2tu, \quad t = \frac{u^2-1}{2u}, \quad t + \sqrt{t^2+1} = u, \quad dt = \frac{1}{2}\frac{u^2+1}{u^2}du$$

$$\int \frac{dt}{t-1+\sqrt{t^2+1}} = \frac{1}{2}\int \frac{(u^2+1)du}{u^2(u-1)} = \int \frac{du}{u-1} - \frac{1}{2}\int \frac{u+1}{u^2}du =$$

$$= \ln|u-1| - \frac{1}{2}\ln|u| + \frac{1}{2u} + C =$$

$$= \ln|\sqrt{t^2+1}+t-1| - \frac{1}{2}\ln|\sqrt{t^2+1}+t| + \frac{1}{2(\sqrt{t^2+1}+t)} + C =$$

$$= \ln|\sqrt{x^2+2x+2}+x| - \frac{1}{2}\ln|\sqrt{x^2+2x+2}+x+1| +$$

$$+ \frac{1}{2(\sqrt{x^2+2x+2}+x+1)} + C. \quad \textbf{Ç.S.}$$

1-*nji bellik.* $\int R(x, \sqrt{ax+b}, \sqrt{cx+d})dx$ görnüşdäki integral $t^2 = ax + b$ çalşyrmanyň kömegi bilen 3-nji görnüşdäki integrala getirilýär.

2-*nji bellik.* $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$ bolýandygy üçin kök astyndaky aňlatma položitel bolanda integraly aşakdaky üç integrallaryň birine getirmek bolar:

$$\int R(t, \sqrt{1-t^2})dt, \quad \int R(t, \sqrt{t^2-1})dt, \quad \int R(t, \sqrt{t^2+1})dt.$$

Olar bolsa $t = \sin u$, $t = \cos u$, $t = \mathrm{tg}\, u$, $t = \mathrm{sh}\, u$, $t = \mathrm{ch}\, u$, $t = \mathrm{th}\, u$ çalşyrmalaryň kömegi bilen aňsat hasaplanylýar.

2. *Binomial differensialyň integrirlenişi*

$$x^m(a + bx^n)^p dx \quad (a \neq 0, \ b \neq 0)$$

görnüşdäki aňlatma binomial differensial diýilýär. Bu ýerde $a, b \in R$ we m, n, p – rasional sanlar. Binomial differensialyň integraly p, $\frac{m+1}{n}$ we $\frac{m+1}{n}+p$ sanlaryň haýsy-da bolsa biri bitin san bolanda integrirlenýär we şol hallarda şeýle orun çalşyrmalar ulanylýar:

a) p – bitin san, $\dfrac{m+1}{n}$ – rasional san, $x^{\frac{n}{s}} = u$, bu ýerde s san $\dfrac{m+1}{n}$ drobuň maýdalawjysy, ýagny $\dfrac{m+1}{n} = \dfrac{r}{s}$;

b) p – rasional san, $\dfrac{m+1}{n}$ – bitin san, $(a+bx^n)^{\frac{1}{s}} = u$, bu ýerde hem s san p sanyň maýdalawjysy, ýagny $p = \dfrac{r}{s}$;

ç) $\dfrac{m+1}{n} + p$ – bitin san, p – rasional san, $(ax^{-n}+b)^{\frac{1}{s}} = u$, bu ýerde hem s san p sanyň maýdalawjysydyr, ýagny $p = \dfrac{r}{s}$.

9-njy mysal. $\displaystyle\int \frac{dx}{x^2\sqrt{a+bx^2}}$ integraly hasaplamaly.

Ç.B. Integraly $\displaystyle\int x^{-2}(a+bx^2)^{-1/2}dx$ görnüşde ýazalyň. Diýmek, $m=-2$, $n=2$, $p=-1/2$, $\dfrac{m+1}{n} = -\dfrac{1}{2}$, $\dfrac{m+1}{n} + p = -\dfrac{1}{2} - \dfrac{1}{2} = -1$ bitin san, ýagny ç) halyň şertleri ýerine ýetýär. Şonuň üçin hem $(ax^{-2}+b)^{1/2}=u$ çalşyrma girizilýär. Bu ýerden

$$ax^{-2}+b=u^2, \qquad x = \frac{\sqrt{a}}{\sqrt{u^2-b}}, \qquad dx = -\frac{\sqrt{a}\,u}{\sqrt{(u^2-b)^3}}du.$$

Şeýlelikde,

$$\int \frac{dx}{x^2\sqrt{a+bx^2}} = \int \frac{dx}{x^3\sqrt{ax^{-2}+b}} = -\int \frac{\sqrt{(u^2-b)^3}}{u\sqrt{a^3}}\frac{u\sqrt{a}}{\sqrt{(u^2-b)^3}}du =$$

$$= \frac{1}{a}\int du = -\frac{1}{a}u + c = -\frac{1}{a}\sqrt{ax^{-2}+b} + c. \qquad \textbf{Ç.S.}$$

Gönükmeler

Integral astyndaky funksiýalary rasional funksiýalara getirip, integrallary tapmaly:

304. $\displaystyle\int \frac{dx}{1+\sqrt{x}}.$

305. $\displaystyle\int \frac{dx}{x(1+2\sqrt{x})+\sqrt[3]{x}}.$

306. $\displaystyle\int \frac{x\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}}dx.$

307. $\displaystyle\int \frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}dx.$

308. $\displaystyle\int \frac{dx}{(1+\sqrt[4]{x})^3\sqrt{x}}.$

309. $\displaystyle\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}dx.$

310. $\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}$.

311. $\int \frac{xdx}{\sqrt[4]{x^3(a-x)}}$ $(a > 0)$.

312. $\int \frac{dx}{\sqrt[n]{(x-a)^{n+1}(x-b)^{n-1}}}$ (n – natural san).

313. $\int \frac{dx}{1+\sqrt{x}+\sqrt{1+x}}$. (*Görkezme*: $x = \left(\frac{u^2-1}{2u}\right)^2$ *almaly*).

314. Eger R rasional funksiýa we p, q, n – bitin sanlar bolup, bitin k san üçin $p+q=kn$ bolsa, onda

$$\int R[x, (x-a)^{p/n}(x-b)^{q/n}]dx$$

integralyň elementar funksiýa bolýandygyny subut etmeli.

Ýönekeý kwadrat irrasionallyklaryň integrallaryny tapmaly:

315. $\int \frac{x^2}{\sqrt{1+x+x^2}}dx$.

316. $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$.

317. $\int \frac{dx}{(1-x)^2\sqrt{1-x^2}}$.

318. $\int \frac{\sqrt{x^2+2x+2}}{x}dx$.

319. $\int \frac{xdx}{(x+1)\sqrt{1-x-x^2}}$.

320. $\int \frac{1-x+x^2}{\sqrt{1+x-x^2}}dx$.

$y = \sqrt{ax^2+bx+c}$, n derejeli $P_n(x)$, $n - 1$ derejeli $Q_{n-1}(x)$ köpagza we hemişelik α san üçin

$$\int \frac{P_n(x)}{y}dx = Q_{n-1}(x)y + \alpha\int\frac{dx}{y}$$

formuladan peýdalanyp, aşakdaky integrallary tapmaly:

321. $\int \frac{x^3}{\sqrt{1+2x-x^2}}dx$.

322. $\int \frac{x^{10}dx}{\sqrt{1+x^2}}$.

323. $\int x^4\sqrt{a^2-x^2}\,dx$.

324. $\int \frac{x^3-6x^2+11x-6}{\sqrt{x^2+4x+3}}dx$.

325. $\int \frac{dx}{x^3\sqrt{x^2+1}}$.

326. $\int \frac{dx}{x^4\sqrt{x^2-1}}$.

327. $\int \frac{dx}{(x-1)^3\sqrt{x^2+3x+1}}$.

328. $\int \frac{dx}{(x+1)^5\sqrt{x^2+2x}}$.

329. Haýsy şertlerde

$$\int \frac{a_1x^2 + b_1x + c_1}{\sqrt{ax^2 + bx + c}}dx$$

integral algebraik funksiýany aňladýar?

$y = \sqrt{ax^2 + bx + c}$ üçin $\frac{P(x)}{Q(x)y}$ rasional funksiýany ýönekeý droblara dagydyp, $\int \frac{P(x)}{Q(x)}dx$ integraly tapmaly:

330. $\int \frac{xdx}{(x-1)^2\sqrt{1+2x-x^2}}$.

331. $\int \frac{xdx}{(x^2-1)\sqrt{x^2-x-1}}$.

332. $\int \frac{\sqrt{x^2+x+1}}{(x+1)^2}dx$.

333. $\int \frac{x^3}{(x+1)\sqrt{1+2x-x^2}}dx$.

334. $\int \frac{xdx}{(x^2-3x+2)\sqrt{x^2-4x+3}}$.

335. $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$.

336. $\int \frac{dx}{(1+x^2)\sqrt{x^2-1}}$.

337. $\int \frac{dx}{(1-x^4)\sqrt{1+x^2}}$.

338. $\int \frac{\sqrt{x^2+2}}{x^2+1}dx$.

Kwadrat üçagzany kanonik görnüşe getirip, integrallary tapmaly:

339. $\int \frac{dx}{(x^2+x+1)\sqrt{x^2+x-1}}$.

340. $\int \frac{x^2dx}{(4-2x+x^2)\sqrt{2+2x-x^2}}$.

341. $\int \frac{(x+1)dx}{(x^2+x+1)\sqrt{x^2+x+1}}$.

342. $x = \frac{\alpha + \beta t}{1+t}$ çyzykly drob çalşyrmany ulanyp,

$$\int \frac{dx}{(x^2-x+1)\sqrt{x^2+x+1}}$$

intergaly hasaplamaly.

343. $\int \frac{dx}{(x^2+2)\sqrt{2x^2-2x+5}}$ integraly tapmaly.

Eýleriň:

1) $\sqrt{ax^2+bx+c} = \pm x\sqrt{a} + t$ eger $a > 0$;

2) $\sqrt{ax^2+bx+c} = \pm\sqrt{c} + xt$, eger $c > 0$;

3) $\sqrt{a(x-x_1)(x-x_2)} = t(x-x_1)$

çalşyrmalaryny ulanyp, aşakdaky integrallary tapmaly:

344. $\int \frac{dx}{x+\sqrt{x^2+x+1}}$.

345. $\int \frac{dx}{1+\sqrt{1-2x-x^2}}$.

346. $\int x\sqrt{x^2-2x+2}\,dx$.

347. $\int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}dx$.

348. $\int \frac{dx}{[1+\sqrt{x(1+x)}]^2}$.

Dürli usullary ulanyp, aşakdaky integrallary tapmaly:

349. $\int \frac{dx}{\sqrt{x^2+1}-\sqrt{x^2-1}}$.

350. $\int \frac{xdx}{(1-x^3)\sqrt{1-x^2}}$.

351. $\int \frac{dx}{\sqrt{2}+\sqrt{1-x}+\sqrt{1+x}}$.

352. $\int \frac{x+\sqrt{1+x+x^2}}{1+x+\sqrt{1+x+x^2}}dx$.

353. $\int \frac{\sqrt{x(x+1)}}{\sqrt{x}+\sqrt{x+1}}dx$.

354. $\int \frac{(x^2-1)dx}{(x^2+1)\sqrt{x^4+1}}$.

355. $\int \frac{(x^2+1)dx}{(x^2-1)\sqrt{x^4+1}}$.

356. $\int \frac{dx}{x\sqrt{x^4+2x^2-1}}$.

357. $\int \frac{(x^2+1)dx}{x\sqrt{x^4+x^2+1}}$.

358. Rasional R funksiýa üçin $\int R(x, \sqrt{ax+b}, \sqrt{cx+d})dx$ integralyň rasional funksiýanyň integralyna getirilýändigini subut etmeli.

Aşakdaky integrallary tapmaly:

359. $\int \sqrt{x^3+x^4}\,dx$.

360. $\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}dx$.

361. $\int \frac{xdx}{\sqrt{1+\sqrt[3]{x^2}}}$.

362. $\int \frac{x^5dx}{\sqrt{1-x^2}}$.

363. $\int \frac{dx}{\sqrt[3]{1+x^3}}$.

364. $\int \frac{dx}{\sqrt[4]{1+x^4}}$.

365. $\int \frac{dx}{x\sqrt[6]{1+x^6}}$.

366. $\int \frac{dx}{x^3\sqrt[5]{1+\frac{1}{x}}}$.

367. $\int \sqrt[3]{3x - x^3}\, dx.$

368. Haýsy hallarda rasional m san üçin

$$\int \sqrt{1 + x^m}\, dx$$

integral elementar funksiýa bolýar?

§4. Trigonometrik funksiýalaryň integrirlenişi

1. $\int R(\sin x, \cos x)dx$ görnüşdäki integralyň astyndaky $R(u,\ \upsilon)$ funksiýanyň üýtgeýän u we υ ululyklara görä rasional bolan halynda $t = \mathrm{tg}(x/2)$ $(-\pi < x < \pi)$ çalşyrma ulanylyp, integraly rasionallaşdyryp bolýar. Bu çalşyrmada $dx = 2\dfrac{dt}{1+t^2}$ we

$$\sin x = \frac{2\mathrm{tg}\dfrac{x}{2}}{1+\mathrm{tg}^2\dfrac{x}{2}} = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-\mathrm{tg}^2\dfrac{x}{2}}{1+\mathrm{tg}^2\dfrac{x}{2}} = \frac{1-t^2}{1+t^2}. \tag{1}$$

10-njy mysal. $\int \dfrac{dx}{1+\sin x}$ integraly hasaplamaly.

Ç.B. (1) formulanyň esasynda alarys:

$$\int \frac{dx}{1+\sin x} = \int \frac{2dt}{\left(1+\dfrac{2t}{1+t^2}\right)(1+t^2)} = 2\int \frac{dt}{(1+t)^2} =$$

$$= -\frac{2}{1+t} + C = -\frac{2}{1+\mathrm{tg}\dfrac{x}{2}} + C. \qquad \textbf{Ç.S.}$$

Integral astyndaky funksiýanyň hususy haly üçin:

1. $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ bolanda $t = \sin x$ çalşyrmany,
2. $R(-\sin x, \cos x) = -R(\sin x, \cos x)$ bolanda $t = \cos x$ çalşyrmany,
3. $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ bolanda $t = \mathrm{tg} x$ çalşyrmany ulanmak amatlydyr.

11-nji mysal. $\int \dfrac{\sin^5 x}{\cos^4 x} dx$ integraly hasaplamaly.

Ç.B. Bu ýerde integral astyndaky $R(\sin x, \cos x) = \dfrac{\sin^5 x}{\cos^4 x}$ funksiýa 2-nji şerti kanagatlandyrýar. Şonuň üçin ony

$$R(\sin x, \cos x) = \frac{\sin^5 x}{\cos^4 x} = \frac{(\sin^2 x)^2}{\cos^4 x}\sin x$$

görnüşde ýazyp, soňra çalşyrmany ulanarys:

$$\int \frac{\sin^5 x}{\cos^4 x} dx = \int \frac{(\sin^2 x)^2}{\cos^4 x} \sin x dx = -\int \frac{(1-\cos^2 x)^2}{\cos^4 x} d\cos x =$$

$$= -\int \frac{(1-t^2)^2}{t^4} dt = -\int t^{-4} dt + 2\int t^{-2} dt - \int dt =$$

$$= \frac{1}{3t^3} - \frac{2}{t} - t + c = \frac{1}{3\cos^3 x} - \frac{2}{\cos x} - \cos x + C \cdot \quad \textbf{Ç.S.}$$

Indi $R(\sin x, \cos x) = \sin^m x \cos^n x$ hala aýratynlykda garap geçeliň. Goý, m we n bitin sanlar bolsun.

a) eger n täk san bolsa, onda 1-nji şert ýerine ýetýär, şonuň üçin hem $t = \sin x$ çalşyrma ulanylýar.

b) eger m täk san bolsa, onda 2-nji şert ýerine ýetýär, şonuň üçin hem $t = \cos x$ çalşyrma ulanylýar.

ç) eger m we n sanlaryň ikisi hem bir wagtda täk ýa-da jübüt san bolsalar, onda 3-nji şert ýerine ýetýär, şonuň üçin hem $t = \operatorname{tg} x$ çalşyrmany ulanmak bolar.

Käbir hallarda

$$\sin 2x = 2\sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x,$$

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

we beýleki trigonometrik formulalardan peýdalanmak trigonometrik aňlatmalaryň integrallaryny hasaplamaklygy aňsatlaşdyrýar. Mysal üçin, m we n görkezijileriň ikisi hem täk we položitel san bolanda ony

$$\int \sin^{2k+1} x \cos^{2l+1} x dx = \frac{1}{2}\int \sin^{2k} x \cos^{2l} x 2\sin x \cos x dx =$$

$$= -\frac{1}{4}\int \left(\frac{1-\cos 2x}{5}\right)^k \left(\frac{1+\cos 2x}{2}\right)^l d(\cos 2x)$$

görnüşde ýazyp, $t = \cos 2x$ çalşyrmany ulanmak amatly bolýar.

12-nji mysal. $\int \sin^2 x \cos^4 x dx$ integraly hasaplamaly.

Ç.B. Integral astyndaky funksiýany

$$\sin^2 x \cos^4 x = \sin^2 x \cos^2 x \cos^2 x = \frac{1}{8}\sin^2 2x(\cos 2x + 1),$$

görnüşde ýazalyň. Onda integral aňsat hasaplanylar:

$$\int \sin^2 x \cos^4 x dx = \frac{1}{16}\int \sin^2 2x d(\sin 2x) + \frac{1}{16}\int (1-\cos 4x) dx =$$

$$= \frac{1}{48}\sin^3 2x + \frac{1}{16}x - \frac{1}{64}\sin 4x + C \cdot \quad \textbf{Ç.S.}$$

Gönükmeler

Integrallary tapmaly:

369. $\int \cos^5 x dx.$

370. $\int \sin^6 x dx.$

371. $\int \cos^6 x dx.$

372. $\int \sin^2 x \cos^4 x dx.$

373. $\int \sin^4 x \cos^5 x dx.$

374. $\int \sin^5 x \cos^5 x dx.$

375. $\int \frac{\sin^3 x}{\cos^4 x} dx.$

376. $\int \frac{\cos^4 x}{\sin^3 x} dx.$

377. $\int \frac{dx}{\sin^3 x}.$

378. $\int \frac{dx}{\cos^3 x}.$

379. $\int \frac{dx}{\sin^4 x \cos^4 x}.$

380. $\int \frac{dx}{\sin^3 x \cos^5 x}.$

381. $\int \frac{dx}{\sin x \cos^4 x}.$

382. $\int \mathrm{tg}^5 x dx.$

383. $\int \mathrm{ctg}^6 x dx.$

384. $\int \frac{\sin^4 x}{\cos^6 x} dx.$

385. $\int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}}.$

386. $\int \frac{dx}{\cos x \sqrt[3]{\sin^2 x}}.$

387. $\int \frac{dx}{\sqrt{\mathrm{tg} x}}.$

388. $\int \frac{dx}{\sqrt[3]{\mathrm{tg} x}}.$

389. Aşakdaky integrallar üçin tertibini kemeltme formulalaryny getirip çykarmaly:

$$\text{a) } I_n = \int \sin^n x dx; \quad \text{b) } K_n = \int \cos^n x dx \quad (n > 2)$$

we ol formulalary ulanyp,

$$\int \sin^6 x dx \quad \text{we} \quad \int \cos^8 x dx$$

integrallary hasaplamaly.

390. Aşakdaky integrallar üçin tertibini kemeltme formulalaryny getirip çykarmaly:

$$\text{a) } I_n = \int \frac{dx}{\sin^n x}; \quad \text{b) } K_n = \int \frac{dx}{\cos^n x} \quad (n > 2)$$

we ol formulalary ulanyp,

$$\int \frac{dx}{\sin^5 x} \quad \text{we} \quad \int \frac{dx}{\cos^7 x}$$

integrallary hasaplamaly.

Aşakdaky integrallar

I. $\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$,

II. $\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$,

III. $\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$

formulalar ulanylyp hasaplanylýar.

Integrallary tapmaly:

391. $\int \sin 5x \cos x dx.$

392. $\int \cos x \cos 2x \cos 3x dx.$

393. $\int \sin x \sin\frac{x}{2} \sin\frac{x}{3} dx.$

394. $\int \sin x \sin(x+a) \sin(x+b) dx.$

395. $\int \cos^2 ax \cos^2 bx dx.$

396. $\int \sin^3 2x \cdot \cos^2 3x dx.$

Aşakdaky integrallar

$$\sin(\alpha-\beta) \equiv \sin[(x+\alpha)-(x+\beta)],$$

$$\cos(\alpha-\beta) \equiv \cos[(x+\alpha)-(x+\beta)]$$

toždestwolary ulanmak bilen hasaplanylýar.

Integrallary tapmaly:

397. $\int \frac{dx}{\sin(x+a)\sin(x+b)}.$

398. $\int \frac{dx}{\sin(x+a)\cos(x+b)}.$

399. $\int \frac{dx}{\cos(x+a)\cos(x+b)}.$

400. $\int \frac{dx}{\sin x - \sin\alpha}.$

401. $\int \frac{dx}{\cos x + \cos\alpha}.$

402. $\int \operatorname{tg}x \operatorname{tg}(x+a) dx.$

Integrallary tapmaly:

403. $\int \frac{dx}{2\sin x - \cos x + 5}.$

404. $\int \frac{dx}{(2+\cos x)\sin x}.$

405. $\int \frac{\sin^2 x}{\sin x + 2\cos x} dx.$

406. $\int \frac{dx}{1+\varepsilon\cos x}$; a) $0<\varepsilon<1$; b) $\varepsilon>1$.

407. $\int \frac{\sin^2 x}{1+\sin^2 x}dx.$

408. $\int \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}.$

409. $\int \frac{\cos^2 x dx}{(a^2\sin^2 x+b^2\cos^2 x)^2}.$

410. $\int \frac{\sin x\cos x}{\sin x+\cos x}dx.$

411. $\int \frac{dx}{(a\sin x+b\cos x)^2}.$

412. $\int \frac{\sin x dx}{\sin^3 x+\cos^3 x}.$

413. $\int \frac{dx}{\sin^4 x+\cos^4 x}.$

414. $\int \frac{\sin^2 x\cos^2 x}{\sin^8 x+\cos^8 x}dx.$

415. $\int \frac{\sin^2 x-\cos^2 x}{\sin^4 x+\cos^4 x}dx.$

416. $\int \frac{\sin x\cos x}{1+\sin^4 x}dx.$

417. $\int \frac{dx}{\sin^6 x+\cos^6 x}.$

418. $\int \frac{dx}{(\sin^2 x+2\cos^2 x)^2}.$

419. Maýdalawjysyny logarifmik görnüşe getirip, integraly tapmaly:

$$\int \frac{dx}{a\sin x+b\cos x}.$$

420. A, B, C hemişelik sanlar üçin

$$\int \frac{a_1\sin x+b_1\cos x}{a\sin x+b\cos x}dx = Ax+B\ln|a\sin x+b\cos x|+C$$

deňligi subut etmeli. (*Görkezme*: $a_1\sin x + b_1\cos x = A(a\sin x + b\cos x) + B(a\cos x - b\sin x)$ *almaly*).

Integrallary tapmaly:

421. $\int \frac{\sin x-\cos x}{\sin x+2\cos x}dx.$

422. $\int \frac{\sin x}{\sin x-3\cos x}dx.$

423. $\int \frac{dx}{3+5\mathrm{tg}x}.$

424. $\int \frac{a_1\sin x+b_1\cos x}{(a\sin x+b\cos x)^2}dx.$

425. $\int \frac{a_1\sin x+b_1\cos x+c_1}{a\sin x+b\cos x+c}dx = Ax+B\ln|a\sin x+b\cos x+c|+$
$+C\int \frac{dx}{a\sin x+b\cos x+c}$

deňligi subut etmeli, bu ýerde A, B, C käbir hemişelik koeffisiýentler.

426. $\int \frac{\sin x + 2\cos x - 3}{\sin x - 2\cos x + 3} dx.$ **427.** $\int \frac{\sin x}{\sqrt{2} + \sin x + \cos x} dx.$

428. $\int \frac{2\sin x + \cos x}{3\sin x + 4\cos x - 2} dx.$

429. $\int \frac{a_1 \sin^2 x + 2b_1 \sin x \cos x + c_1 \cos^2 x}{a \sin x + b \cos x} dx =$

$= A\sin x + B\cos x + C\int \frac{dx}{a\sin x + b\cos x}$

deňligi subut etmeli, bu ýerde A, B, C hemişelik koeffisiýentler.

430. $\int \frac{\sin^2 x - 4\sin x \cos x + 3\cos^2 x}{\sin x + 2\cos x} dx.$

431. $\int \frac{\sin^2 x - \sin x \cos x + 2\cos^2 x}{\sin x + 2\cos x} dx.$

432. $(a - c)^2 + b^2 \neq 0$ bolanda

$$\int \frac{a_1 \sin x + b_1 \cos x}{a\sin^2 x + 2b\sin x \cos x + c\cos^2 x} dx = A\int \frac{du_1}{k_1 u_1^2 + \lambda_1} + B\int \frac{du_2}{k_2 u_2^2 + \lambda_2}$$

deňligi subut etmeli, bu ýerde A, B – näbelli koeffisiýentler, λ_1, λ_2

$$\begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix} = 0 \quad (\lambda_1 \neq \lambda_2)$$

deňlemäniň kökleri, $u_i = (a - \lambda_i)\sin x + b\cos x$, $k_i = \frac{1}{a - \lambda_1}$ $(i = 1, 2)$.

Integrallary tapmaly:

433. $\int \frac{2\sin x - \cos x}{3\sin^2 x + 4\cos^2 x} dx.$

434. $\int \frac{(\sin x + \cos x)dx}{2\sin^2 x - 4\sin x \cos x + 5\cos^2 x}.$

435. $\int \frac{\sin x - 2\cos x}{1 + 4\sin x \cos x} dx.$

436. Deňligi subut etmeli:

$$\int \frac{dx}{(a\sin x + b\cos x)^n} = \frac{A\sin x + B\cos x}{(a\sin x + b\cos x)^{n-1}} + C\int \frac{dx}{(a\sin x + b\cos x)^{n-2}},$$

bu ýerde A, B, C – näbelli koeffisiýentler.

437. $\int \frac{dx}{(\sin x + 2\cos x)^3}$ integraly tapmaly.

438. Deňligi subut etmeli:

$$\int \frac{dx}{(a + b\cos x)^n} =$$

$$= \frac{A\sin x}{(a + b\cos x)^{n-1}} + B\int \frac{dx}{(a + b\cos x)^{n-1}} + C\int \frac{dx}{(a + b\cos x)^{n-2}} \qquad (|a| \neq |b|)$$

we natural $n > 1$ san üçin A, B we C koeffisiýentleri kesgitlemeli.

Integrallary tapmaly:

439. $\int \frac{\sin x dx}{\cos x\sqrt{1 + \sin^2 x}}$.

440. $\int \frac{\sin^2 x}{\cos^2 x\sqrt{\operatorname{tg} x}} dx$.

441. $\int \frac{\sin x dx}{\sqrt{2 + \sin 2x}}$.

442. $\int \frac{dx}{(1 + \varepsilon\cos x)^2}$ $(0 < \varepsilon < 1)$.

443. $\int \frac{\cos^{n-1}\frac{x+a}{2}}{\sin^{n+1}\frac{x-a}{2}} dx$. (*Görkezme*: $t = \frac{\cos\frac{x+a}{2}}{\sin\frac{x-a}{2}}$ *almaly*).

444. Integraly peseltmek formulasyny getirip çykarmaly:

$$I_n = \int \left(\frac{\sin\frac{x-a}{2}}{\sin\frac{x+a}{2}} \right)^n dx \qquad (n \text{ – natural san}).$$

§5. Dürli transsendent funksiýalaryň integrirlenişi

Gönükmeler

445. n derejeli $P(x)$ köpagza üçin deňligi subut etmeli:

$$\int P(x)e^{ax} dx = e^{ax}\left[\frac{P(x)}{a} - \frac{P'(x)}{a^2} + \ldots + (-1)^n \frac{P^{(n)}(x)}{a^{n+1}} \right] + C.$$

446. n derejeli $P(x)$ köpagza üçin deňlikleri subut etmeli:

$$\int P(x)\cos ax dx = \frac{\sin ax}{a}\left[P(x) - \frac{P''(x)}{a^2} + \frac{P^{IV}(x)}{a^4} - \ldots \right] +$$

$$+ \frac{\cos ax}{a^2}\left[P'(x) - \frac{P'''(x)}{a^2} + \frac{P^{V}(x)}{a^4} - \ldots \right] + C,$$

$$\int P(x)\sin ax dx = -\frac{\cos ax}{a}\left[P(x) - \frac{P''(x)}{a^2} + \frac{P^{IV}(x)}{a^4} - \ldots\right] +$$

$$+\frac{\sin ax}{a^2}\left[P'(x) - \frac{P'''(x)}{a^2} + \frac{P^{V}(x)}{a^4} - \ldots\right] + C.$$

Integrallary tapmaly:

447. $\int x^3 e^{3x} dx.$ **448.** $\int (x^2 - 2x + 2)e^{-x} dx.$

449. $\int x^5 \sin 5x dx.$ **450.** $\int (1 + x^2)^2 \cos x dx.$

451. $\int x^7 e^{-x^2} dx.$ **452.** $\int x^2 e^{\sqrt{x}} dx.$

453. $\int e^{ax} \cos^2 bx dx.$ **454.** $\int e^{ax} \sin^3 bx dx.$

455. $\int x e^x \sin x dx.$ **456.** $\int x^2 e^x \cos x dx.$

457. $\int x e^x \sin^2 x dx.$ **458.** $\int (x - \sin x)^3 dx.$

459. $\int \cos^2 \sqrt{x}\, dx.$

460. Rasional R funksiýa we ölçegdeş $a_1, a_2, \ldots, a_n$ sanlar üçin

$$\int R(e^{a_1 x}, e^{a_2 x}, \ldots, e^{a_n x}) dx$$

integralyň elementar funksiýa bolýandygyny subut etmeli.

Aşakdaky integrallary tapmaly:

461. $\int \frac{dx}{(1 + e^x)^2}.$ **462.** $\int \frac{e^{2x}}{1 + e^x} dx.$

463. $\int \frac{dx}{e^{2x} + e^x - 2}.$ **464.** $\int \frac{dx}{1 + e^{x/2} + e^{x/3} + e^{x/6}}.$

465. $\int \frac{1 + e^{x/2}}{(1 + e^{x/4})^2} dx.$ **466.** $\int \frac{dx}{\sqrt{e^x - 1}}.$

467. $\int \sqrt{\frac{e^x - 1}{e^x + 1}}\, dx.$ **468.** $\int \sqrt{e^{2x} + 4e^x - 1}\, dx.$

469. $\int \frac{dx}{\sqrt{1 + e^x} + \sqrt{1 - e^x}}.$

470. Rasional R funksiýanyň maýdalawjysynyň diňe hakyky kökleri bar halynda $\int R(x)e^{ax}dx$ integralyň elementar funksiýalar we transsendent $\int \frac{e^{ax}}{x}dx = li(e^{ax}) + C$ funksiýa arkaly aňladylýandygyny subut etmeli, bu ýerde $lix = \int \frac{dx}{\ln x}$.

471. Haýsy halda $P\left(\frac{1}{x}\right) = a_0 + \frac{a_1}{x} + \ldots + \frac{a_n}{x^n}$ we $a_0, a_1, \ldots, a_n$ – hemişelik sanlar üçin $\int P\left(\frac{1}{x}\right)e^x dx$ integral elementar funksiýa bolýar?

Integrallary tapmaly:

472. $\int \left(1 - \frac{2}{x}\right)^2 e^x dx.$

473. $\int \left(1 - \frac{1}{x}\right)e^{-x} dx.$

474. $\int \frac{e^{2x}}{x^2 - 3x + 2} dx.$

475. $\int \frac{xe^x}{(x+1)^2} dx.$

476. $\int \frac{x^4 e^{2x}}{(x-2)^2} dx.$

Algebraik $f(x)$ funksiýa üçin $\ln f(x)$, arctg$f(x)$, arcsin$f(x)$, arccos$f(x)$ funksiýalary özünde saklaýan integrallary tapmaly:

477. $\int \ln^n x dx$ (n – natural san).

478. $\int x^3 \ln^3 x dx.$

479. $\int \left(\frac{\ln x}{x}\right)^3 dx.$

480. $\int \ln[(x+a)^{x+a}(x+b)^{x+b}] \cdot \frac{dx}{(x+a)(x+b)}.$

481. $\int \ln^2(x + \sqrt{1+x^2})dx.$

482. $\int \ln(\sqrt{1-x} + \sqrt{1+x})dx.$

483. $\int \frac{\ln x}{(1+x^2)^{3/2}} dx.$

484. $\int x \operatorname{arctg}(x+1)dx.$

485. $\int \sqrt{x} \operatorname{arctg}\sqrt{x}\, dx.$

486. $\int x \arcsin(1-x)dx.$

487. $\int \arcsin\sqrt{x}\, dx.$

488. $\int x \arccos\frac{1}{x} dx.$

489. $\int \arcsin\frac{2\sqrt{x}}{1+x} dx.$

490. $\int \frac{\arccos x}{(1-x^2)^{3/2}} dx.$

491. $\int \frac{x \arccos x}{(1-x^2)^{3/2}} dx.$

492. $\int x \operatorname{arctg} x \ln(1+x^2) dx.$

493. $\int x \ln \frac{1+x}{1-x} dx.$

494. $\int \frac{\ln(x+\sqrt{1+x^2})dx}{(1+x^2)^{3/2}}.$

Giperbolik funksiýalary özünde saklaýan integrallary tapmaly:

495. $\int \mathrm{sh}^2 x \mathrm{ch}^2 x dx.$

496. $\int \mathrm{ch}^4 x dx.$

497. $\int \mathrm{sh}^3 x dx.$

498. $\int \mathrm{sh} x \mathrm{sh} 2x \mathrm{sh} 3x dx.$

499. $\int \mathrm{th} x dx.$

500. $\int \mathrm{cth}^2 x dx.$

501. $\int \sqrt{\mathrm{th} x}\, dx.$

502. $\int \frac{dx}{\mathrm{sh} x + 2\mathrm{ch} x}.$

503. $\int \frac{dx}{\mathrm{sh}^2 x - 4\mathrm{sh} x \mathrm{ch} x + 9\mathrm{ch}^2 x}.$

504. $\int \frac{dx}{0,1 + \mathrm{ch} x}.$

505. $\int \frac{\mathrm{ch} x dx}{3\mathrm{sh} x - 4\mathrm{ch} x}.$

506. $\int \mathrm{sh} ax \sin bx dx.$

507. $\int \mathrm{sh} ax \cos bx dx.$

§6. Dürli görnüşdäki funksiýalary integrirlemegiň mysallary

Gönükmeler

Integrallary tapmaly:

508. $\int \frac{dx}{x^6(1+x^2)}.$

509. $\int \frac{x^2 dx}{(1-x^2)^3}.$

510. $\int \frac{dx}{1+x^4+x^8}.$

511. $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}.$

512. $\int x^2 \sqrt{\frac{x}{1-x}} dx.$

513. $\int \frac{x+2}{x^2\sqrt{1-x^2}} dx.$

514. $\int \sqrt{\frac{x}{1-x\sqrt{x}}} dx.$

515. $\int \frac{x^5 dx}{\sqrt{1+x^2}}.$

516. $\int \frac{dx}{\sqrt[3]{x^2(1-x)}}.$

517. $\int \frac{dx}{x\sqrt{1+x^3+x^6}}.$

518. $\int \frac{dx}{x\sqrt{x^4-2x^2-1}}$.

519. $\int \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}dx$.

520. $\int \frac{(1+x)dx}{x+\sqrt{x+x^2}}$.

521. $\int \frac{\ln(1+x+x^2)}{(1+x)^2}dx$.

522. $\int (2x+3)\arccos(2x-3)dx$.

523. $\int x\ln(4+x^4)dx$.

524. $\int \frac{\arcsin x}{x^2}\cdot\frac{1+x^2}{\sqrt{1-x^2}}dx$.

525. $\int \frac{x\ln(1+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx$.

526. $\int x\sqrt{x^2+1}\ln\sqrt{x^2-1}\,dx$.

527. $\int \frac{x}{\sqrt{1-x^2}}\ln\frac{x}{\sqrt{1-x}}dx$.

528. $\int \frac{dx}{(2+\sin x)^2}$.

529. $\int \frac{\sin 4x}{\sin^8 x+\cos^8 x}dx$.

530. $\int \frac{dx}{\sin x\sqrt{1+\cos x}}$.

531. $\int \frac{ax^2+b}{x^2+1}\mathrm{arctg}x dx$.

532. $\int \frac{ax^2+b}{x^2-1}\ln\left|\frac{x-1}{x+1}\right|dx$.

533. $\int \frac{x\ln x}{(1+x^2)^2}dx$.

534. $\int \frac{x\,\mathrm{arctg}\,x}{\sqrt{1+x^2}}dx$.

535. $\int \frac{\sin 2x}{\sqrt{1+\cos^4 x}}dx$.

536. $\int \frac{x^3\arccos x}{\sqrt{1-x^2}}dx$.

537. $\int \frac{x^4\mathrm{arctg}x}{1+x^2}dx$.

538. $\int \frac{x\,\mathrm{arcctg}\,x}{(1+x^2)^2}dx$.

539. $\int \frac{x\ln(x+\sqrt{1+x^2})}{(1-x^2)^2}dx$.

540. $\int \sqrt{1-x^2}\arcsin x dx$.

541. $\int x(1+x^2)\mathrm{arcctg}x dx$.

542. $\int x^x(1+\ln x)dx$.

543. $\int \frac{\arcsin e^x}{e^x}dx$.

544. $\int \frac{\mathrm{arctg}e^{x/2}}{e^{x/2}(1+e^x)}dx$.

545. $\int \frac{dx}{(e^{x+1}+1)^2-(e^{x-1}+1)^2}$.

546. $\int \sqrt{\mathrm{th}^2 x+1}\,dx$.

547. $\int \frac{1+\sin x}{1+\cos x}\cdot e^x dx$.

548. $\int |x|dx$.

549. $\int x|x|dx$.

550. $\int (x + |x|)^2 dx.$

551. $\int \{|1 + x| - |1 - x|\} dx.$

552. $\int e^{-|x|} dx.$

553. $\int \max(1, x^2) dx.$

554. $\int \varphi(x) dx$, bu ýerde $\varphi(x)$ funksiýa x-iň iň ýakyn bitin sana çenli uzaklygy.

555. $\int [x]|\sin \pi x| dx \quad (x \geq 0).$

556. $\int f(x) dx$, bu ýerde $f(x) = \begin{cases} 1 - x^2, & |x| \leq 1 \text{ bolanda;} \\ 1 - |x|, & |x| > 1 \text{ bolanda.} \end{cases}$

557. $\int f(x) dx$, bu ýerde $f(x) = \begin{cases} 1, & \text{eger } -\infty < x < 0; \\ x + 1, & \text{eger } 0 \leq x \leq 1; \\ 2x, & \text{eger } 1 < x < +\infty. \end{cases}$

Integrallary tapmaly:

558. $\int x f''(x) dx.$

559. $\int f'(2x) dx.$

560. Berlen $f'(x^2) = \frac{1}{x}$ $(x > 0)$ boýunça $f(x)$ funksiýany tapmaly.

561. Berlen $f'(\sin^2 x) = \cos^2 x$ boýunça $f(x)$ funksiýany tapmaly.

562. Berlen $f'(\ln x) = \begin{cases} 1, & \text{eger } 0 < x \leq 1; \\ x, & \text{eger } 1 < x < +\infty \end{cases}$ we $f(0) = 0$ boýunça $f(x)$ funksiýany tapmaly.

563. Goý, $f(x)$ üznüksiz monoton funksiýa we $f^{-1}(x)$ onuň ters funksiýasy bolsun. Eger $\int f(x) dx = F(x) + C$ bolsa, onda deňligi subut etmeli:

$$\int f^{-1}(x) dx = x f^{-1}(x) - F(f^{-1}(x)) + C.$$

Bu deňligiň esasynda aşakdaky funksiýalary derňemeli:

a) $f(x) = x^n \quad (n > 0)$; b) $f(x) = e^x$; ç) $f(x) = \arcsin x$; d) $f(x) = \mathrm{Arth}\, x$.

VII. KESGITLI INTEGRAL

§1. Kesgitli integral we integrirlemek usullary

1. ***Kesgitli integral düşünjesi.*** Goý, $f(x)$ funksiýa $[a, b]$ kesimde kesgitlenen we çäkli bolsun. $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ deňsizlikleri kanagatlandyrýan nokatlaryň $\{x_i\}_{i=0}^{n}$ köplügine $[a, b]$ kesimiň bölünmesi diýilýär we P bilen belgilenýär. $[a, b]$ kesimiň P bölünmesi we erkin $t_i \in [x_{i-1}, x_i]$ üçin düzülen

$$S_P(f) = \sum_{i=1}^{n} f(t_i)\Delta x_i \quad (\Delta x_i = x_i - x_{i-1}) \tag{1}$$

jeme f funksiýanyň $[a, b]$ kesim boýunça integral jemi diýilýär.

Eger $d = \max\limits_{i=1,\ldots,n} \Delta x_i \to 0$ bolanda (1) integral jemiň $[a, b]$ kesimiň P bölünmesine we $t_i \in [x_{i-1}, x_i]$ nokatlara bagly bolmadyk

$$I = \lim_{d\to 0} S_P(f) = \lim_{d\to 0}\sum_{i=1}^{n} f(t_i)\Delta x_i \tag{2}$$

predeli bar bolsa, onda şol predele f funksiýanyň $[a, b]$ kesim boýunça Rimanyň kesgitli integraly diýilýär we ol $\int_a^b f(x)dx$ bilen belgilenýär. Şeýlelikde,

$$\int_a^b f(x)dx = \lim_{d\to 0}\sum_{i=1}^{n} f(t_i)\Delta x_i. \tag{3}$$

2. ***Integrirlemegiň şertleri we integrirlenýän funksiýalar***

$m_i = \inf\limits_{x_{i-1}\le x\le x_i} f(x)$ we $M_i = \sup\limits_{x_{i-1}\le x\le x_i} f(x)$ üçin düzülen

$$\underline{S_P}(f) = \sum_{i=1}^{n} m_i\Delta x_i \quad \text{we} \quad \overline{S_P}(f) = \sum_{i=1}^{n} M_i\Delta x_i \tag{4}$$

jemlere degişlilikde f funksiýanyň $[a, b]$ kesim boýunça Darbunyň aşaky we ýokarky jemleri diýilýär.

f funksiýanyň $[a, b]$ kesimde integrirlenmegi üçin

$$\lim_{d\to 0}\sum_{i=1}^{n} \omega_i\Delta x_i = 0$$

deňligiň ýerine ýetmegi zerur we ýeterlikdir, bu ýerde $\omega_i = M_i - m_i$.

Integrirlenýän funksiýalar:

1) $[a, b]$ kesimde üznüksiz funksiýa.

2) $[a, b]$ kesimde çäkli we tükenikli sany üzülme nokatlary bolan funksiýa.

3) $[a, b]$ kesimde çäkli we monoton funksiýa.

3. *Kesgitli integralyň häsiýetleri*

1. Eger f funksiýa $[a, b]$ kesimde integrirlenýän bolsa, onda

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

2. Eger f funksiýa $[a, b]$ we $[b, c]$ kesimlerde integrirlenýän bolsa, onda ol $[a, c]$ kesimde integrirlenýär we

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx.$$

4. *Kesgitli integraly hasaplamagyň usullary*

1. ***Nýuton-Leýbnisiň formulasy.*** Eger f funksiýa $[a, b]$ kesimde üznüksiz we F onuň asyl funksiýasy bolsa, onda kesgitli integraly hasaplamak üçin Nýuton-Leýbnisiň formulasy dogrudyr:

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b.$$

2. ***Bölekleýin integrirleme usuly.*** Eger $u = u(x)$ we $v = v(x)$ funksiýalar $[a, b]$ kesimde üznüksiz differensirlenýän bolsalar, onda bölekleýin integrirlemegiň

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx$$

formulasy dogrudyr.

3. ***Üýtgeýän ululygy çalşyrma usuly.*** Eger f funksiýa $[a, b]$ kesimde üznüksiz we φ funksiýa $[\alpha, \beta]$ kesimde üznüksiz differensirlenýän bolup, $\varphi(\alpha) = a$, $\varphi(\beta) = b$ we $\forall t \in [\alpha, \beta]$ üçin $\varphi(t)$ funksiýanyň bahalary $[a, b]$ kesimine degişli bolsa, onda

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$$

formula dogrudyr.

Gönükmeler

1. $f(x) = 1 + x$ funksiýa üçin $[-1, 4]$ kesimi deň n böleklere bölüp we argumentiň t_i $(i = 1, 2, ..., n)$ bahasyny şol bölekleriň ortasynda alyp, S_n integral jemi tapmaly.

2. Berlen $f(x)$ funksiýa üçin degişli kesimi deň n böleklere bölüp, Darbunyň aşaky $\underline{S}_n$ we ýokarky $\overline{S}_n$ jemlerini düzmeli:

a) $f(x) = x^3\,[-2 \le x \le 3]$; b) $f(x) = \sqrt{x}\,[0 \le x \le 1]$;

ç) $f(x) = 2^x\,[0 \le x \le 10]$.

3. $f(x) = x^4$ funksiýa üçin $[1, 2]$ kesimi uzynlyklary geometrik progressiýany emele getirýän n böleklere bölüp, Darbunyň aşaky integral jemini tapmaly.

4. Kesgitli integralyň kesgitlemesinden peýdalanyp, hemişelik ϑ_0 we g üçin $\int\limits_0^T (\vartheta_0 + gt)dt$ integraly tapmaly.

Kesgitli integrallara degişli integral jemleriň predelleri hökmünde garap we integrirleme aralyklary görkezilişi ýaly böleklere bölüp, kesgitli integrallary hasaplamaly:

5. $\int\limits_{-1}^{2} x^2 dx$. **6.** $\int\limits_0^1 a^x dx \quad (a > 0)$.

7. $\int\limits_0^{\pi/2} \sin x dx$. **8.** $\int\limits_0^x \cos t dt$.

9. $\int\limits_a^b \frac{dx}{x^2}$ $(0 < a < b)$. (*Görkezme*: *Bölünme nokatlary* $t_i = \sqrt{x_i x_{i+1}}$ $(i = 0, 1, ..., n)$ *görnüşde almaly*).

10. $\int\limits_a^b x^m dx$ $(0 < a < b;\ m \ne -1)$. (*Görkezme*: *Bölünme nokatlaryň* x_i *absissasyny geometrik progressiýany emele getirer ýaly saýlap almaly*).

11. $\int\limits_a^b \frac{dx}{x}$ $(0 < a < b)$.

12. Puassonyň

$$\int\limits_0^{\pi} \ln(1 - 2\alpha \cos x + \alpha^2)dx$$

integralyny: a) $|\alpha| < 1$; b) $|\alpha| > 1$ bolanda hasaplamaly.

(*Görkezme*: $\alpha^{2n} - 1$ *köpagzanyň kwadrat köpeldijilere dargamasyny ulanmaly*).

13. Goý, $f(x)$ we $\varphi(x)$ funksiýalar $[a, b]$ kesimde üznüksiz bolsun. Deňligi subut etmeli:

$$\lim_{d\to 0}\sum_{i=1}^{n} f(t_i)\varphi(\tau_i)\Delta x_i = \int_a^b f(x)\varphi(x)dx,$$

bu ýerde $x_{i-1}\le t_i\le x_i$, $x_{i-1}\le \tau_i\le x_i$ $(i = 1, 2, ..., n)$ we $\Delta x_i = x_i - x_{i-1}$ $(x_0=a, x_n = b)$.

14. Goý, $f(x)$ funksiýa [0, 1] kesimde monoton we çäkli bolsun. Deňligi subut etmeli:

$$\int_0^1 f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = O\left(\frac{1}{n}\right).$$

15. Goý, $f(x)$ funksiýa $[a, b]$ kesimde çäkli we ýokarlygyna güberçek bolsun. Deňsizligi subut etmeli:

$$(b-a)\frac{f(a)+f(b)}{2} \le \int_a^b f(x)dx \le (b-a)f\left(\frac{a+b}{2}\right).$$

16. Goý, $x \in [1, +\infty)$ bolanda $f(x) \in C^{(2)}[1, +\infty)$ we $f(x) \ge 0$, $f'(x) \ge 0$, $f''(x) \le 0$ bolsun. $n \to \infty$ bolanda deňligi subut etmeli:

$$\sum_{k=1}^{n} f(k) = \frac{1}{2}f(n) + \int_1^n f(x)dx + O(1).$$

17. Goý, $f(x)\in C^{(1)}[a,b]$ we $\Delta_n = \int_a^b f(x)dx - \frac{b-a}{n}\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)$ bolsun. $\lim_{n\to\infty} n\Delta_n$ predeli tapmaly.

18. Üznükli $f(x) = \operatorname{sgn}\left(\sin\frac{\pi}{x}\right)$ funksiýanyň [0, 1] kesimde integrirlenýändigini subut etmeli.

19. Rimanyň $\varphi(x) = \begin{cases} 0, & x - \text{irrasional;} \\ \frac{1}{n}, & x = \frac{m}{n} \end{cases}$ funksiýasynyň islendik tükenikli aralykda integrirlenýändigini subut etmeli, bu ýerde m we n $(n \ge 1)$ özara ýönekeý bitin sanlar.

20. $f(x) = \frac{1}{x} - \left[\frac{1}{x}\right]$, $x \ne 0$ we $f(0) = 0$ funksiýanyň [0, 1] kesimde integrirlenýändigini subut etmeli.

21. Dirihläniň $D(x) = \begin{cases} 0, & x - \text{irrasional;} \\ 1, & x - \text{rasional} \end{cases}$ funksiýasynyň islendik aralykda integrirlenmeýändigini subut etmeli.

22. Goý, $f(x)$ funksiýa $[a, b]$ kesimde integrirlenýän we

$$f_n(x) = \sup_{x_i \le x < x_{i+1}} f(x)$$

bolsun, bu ýerde

$$x_i = a + \frac{i}{n}(b-a) \qquad (i = 0, 1, ..., n;\ \ n = 1, 2, ...).$$

Deňligi subut etmeli:

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

23. Eger $f(x)$ funksiýa $[a, b]$ kesimde integrirlenýän bolsa, onda üznüksiz $f_n(x)$ $(n = 1, 2, ...)$ funksiýalaryň şeýle yzygiderligi bar bolup,

$$\int_a^c f(x)dx = \lim_{n\to\infty}\int_a^c f_n(x)dx \qquad (a \le c \le b)$$

deňligiň ýerine ýetýändigini subut etmeli.

24. Eger çäkli $f(x)$ funksiýa $[a, b]$ kesimde integrirlenýän bolsa, onda onuň $|f(x)|$ absolýut ululygynyň hem şol kesimde integrirlenýändigini we

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$$

deňsizligiň dogrudygyny subut etmeli.

25. Goý, $f(x)$ funksiýa $[a, b]$ kesimde absolýut integrirlenýän bolsun, ýagny $\int_a^b |f(x)|dx$ integral bar bolsun. Ol funksiýa $[a, b]$ kesimde integrirlenýärmi? Aşakdaky funksiýa hem bu şertlerde integrirlenýärmi?

$$f(x) = \begin{cases} -1, & x - \text{irrasional}; \\ \ \ 1, & x - \text{rasional}. \end{cases}$$

26. Goý, $f(x)$ funksiýa $[a, b]$ kesimde integrirlenýän we şol kesimde $A \le f(x) \le B$ bolsun, $\varphi(x)$ funksiýa bolsa $[A, B]$ kesimde kesgitlenen we üznüksiz bolsun. $\varphi(f(x))$ funksiýanyň $[a, b]$ kesimde integrirlenýändigini subut etmeli.

27. Eger $f(x)$ we $\varphi(x)$ funksiýalar integrirlenýän bolsa, onda $f(\varphi(x))$ funksiýa hökman integrirlenýärmi? Aşakdaky funksiýa hem şu şertlerde hökman integrirlenýärmi?

$$f(x) = \begin{cases} 0, & \text{eger } x = 0; \\ 1, & \text{eger } x \ne 0 \end{cases}$$

we $\varphi(x)$ – Rimanyň funksiýasy (*Görkezme*: *19-njy mysala seret*).

28. Goý, $f(x)$ funksiýa $[A, B]$ kesimde integrirlenýän bolsun. $f(x)$ funksiýanyň integral üznüksizlik häsiýetini, ýagny

$$\lim_{h\to 0}\int_a^b |f(x+h)-f(x)|dx = 0$$

deňligi subut etmeli.

29. Goý, $f(x)$ funksiýa $[a, b]$ kesimde integrirlenýän bolsun.

$$\int_a^b f^2(x)dx = 0$$

deňligiň $f(x)$ funksiýanyň $[a, b]$ kesime degişli bolan ähli üznüksiz nokatlarynda $f(x) = 0$ bolanda we diňe şol nokatlarda ýerine ýetýändigini subut etmeli.

Nýuton-Leýbnisiň formulasyny ulanyp, aşakdaky kesgitli integrallary tapmaly we degişli egri çyzykly meýdanlary çyzmaly:

30. $\int_{-1}^{8} \sqrt[3]{x}\, dx.$

31. $\int_0^{\pi} \sin x dx.$

32. $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}.$

33. $\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}}.$

34. $\int_{\mathrm{sh}1}^{\mathrm{sh}2} \frac{dx}{\sqrt{1+x^2}}.$

35. $\int_0^2 |1-x|dx.$

36. $\int_{-1}^{1} \frac{dx}{x^2-2x\cos\alpha+1}$ $(0<\alpha<\pi)$.

37. $\int_0^{2\pi} \frac{dx}{1+\varepsilon\cos x}$ $(0\le\varepsilon<1)$.

38. $\int_{-1}^{1} \frac{dx}{\sqrt{(1-2ax+a^2)(1-2bx+b^2)}}$ $(|a|<1,\ |b|<1,\ ab>0)$.

39. $\int_0^{\pi/2} \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}$ $(ab\ne 0)$.

40. Berlen

a) $\int_{-1}^{1} \frac{dx}{x}$;   b) $\int_0^{2\pi} \frac{\sec^2 x dx}{2+\mathrm{tg}^2 x}$;   ç) $\int_{-1}^{1} \frac{d}{dx}\left(\mathrm{arctg}\frac{1}{x}\right)dx$

integrallarda Nýuton-Leýbnisiň formulasynyň ulanylyşynyň näme üçin nädogry netijelere getirýändigini düşündiriň.

41. Tapmaly: $\int\limits_{-1}^{1} \frac{d}{dx}\left(\frac{1}{1+2^{1/x}}\right)dx$.

42. Tapmaly: $\int\limits_{0}^{100\pi} \sqrt{1-\cos 2x}\, dx$.

Kesgitli integrallaryň kömegi bilen aşakdaky jemleriň predellerini tapmaly:

43. $\lim\limits_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n-1}{n^2}\right)$.

44. $\lim\limits_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+n}\right)$.

45. $\lim\limits_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+n^2}\right)$.

46. $\lim\limits_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\ldots+\sin\frac{(n-1)\pi}{n}\right)$.

47. $\lim\limits_{n\to\infty}\frac{1^p+2^p+\ldots+n^p}{n^{p+1}}$ $(p>0)$.

48. $\lim\limits_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\ldots+\sqrt{1+\frac{n}{n}}\right)$.

Predelleri tapmaly:

49. $\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!}}{n}$.

50. $\lim\limits_{n\to\infty}\left[\frac{1}{n}\sum\limits_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)\right]$.

Ýokary tertipli, deňölçegli tükeniksiz kiçi ululyklary taşlap, aşakdaky jemleriň predellerini tapmaly:

51. $\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\sin\frac{\pi}{n^2}+\left(1+\frac{2}{n}\right)\sin\frac{2\pi}{n^2}+\ldots+\left(1+\frac{n-1}{n}\right)\sin\frac{(n-1)\pi}{n^2}\right]$.

52. $\lim\limits_{n\to\infty}\sin\frac{\pi}{n}\cdot\sum\limits_{k=1}^{n}\frac{1}{2+\cos\frac{k\pi}{n}}$.

53. $\lim\limits_{n\to\infty}\frac{\sum\limits_{k=1}^{n}\sqrt{(nx+k)(nx+k+1)}}{n^2}$, $(x>0)$.

54. $\lim\limits_{n\to\infty}\left(\frac{2^{1/n}}{n+1}+\frac{2^{2/n}}{n+\frac{1}{2}}+\ldots+\frac{2^{n/n}}{n+\frac{1}{n}}\right)$.

55. Tapmaly:

a) $\frac{d}{dx}\int_a^b \sin x^2\,dx$; b) $\frac{d}{da}\int_a^b \sin x^2\,dx$; ç) $\frac{d}{db}\int_a^b \sin x^2\,dx$.

56. Tapmaly:

a) $\frac{d}{dx}\int_0^{x^2}\sqrt{1+t^2}\,dt$; b) $\frac{d}{dx}\int_{x^2}^{x^3}\frac{dt}{\sqrt{1+t^4}}$; ç) $\frac{d}{dx}\int_{\sin x}^{\cos x}\cos(\pi t^2)\,dt$.

57. Tapmaly:

a) $\lim\limits_{x\to 0}\frac{\int_0^x \cos x^2\,dx}{x}$; b) $\lim\limits_{x\to+\infty}\frac{\int_0^x (\operatorname{arctg} x)^2\,dx}{\sqrt{x^2+1}}$; ç) $\lim\limits_{x\to+\infty}\frac{\left(\int_0^x e^{x^2}\,dx\right)^2}{\int_0^x e^{2x^2}\,dx}$.

58. Goý, $x \to +\infty$ bolanda $f(x) \in C[0, +\infty]$ we $f(x) \to A$ bolsun. $\lim\limits_{x\to\infty}\int_0^1 f(nx)\,dx$ predeli tapmaly.

59. $x \to \infty$ bolanda $\int_0^x e^{x^2}\,dx \sim \frac{1}{2x}e^{x^2}$ subut etmeli.

60. $\lim\limits_{x\to+0}\frac{\int_0^{\sin x}\sqrt{\operatorname{tg} x}\,dx}{\int_0^{\operatorname{tg} x}\sqrt{\sin x}\,dx}$ predeli tapmaly.

61. Goý, $f(x)$ üznüksiz položitel funksiýa bolsun. $x \geq 0$ bolanda $\varphi(x) = \frac{\int_0^x tf(t)\,dt}{\int_0^x f(t)\,dt}$ funksiýanyň artýandygyny subut etmeli.

62. Berlen funksiýalar boýunça integrallary tapmaly:

a) $\int_0^2 f(x)\,dx$, $f(x) = \begin{cases} x^2, & 0 \leq x \leq 1, \\ 2-x, & 1 < x \leq 2; \end{cases}$

b) $\int_0^1 f(x)\,dx$, $f(x) = \begin{cases} x, & 0 \leq x \leq t, \\ t\cdot\frac{1-x}{1-t}, & t \leq x \leq 1. \end{cases}$

63. Berlen integrallary hasaplamaly we $I = I(\alpha)$ integrala α parametriň funksiýasy hökmünde garap, onuň grafigini gurmaly:

a) $I = \int\limits_0^1 x|x - \alpha|dx$;

b) $I = \int\limits_0^{\pi} \frac{\sin^2 x}{1 + 2\alpha\cos x + \alpha^2}dx$;

ç) $I = \int\limits_0^{\pi} \frac{\sin x}{\sqrt{1 - 2\alpha\cos x + \alpha^2}}dx$.

Bölekleýin integrirlemek formulasyny ulanyp, kesgitli integrallary tapmaly:

64. $\int\limits_0^{\ln 2} xe^{-x}dx$.

65. $\int\limits_0^{\pi} x\sin x dx$.

66. $\int\limits_0^{2\pi} x^2\cos x dx$.

67. $\int\limits_{1/e}^{e} |\ln x|dx$.

68. $\int\limits_0^1 \arccos x dx$.

69. $\int\limits_0^{\sqrt{3}} x\,\mathrm{arctg}\,x dx$.

Amatly orun çalşyrmalary ulanyp, kesgitli integrallary tapmaly:

70. $\int\limits_{-1}^{1} \frac{xdx}{\sqrt{5 - 4x}}$.

71. $\int\limits_0^{a} x^2\sqrt{a^2 - x^2}\,dx$.

72. $\int\limits_0^{0,75} \frac{dx}{(x + 1)\sqrt{x^2 + 1}}$.

73. $\int\limits_0^{\ln 2} \sqrt{e^x - 1}\,dx$.

74. $\int\limits_0^1 \frac{\arcsin\sqrt{x}}{\sqrt{x(1 - x)}}dx$.

75. $x - \frac{1}{x} = t$ alyp, $\int\limits_{-1}^{1} \frac{1 + x^2}{1 + x^4}dx$ integraly hasaplamaly.

76. Berlen integrallarda görkezilen orun çalşyrmalaryň näme üçin ýalňyş netijä getirýändigini düşündirmeli:

a) $\int\limits_{-1}^{1} dx,\ t = x^{2/3}$; b) $\int\limits_{-1}^{1} \frac{dx}{1 + x^2},\quad x = \frac{1}{t}$; ç) $\int\limits_0^{\pi} \frac{dx}{1 + \sin^2 x},\quad \mathrm{tg}x = t$.

77. $\int\limits_0^3 x^3\sqrt{1-x^2}\,dx$ integralda $x = \sin t$ alyp bolarmy?

78. $\int\limits_0^1 \sqrt{1-x^2}\,dx$ integralda $x = \sin t$ çalşyrma ulanylanda integralyň täze çäkleri hökmünde π we $\pi/2$ sanlary alyp bolarmy?

79. Eger $f(x)$ funksiýa $[a, b]$ kesimde üznüksiz bolsa, onda

$$\int\limits_a^b f(x)dx = (b-a)\int\limits_0^1 f(a+(b-a)x)dx$$

deňligi subut etmeli.

80. Deňligi subut etmeli:

$$\int\limits_0^a x^3 f(x^2)dx = \frac{1}{2}\int\limits_0^{a^2} xf(x)dx \qquad (a > 0).$$

81. Eger $f(x)$ funksiýa $[a, b] \subset [A, B]$ kesimde üznüksiz bolsa, onda $[a-x, b-x] \subset [A, B]$ bolanda $\frac{d}{dx}\int\limits_a^b f(x+y)dy$ tapmaly.

82. Eger $f(x)$ funksiýa $[0, 1]$ kesimde üznüksiz bolsa, onda aşakdakylary subut etmeli:

a) $\int\limits_0^{\pi/2} f(\sin x)dx = \int\limits_0^{\pi/2} f(\cos x)dx;$ b) $\int\limits_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int\limits_0^{\pi} f(\sin x)dx.$

83. Eger $f(x)$ funksiýa $[-l, l]$ kesimde üznüksiz bolsa, onda

1) jübüt $f(x)$ funksiýa üçin $\int\limits_{-l}^{l} f(x)dx = 2\int\limits_0^{l} f(x)dx$ deňligi we

2) täk $f(x)$ funksiýa üçin $\int\limits_{-l}^{l} f(x)dx = 0$ deňligi subut etmeli.

Bu deňliklere geometrik taýdan düşündiriş bermeli.

84. Jübüt funksiýalaryň asyl funksiýalarynyň biriniň täk funksiýa we täk funksiýalaryň asyl funksiýalarynyň ählisiniň jübüt funksiýa bolýandygyny subut etmeli.

85. $t = x + \frac{1}{x}$ orun çalşyrmany ulanyp, $\int\limits_{1/2}^{2}\left(1 + x - \frac{1}{x}\right)e^{x+1/x}dx$ integraly hasaplamaly.

86. $\int_{0}^{2\pi} f(x)\cos x dx$ integralda $\sin x = t$ çalşyrmany ýerine ýetirmeli.

87. $\int_{e^{-2\pi n}}^{1} \left|\left[\cos\left(\ln \frac{1}{x}\right)\right]'\right| dx$ integraly hasaplamaly, bu ýerde n natural san.

88. $\int_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$ integraly tapmaly.

89. Eger $f(x) = \frac{(x+1)^2(x-1)}{x^3(x-2)}$ bolsa, onda $\int_{-1}^{3} \frac{f'(x)}{1+f^2(x)} dx$ integraly tapmaly.

90. Eger $f(x)$ funksiýa $-\infty < x < +\infty$ interwalda kesgitlenen, üznüksiz hem-de T periodik funksiýa bolsa, onda erkin a san üçin

$$\int_{a}^{a+T} f(x)dx = \int_{0}^{T} f(x)dx$$

deňligi subut etmeli.

91. Täk n üçin

$$F(x) = \int_{0}^{x} \sin^n x dx \quad \text{we} \quad G(x) = \int_{0}^{x} \cos^n x dx$$

funksiýalaryň 2π – periodik, jübüt n üçin bolsa ol funksiýalaryň her biriniň çyzykly we periodik funksiýalaryň jemi bolýandygyny subut etmeli.

92. Üznüksiz T periodik $f(x)$ funksiýa üçin

$$F(x) = \int_{a}^{x} f(x)dx$$

funksiýanyň, umumy halda, çyzykly we T periodik funksiýanyň jemi bolýandygyny subut etmeli.

Integrallary hasaplamaly:

93. $\int_{0}^{1} x(2-x^2)^{12} dx.$

94. $\int_{-1}^{1} \frac{xdx}{x^2+x+1}.$

95. $\int_{1}^{e} (x\ln x)^2 dx.$

96. $\int_{1}^{9} x\sqrt[3]{1-x}\, dx.$

97. $\int_{-2}^{-1} \frac{dx}{x\sqrt{x^2-1}}.$

98. $\int_{0}^{1} x^{15}\sqrt{1+3x^8}\, dx.$

99. $\int_0^3 \arcsin\sqrt{\frac{x}{1+x}}\,dx.$

100. $\int_0^{2\pi} \frac{dx}{(2+\cos x)(3+\cos x)}.$

101. $\int_0^{2\pi} \frac{dx}{\sin^4 x + \cos^4 x}.$

102. $\int_0^{\pi/2} \sin x \sin 2x \sin 3x dx.$

103. $\int_0^{\pi} (x\sin x)^2 dx.$

104. $\int_0^{\pi} e^x \cos^2 x dx.$

105. $\int_0^{\ln 2} \mathrm{sh}^4 x dx.$

Bitin položitel bahalary alýan n parametre bagly bolan integrallary, peseltmek formulalarynyň kömegi bilen hasaplamaly:

106. $I_n = \int_0^{\pi/2} \sin^n x dx.$

107. $I_n = \int_0^{\pi/2} \cos^n x dx.$

108. $I_n = \int_0^{\pi/4} \mathrm{tg}^{2n} x dx.$

109. $I_n = \int_0^{1} (1-x^2)^n dx.$

110. $I_n = \int_0^{1} \frac{x^n dx}{\sqrt{1-x^2}}.$

111. $I_n = \int_0^{1} x^m (\ln x)^n dx.$

112. $I_n = \int_0^{\pi/4} \left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)^{2n+1} dx.$

113. $e^{ix} = \cos x + i \sin x$ Eýleriň formulasyny ulanyp, bitin m we n sanlar üçin

$$\int_0^{2\pi} e^{inx} e^{-imx} dx = \begin{cases} 0, & m \neq n, \\ 2\pi, & m = n \end{cases}$$

deňligi subut etmeli.

114. Hemişelik α we β üçin

$$\int_a^b e^{(\alpha + i\beta)x} dx = \frac{e^{b(\alpha+i\beta)} - e^{a(\alpha+i\beta)}}{\alpha + i\beta}$$

deňligi subut etmeli.

Eýleriň $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$, $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$ formulalaryny ulanyp, aşakdaky integrallary hasaplamaly (m we n – bitin položitel sanlar):

115. $\int_0^{\pi/2} \sin^{2m} x \cos^{2n} x dx.$

116. $\int_0^{\pi} \frac{\sin nx}{\sin x} dx.$

117. $\int_0^{\pi} \frac{\cos(2n+1)x}{\cos x} dx.$

118. $\int_0^{\pi} \cos^n x \cos nx dx.$

119. $\int_0^{\pi} \sin^n x \sin nx dx.$

Integrallary tapyň (n – natural san):

120. $\int_0^{\pi} \sin^{n-1} x \cos(n+1)x dx.$

121. $\int_0^{\pi} \cos^{n-1} x \sin(n+1)x dx.$

122. $\int_0^{2\pi} e^{-ax} \cos^{2n} x dx.$

123. $\int_0^{\pi/2} \ln\cos x \cdot \cos 2nx dx.$

124. Bölekleýin integrirlemek usulyny köp gezek ulanyp, Eýleriň

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

integralyny hasaplamaly, bu ýerde m we n – bitin položitel sanlar.

125. Ležandryň $P_n(x)$ köpagzasy

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2-1)^n] \quad (n = 0, 1, 2, ...)$$

formula bilen kesgitlenýär. Deňligi subut etmeli:

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \begin{cases} 0, & \text{eger } m \neq n \text{ bolsa}, \\ \frac{2}{2n+1}, & \text{eger } m = n \text{ bolsa}. \end{cases}$$

126. Goý, $f(x)$ funksiýanyň $[a, b]$ kesimde hususy integraly bar bolsun we $F(x)$ şeýle funksiýa bolup, ol funksiýanyň $[a, b]$ kesimiň 1-nji görnüşdäki tükenikli sany içki c_i ($i = 1, ..., p$) we a, b üzülme nokatlaryndan beýleki ähli nokatlarynda $F'(x) = f(x)$ bolsun. Deňligi subut etmeli:

$$\int_a^b f(x) dx = F(b-0) - F(a+0) - \sum_{i=1}^{p} [F(c_i+0) - F(c_i-0)].$$

127. Goý, $f(x)$ funksiýanyň $[a, b]$ kesimde hususy integraly bar we

$$F(x) = C + \int_a^x f(\xi)d\xi$$

onuň kesgitsiz integraly bolsun. $F(x)$ funksiýanyň üznüksizdigini we $f(x)$ funksiýanyň üznüksiz bolan ähli nokatlarynda $F'(x) = f(x)$ deňligi subut etmeli. $f(x)$ funksiýanyň üzülme nokatlarynda $F(x)$ funksiýanyň önümi barada näme aýtmak bolar? Aşakdaky mysallaryň önümleri barada näme aýtmak bolar?

a) $f\left(\frac{1}{n}\right) = 1$ $(n = \pm 1, \pm 2, ...)$ we $f(x) = 0,\ x \neq \frac{1}{n}$;

b) $f(x) = \operatorname{sgn} x$.

Çäkli üznükli funksiýalaryň kesgitsiz integrallaryny tapmaly:

128. $\int \operatorname{sgn} x dx$.

129. $\int \operatorname{sgn}(\sin x)dx$.

130. $\int [x]dx \quad (x \geq 0)$.

131. $\int x[x]dx \quad (x \geq 0)$.

132. $\int (-1)^{[x]}dx$.

133. $\int_0^x f(x)dx$, bu ýerde $f(x) = \begin{cases} 1, & |x| < l \text{ bolanda,} \\ 0, & |x| > l \text{ bolanda.} \end{cases}$

Çäkli üznükli funksiýalaryň kesgitli integrallaryny hasaplamaly:

134. $\int_0^3 \operatorname{sgn}(x - x^3)dx$.

135. $\int_0^2 [e^x]\,dx$.

136. $\int_0^6 [x]\sin\frac{\pi x}{6}dx$.

137. $\int_0^x x\operatorname{sgn}(\cos x)dx$.

138. $\int_1^{n+1} \ln[x]\,dx$, bu ýerde n – natural san.

139. $\int_0^1 \operatorname{sgn}[\sin(\ln x)]\,dx$.

140. $\int_E |\cos x|\sqrt{\sin x}\,dx$ integraly tapmaly, bu ýerde E köplük $[0, 4\pi]$ kesimiň integral astyndaky aňlatmanyň manyly köplügi.

§2. Orta baha hakyndaky teoremalar

1. ***Funksiýanyň orta bahasy.*** $[a, b]$ kesimde integrirlenýän f funksiýa üçin

$$\mu(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

sana f funksiýanyň $[a, b]$ kesimdäki orta bahasy diýilýär. Eger f funksiýa $[a, b]$ kesimde üznüksiz bolsa, onda $[a, b] \ni c$ nokat tapylyp, $\mu(f) = f(c)$ bolar.

2. ***Orta baha hakyndaky birinji teorema.*** Eger f we g funksiýalar $[a, b]$ kesimde integrirlenýän bolup, $\forall x \in [a, b]$ üçin $g(x) \geq 0$ ýa-da $g(x) \leq 0$ bolsa, onda $m = \inf\limits_{a \leq x \leq b} f(x)$ we $M = \sup\limits_{a \leq x \leq b} f(x)$ sanlar üçin $m \leq \mu \leq M$ şerti kanagatlandyrýan μ san tapylyp,

$$\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx$$

deňlik ýerine ýetýär. Hususan-da, eger f funksiýa $[a, b]$ kesimde üznüksiz bolsa, onda $[a, b] \ni c$ nokat tapylyp, $\mu = f(c)$ bolar.

3. ***Orta baha hakyndaky ikinji teorema.*** Eger f we g funksiýalar $[a, b]$ kesimde integrirlenýän bolup, g funksiýa $[a, b]$ kesimde monoton bolsa, onda $[a, b] \ni c$ tapylyp,

$$\int_a^b f(x)g(x)dx = g(a)\int_a^c f(x)dx + g(b)\int_c^b f(x)dx$$

formula dogrudyr. Hususan-da, eger g funksiýa $[a, b]$ kesimde otrisatel däl we kemelýän (artýan) bolsa, onda

$$\int_a^b f(x)g(x)dx = g(a)\int_a^c f(x)dx \qquad \left(\int_a^b f(x)g(x)dx = g(b)\int_c^b f(x)dx\right)$$

formula dogrudyr.

Gönükmeler

141. Aşakdaky kesgitli integrallaryň alamatlaryny kesgitlemeli:

a) $\int_0^{2\pi} x\sin x dx$; b) $\int_0^{2\pi} \frac{\sin x}{x}dx$; ç) $\int_{-2}^{2} x^3 2^x dx$; d) $\int_{1/2}^{1} x^2 \ln x dx$.

142. Integrallaryň haýsysy uly:

a) $\int_0^{\frac{\pi}{2}} \sin^{10} x dx$ ýa-da $\int_0^{\frac{\pi}{2}} \sin^2 x dx$;

b) $\int_0^1 e^{-x}dx$ ýa-da $\int_0^1 e^{-x^2}dx$;

ç) $\int_0^{\pi} e^{-x^2}\cos^2 xdx$ ýa-da $\int_{\pi}^{2\pi} e^{-x^2}\cos^2 xdx$?

143. Görkezilen aralyklarda berlen funksiýalaryň orta bahalaryny kesgitlemeli:

a) $f(x) = x^2$ $[0, 1]$;

ç) $f(x) = 10 + 2\sin x + 3\cos x$ $[0, 2\pi]$;

b) $f(x) = \sqrt{x}$ $[0, 100]$;

d) $f(x) = \sin x\sin(x + \varphi)$ $[0, 2\pi]$.

144. $r = \dfrac{p}{1-\varepsilon\cos\varphi}$ $(0 < \varepsilon < 1)$ ellipsiň fokal radius-wektorynyň uzynlygynyň orta bahasyny tapmaly.

145. Başlangyç tizligi ϑ_0 bolan erkin gaçýan jisimiň tizliginiň orta bahasyny tapmaly.

146. Üýtgeýän toguň güýji

$$i = i_0\sin\left(\frac{2\pi t}{T} + \varphi\right)$$

düzgün boýunça üýtgeýär: bu ýerde: i_0 – amplituda, t – wagt, T – period, φ – başlangyç faza. Toguň güýjüniň kwadratynyň orta bahasyny tapmaly.

147. Goý, $f(x) \in C[0, +\infty)$ we $\lim\limits_{x\to+\infty} f(x) = A$ bolsun, onda $\lim\limits_{x\to+\infty}\dfrac{1}{x}\int_0^x f(x)dx$ predeli tapmaly.

$f(x) = \text{arctg}x$ mysala serediň.

148. Goý, $\int_0^x f(t)dt = xf(\theta x)$ bolsun. Onda

a) $f(t) = t^n$ $(n > -1)$;

b) $f(t) = \ln t$;

ç) $f(t) = e^t$

funksiýalar üçin θ-ni tapmaly?

$\lim\limits_{x\to+0}\theta$ we $\lim\limits_{x\to+\infty}\theta$ predeller näçä deň?

Orta baha baradaky 1-nji teoremany ulanyp, integrallary bahalandyrmaly:

149. $\int_0^{2\pi}\dfrac{dx}{1+0{,}5\cos x}$.

150. $\int_0^1 \dfrac{x^9}{\sqrt{1+x}}dx$.

151. $\int\limits_{0}^{100} \frac{e^{-x}}{x+100}dx.$

152. Deňlikleri subut etmeli:

a) $\lim\limits_{n\to\infty}\int\limits_{0}^{1} \frac{x^n}{1+x}dx = 0;$ b) $\lim\limits_{n\to\infty}\int\limits_{0}^{\pi/2} \sin^n x dx = 0.$

153. Tapmaly:

a) $\lim\limits_{\varepsilon\to 0}\int\limits_{0}^{1} \frac{dx}{\varepsilon x^3+1};$ b) $\lim\limits_{\varepsilon\to +0}\int\limits_{a\varepsilon}^{b\varepsilon} f(x)\frac{dx}{x},$

bu ýerde $a > 0$, $b > 0$ we $f(x) \in C[0, 1]$.

154. Goý, $f(x)$ funksiýa $[a, b]$ kesimde üznüksiz we $\varphi(x)$ funksiýa $[a, b]$ kesimde üznüksiz, (a, b) interwalda differensirlenýän bolsun. Şeýle-de, şol interwalda $\varphi'(x) \geq 0$. Bölekleýin integrirleme usulyny peýdalanyp we orta baha baradaky birinji teoremany ulanyp, orta baha baradaky ikinji teoremany subut etmeli.

Orta baha baradaky ikinji teoremany ulanyp, integrallary bahalandyryň:

155. $\int\limits_{100\pi}^{200\pi} \frac{\sin x}{x}dx.$ **156.** $\int\limits_{a}^{b} \frac{e^{-\alpha x}}{x}\sin x dx$ $(\alpha \geq 0;\ 0 < a < b).$

157. $\int\limits_{a}^{b} \sin x^2 dx$ $(0 < a < b).$

158. Goý, $\varphi(x)$ we $\psi(x)$ funksiýalar $[a, b]$ kesimde kwadratlary bilen integrirlenýän bolsun. Koşi-Bunýakowskiniň deňsizligini subut etmeli:

$$\left\{\int\limits_{a}^{b} \varphi(x)\psi(x)dx\right\}^2 \leq \int\limits_{a}^{b} \varphi^2(x)dx \int\limits_{a}^{b} \psi^2(x)dx.$$

159. Goý, $f(x)$ funksiýa $[a, b]$ kesimde üznüksiz differensirlenýän we $f(a) = 0$ bolsun. Deňsizligi subut etmeli:

$$M^2 \leq (b-a)\int\limits_{a}^{b} f'^2(x)dx$$

bu ýerde $M = \sup\limits_{a\leq x\leq b} |f(x)|$.

160. Deňligi subut etmeli:

$$\lim_{n\to\infty}\int\limits_{n}^{n+p} \frac{\sin x}{x}dx = 0 \quad (p > 0).$$

§3. Hususy däl integrallar

1. *Çäksiz aralygyň hususy däl integrallary.* Eger f funksiýa $[a,+\infty)$ aralykda kesgitlenen we $\forall B \in [a, +\infty)$ üçin $[a, B]$ kesimde integrirlenýän bolsa, onda

$$\lim_{B\to+\infty}\int_a^B f(x)dx$$

predele f funksiýanyň $[a, +\infty)$ aralykdaky hususy däl integraly diýilýär:

$$\int_a^{+\infty} f(x)dx = \lim_{B\to+\infty}\int_a^B f(x)dx. \tag{1}$$

Eger f funksiýa $(-\infty, b]$ aralykda kesgitlenen we $\forall A \in (-\infty, b)$ üçin $[A, b]$ kesimde integrirlenýän bolsa, onda $\int_{-\infty}^{b} f(x)dx$ hususy däl integral

$$\int_{-\infty}^{b} f(x)dx = \lim_{A\to-\infty}\int_A^b f(x)dx \tag{2}$$

predel arkaly kesgitlenýär.

(1) we (2) hususy däl integrallaryň degişlilikde ýokarky we aşaky çäklerine ol integrallaryň aýratyn nokatlary diýilýär.

Eger-de f funksiýa $(-\infty,+\infty)$ aralykda kesgitlenen we $\forall A,B\in(-\infty,+\infty)$ üçin $[A, B]$ kesimde integrirlenýän bolsa, onda $\int_{-\infty}^{+\infty} f(x)dx$ hususy däl integral

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{\substack{A\to-\infty\\ B\to+\infty}}\int_A^B f(x)dx \tag{3}$$

predel arkaly kesgitlenýär.

1-nji mysal. $I(\alpha) = \int_1^{+\infty}\frac{dx}{x^\alpha}$ integraly α parametre baglylykda derňemeli.

Ç.B. Bu integral (1) görnüşdäki integraldyr. Şonuň üçin ilki bilen aşakdaky integraly hasaplalyň:

$$\int_1^B \frac{dx}{x^\alpha} = \begin{cases} \ln B, & \alpha = 1 \quad \text{bolanda;} \\ \dfrac{B^{1-\alpha}}{1-\alpha} - \dfrac{1}{\alpha-1}, & \alpha \neq 1 \quad \text{bolanda.} \end{cases}$$

Bu deňlikleriň esasynda $\alpha > 1$ bolanda

$$\lim_{B\to+\infty}\int_1^B \frac{dx}{x^\alpha} = \frac{1}{\alpha-1},$$

$\alpha \le 1$ bolanda bolsa

$$\lim_{B \to +\infty} \int_1^B \frac{dx}{x^\alpha} = +\infty.$$

Diýmek, garalýan integral kesgitleme esasynda $a > 1$ bolanda ýygnanýar, $\alpha \le 1$ bolanda bolsa dargaýar. **Ç.S.**

2. *Çäkli aralygyň hususy däl integrallary.* Eger çäkli $[a, b)$ aralykda kesgitlenen f funksiýa b nokadyň käbir golaý töwereginde çäksiz bolup, $\forall A, B \in [a, b)$ üçin $[a, B]$ kesimde integrirlenýän bolsa, onda

$$\lim_{B \to b-o} \int_a^B f(x)dx$$

predele f funksiýanyň $[a, b)$ aralykdaky hususy däl integraly diýilýär:

$$\int_a^b f(x)dx = \lim_{B \to b-o} \int_a^B f(x)dx. \tag{4}$$

Eger $(a, b]$ aralykda kesgitlenen f funksiýa a nokadyň käbir golaý töwereginde çäksiz bolup, $\forall A, B \in (a, b]$ üçin $[A, b]$ kesimde integrirlenýän bolsa, onda f funksiýanyň $(a, b]$ aralykdaky hususy däl integraly

$$\int_a^b f(x)dx = \lim_{A \to a} \int_A^b f(x)dx \tag{5}$$

predel arkaly kesgitlenýär.

2-nji mysal. $I(\alpha) = \int_3^4 \frac{dx}{(x-3)^\alpha}$ integraly α parametre baglylykda derňemeli.

Ç.B. Bu integral (5) görnüşdäki integraldyr. Şonuň üçin, ilki bilen, aşakdaky integraly hasaplalyň:

$$\int_A^4 \frac{dx}{(x-3)^\alpha} = \begin{cases} -\ln|A-3|, & \alpha = 1 \ \text{bolanda;} \\ \dfrac{1}{1-\alpha} - \dfrac{(A-3)^{1-\alpha}}{1-\alpha}, & \alpha \ne 1 \ \text{bolanda.} \end{cases}$$

Bu deňligiň esasynda $\alpha < 1$ bolanda

$$\lim_{A \to 3} \int_A^4 f(x)dx = \frac{1}{1-\alpha},$$

$\alpha \ge 1$ bolanda bolsa bu predel tükeniksizlige deňdir. Diýmek, garalýan integral $\alpha < 1$ bolanda ýygnanýan integral, $\alpha \ge 1$ bolanda bolsa dargaýan integraldyr. **Ç.S.**

(4) we (5) hususy däl integrallaryň degişlilikde ýokarky we aşaky çäklerine ol integrallaryň aýratyn nokatlary diýilýär.

Derňelişi birmeňzeş bolany üçin diňe ýokarky çägi aýratyn nokat bolan hususy däl integrallara garalyň we olary şeýle kesgitläliň.

3. ***Umumy görnüşdäki hususy däl integral.*** Eger f funksiýa $[a, b)$ aralykda kesgitlenen we $\forall B \in [a, b)$ üçin $[a, B]$ kesimde integrirlenýän hem-de b onuň aýratyn nokady ($b = +\infty$ ýa-da b tükenikli bolup, şol nokadyň käbir golaý töwereginde f çäksiz) bolsa, onda $\int_a^b f(x)dx$ hususy däl integral

$$\int_a^b f(x)dx = \lim_{B \to b} \int_a^B f(x)dx \tag{6}$$

predel arkaly kesgitlenýär.

Bu predel bar bolanda integrala ýygnanýan hususy däl integral, beýleki hallarda oňa dargaýan hususy däl integral diýilýär.

4. ***Hususy däl integrallaryň ýygnanma ölçegleri***

1) Koşiniň ölçegleri. $\forall \varepsilon > 0$ üçin $[a, b) \in B$ $(\delta > 0)$ san tapylyp, $\forall B'$, $B'' \in (B, b)$ $(\forall B', B'' \in (b - \delta, b))$ üçin

$$\left| \int_{B'}^{B''} f(x)dx \right| < \varepsilon$$

deňsizligiň ýerine ýetmegi (6) integralyň ýygnanmagy üçin zerur we ýeterlikdir.

2) otrisatel däl funksiýanyň hususy däl integralynyň ýygnanma ölçegleri. Otrisatel däl f funksiýanyň hususy däl (6) integralynyň ýygnanmagy üçin

$$F(B) = \int_a^B f(x)dx$$

funksiýanyň ýokardan çäkli bolmagy zerur we ýeterlikdir.

5. ***Hususy däl integrallaryň ýygnanma nyşanlary***

Deňeşdirme nyşanlary

1. Eger $\forall x \in [a, b)$ üçin $0 \le f(x) \le g(x)$ bolsa, onda $I_1 = \int_a^b g(x)dx$ integralyň ýygnanmagyndan $I = \int_a^b f(x)dx$ integralyň ýygnanmagy we I integralyň dargamagyndan I_1 integralyň dargamagy gelip çykýar.

2. Eger $[a, b)$ aralykda položitel f we g funksiýalar üçin

$$\lim_{x \to b} \frac{f(x)}{g(x)} = k$$

predel bar bolsa, onda

a) $0 \le k < +\infty$ bolanda I_1 integralyň ýygnanmagyndan I integralyň ýygnanmagy gelip çykýar;

b) $0 < k \le +\infty$ bolanda I_1 integralyň dargamagyndan I integralyň dargamagy gelip çykýar;

ç) $0 < k < +\infty$ bolanda I_1 we I integrallar bir wagtda ýygnanýar ýa-da dargaýar.

3. Eger $x \to +\infty$ bolanda $0 \le f(x) \le \frac{C}{x^p}$, $C > 0$ we $p > 1$ bolsa, onda I integral ($b = +\infty$ üçin) ýygnanýar, eger-de $f(x) \le \frac{C}{x^p}$, $C > 0$ we $p \le 1$ bolsa, onda I integral ($b = +\infty$ üçin) dargaýar.

4. Eger $x \to b$ bolanda $0 \le f(x) \le \frac{C}{(x-b)^p}$, $C>0$ we $p<1$ bolsa, onda I integral ýygnanýar, eger-de $f(x) \le \frac{C}{(x-b)^p}$, $C>0$ we $p \ge 1$ bolsa, onda I integral dargaýar.

Abeliň we Dirihläniň nyşanlary:

1. *Abeliň nyşany.* Eger I integral ýygnanýan we g funksiýa $[a, b)$ aralykda monoton we çäkli bolsa, onda

$$\int_a^b f(x)g(x)dx \tag{7}$$

integral ýygnanýar.

2. *Dirihläniň nyşany.* Eger $F(x) = \int_a^x (x)dx$ funksiýa $[a, b)$ aralykda çäkli we g funksiýa şol aralykda monoton we $x \to b$ bolanda nola ymtylýan bolsa, onda (7) integral ýygnanýar.

Bellik. Şular ýaly ölçegler we nyşanlar aşaky çägi aýratyn nokat bolan hususy däl integrallar üçin hem bardyr.

3-nji mysal. $\int_1^3 \frac{dx}{\sqrt{4x - x^2 - 3}}$ integraly derňemeli.

Ç.B. Bu integralyň çükleriniň ikisi hem şol integralyň aýratyn nokatlarydyr. Şonuň üçin ol integraly derňemek üçin ilki ony

$$\int_1^3 \frac{dx}{\sqrt{4x - x^2 - 3}} = \int_1^2 \frac{dx}{\sqrt{4x - x^2 - 3}} + \int_2^3 \frac{dx}{\sqrt{4x - x^2 - 3}}$$

görnüşde ýazyp, olary aýratynlykda derňäliň:

$$\lim_{A\to 1+0}\int_A^2 \frac{dx}{\sqrt{4x-x^2-3}} = \lim_{A\to 1+0}\int_A^2 \frac{d(x-2)}{\sqrt{1-(x-2)^2}} =$$

$$= \lim_{A\to 1+0} \arcsin(x-2)\Big|_A^2 - \lim_{A\to 1+0} \arcsin(A-2) = \frac{\pi}{2},$$

$$\lim_{B\to 3-0}\int_2^B \frac{dx}{\sqrt{4x-x^2-3}} = \lim_{B\to 3-0} \arcsin(x-2)\Big|_2^B = \frac{\pi}{2}.$$

Diýmek, predelleriň ikisi hem bardyr. Şonuň üçin garalýan integral ýygnanýar we ol integral predelleriň jemine deňdir:

$$\int_1^3 \frac{dx}{\sqrt{4x-x^2-3}} = \frac{\pi}{2} + \frac{\pi}{2} = \pi. \quad \textbf{Ç.S.}$$

Çäkli aralyk üçin ýokarda garalan hallarda hususy däl integrallaryň integrirleme çäkleriniň iň bolmanda birisi integralyň aýratyn nokadydyr. Ýöne aýratyn nokadynyň integrirleme aralygyň içinde ýerleşýän haly hem duş gelýär.

Eger aýratyn nokady bolan $\tilde{N} \in (a, b)$ nokatdan başga $[a, b]$ kesimiň ähli nokatlarynda kesgitlenen f funksiýa, $a \le A < C < B \le b$ şerti kanagatlandyrýan $\forall A, B$ üçin $[a, A]$ we $[B, b]$ kesimlerde integrirlenýän bolsa, onda ol funksiýanyň $[a, b]$ kesimdäki hususy däl integraly

$$\int_a^b f(x)dx = \lim_{A\to C-o}\int_a^A f(x)dx + \lim_{B\to C+o}\int_B^b f(x)dx \tag{8}$$

deňlik boýunça kesgitlenýär. Şunlukda, bu predelleriň ikisi hem bar bolanda hususy däl integral ýygnanýar diýilýär we integral

$$\int_a^b f(x)dx = \int_a^C f(x)dx + \int_C^b f(x)dx$$

deňlik boýunça kesgitlenýär.

4-nji mysal. $\displaystyle\int_0^2 \frac{dx}{\sqrt{|1-x^2|}}$ integraly derňemeli.

Bu integralyň aýratyn nokady $C = 1$ nokatdyr. Şonuň üçin ilki aşakdaky predelleri tapalyň:

$$\lim_{A\to 1-0}\int_0^A \frac{dx}{\sqrt{|1-x^2|}} = \lim_{A\to 1-0}\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{A\to 1-0} \arcsin x\Big|_0^A = \frac{\pi}{2},$$

$$\lim_{B\to 1+0}\int_B^2 \frac{dx}{\sqrt{x^2-1}} = \lim_{B\to 1+0} \ln\left(x+\sqrt{x^2-1}\right)\Big|_B^2 = \ln(2+\sqrt{3}).$$

Bu predelleriň barlygy üçin berlen integral bu predelleriň jemi görnüşinde kesgitlenýär:

$$\int_0^2 \frac{dx}{\sqrt{|1-x^2|}} = \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \int_1^2 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} + \ln|2+\sqrt{3}\,|.$$

6. ***Hususy däl integrallaryň baş bahasy.*** Eger aýratyn nokady bolan $\tilde{N} \in (a, b)$ nokatdan başga $[a, b]$ kesimiň ähli nokatlarynda kesgitlenen f funksiýa, $a \leq A < C < B \leq b$ şerti kanagatlandyrýan $\forall A$, B üçin $[a, A]$ we $[B, b]$ kesimlerde integrirlenýän bolsa, onda ol funksiýanyň $\int_a^b f(x)dx$ hususy däl integralynyň dargaýan halynda $\forall \delta > 0$ üçin

$$\lim_{\delta \to 0}\left[\int_a^{\tilde{N}-\delta} f(x)dx + \int_{\tilde{N}+\delta}^{b} f(x)dx\right]$$

predeli bar bolsa, onda şol predele hususy däl integralyň baş bahasy diýilýär we $V.P. \int_a^b f(x)dx$ bilen belgilenýär, ýagny

$$V.P. \int_a^b f(x)dx = \lim_{\delta \to 0}\left[\int_a^{\tilde{N}-\delta} f(x)dx + \int_{\tilde{N}+\delta}^{b} f(x)dx\right].$$

Şoňa meňzeşlikde, dargaýan $\int_{-\infty}^{+\infty} f(x)dx$ hususy däl integral üçin eger $[-A, A] \subset (-\infty, +\infty)$ bolanda $\lim\limits_{A \to +\infty} \int_{-A}^{A} f(x)dx$ predel bar bolsa, onda şol predele $\int_{-\infty}^{+\infty} f(x)dx$ hususy däl integralyň baş bahasy diýilýär we $V.P. \int_{-\infty}^{+\infty} f(x)dx$ bilen belgilenýär. Şeýlelikde,

$$V.P. \int_{-\infty}^{+\infty} f(x)dx = \lim_{A \to +\infty} \int_{-A}^{A} f(x)dx.$$

5-nji mysal. $\int_{-\infty}^{+\infty} \frac{1+x}{1+x^2}dx$ hususy däl integralyň baş bahasyny tapmaly.

Ç.B. Kesgitleme boýunça

$$V.P.\int_{-\infty}^{+\infty}\frac{1+x}{1+x^2}dx=\lim_{A\to+\infty}\int_{-A}^{+A}\frac{1+x}{1+x^2}dx=\lim_{A\to+\infty}\left(\operatorname{arctg}x+\frac{1}{2}\ln(1+x^2)\right)\Big|_{-A}^{A}=$$

$$=\lim_{A\to+\infty}\left(\operatorname{arctg}A-\operatorname{arctg}(-A)+\frac{1}{2}\ln\frac{1+A^2}{1+A^2}\right)=2\lim_{A\to+\infty}\operatorname{arctg}A=\pi \quad \textbf{Ç.S.}$$

Gönükmeler

Integrallary hasaplamaly:

161. $\int_{a}^{+\infty}\frac{dx}{x^2}$ $(a>0)$.

162. $\int_{0}^{1}\ln x dx$.

163. $\int_{-\infty}^{+\infty}\frac{dx}{1+x^2}$.

164. $\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}$.

165. $\int_{2}^{+\infty}\frac{dx}{x^2+x-2}$.

166. $\int_{-\infty}^{+\infty}\frac{dx}{(x^2+x+1)^2}$.

167. $\int_{0}^{+\infty}\frac{dx}{1+x^3}$.

168. $\int_{0}^{+\infty}\frac{x^2+1}{x^4+1}dx$.

169. $\int_{0}^{1}\frac{dx}{(2-x)\sqrt{1-x}}$.

170. $\int_{1}^{+\infty}\frac{dx}{x\sqrt{1+x^5+x^{10}}}$.

171. $\int_{0}^{+\infty}\frac{x\ln x}{(1+x^2)^2}dx$.

172. $\int_{0}^{+\infty}\frac{\operatorname{arctg}x}{(1+x^2)^{3/2}}dx$.

173. $\int_{0}^{+\infty}e^{-ax}\cos bx dx$ $(a>0)$.

174. $\int_{0}^{+\infty}e^{-ax}\sin bx dx$ $(a>0)$.

Peseltmek formulalarynyň kömegi bilen aşakdaky hususy däl integrallary hasaplamaly (n – natural san):

175. $I_n=\int_{0}^{+\infty}x^n e^{-x}dx$.

176. $I_n=\int_{-\infty}^{+\infty}\frac{dx}{(ax^2+2bx+c)^n}$ $(ac-b^2>0)$.

177. $I_n=\int_{1}^{+\infty}\frac{dx}{x(x+1)...(x+n)}$.

178. $I_n=\int_{0}^{1}\frac{x^n dx}{\sqrt{(1-x)(1+x)}}$.

179. $I_n = \int\limits_0^{+\infty} \frac{dx}{\mathrm{ch}^{n+1} x}$.

180. a) $\int\limits_0^{\pi/2} \ln \sin x dx$. b) $\int\limits_0^{\pi/2} \ln \cos x dx$.

181. $\int\limits_E e^{-\frac{x}{2}} \frac{|\sin x - \cos x|}{\sqrt{\sin x}} dx$ integraly tapmaly, bu ýerde E $(0, +\infty)$ köplük interwalyň integral astyndaky aňlatmanyň manyly nokatlarynyň köplügi.

182. Deňligi subut etmeli:

$$\int\limits_0^{+\infty} f\left(ax + \frac{b}{x}\right) dx = \frac{1}{a} \int\limits_0^{+\infty} f\left(\sqrt{x^2 + 4ab}\right) dx,$$

bu ýerde $a > 0$, $b > 0$ we deňligiň çep bölegindäki integral bar hasap edilýär.

183. $\mu(f) = \lim\limits_{x \to +\infty} \frac{1}{x} \int\limits_0^x f(t) dt$ sana $f(x)$ funksiýanyň $(0, +\infty)$ interwaldaky orta bahasy diýilýär.

Funksiýalaryň orta bahalaryny tapmaly:

a) $f(x) = \sin^2 x + \cos^2 (x\sqrt{2})$;

b) $f(x) = \mathrm{arctg} x$;

ç) $f(x) = \sqrt{x} \sin x$.

184. Tapmaly:

a) $\lim\limits_{x \to 0} x \int\limits_x^1 \frac{\cos t}{t^2} dt$;

ç) $\lim\limits_{x \to 0} \frac{\int\limits_x^{+\infty} t^{-1} e^{-t} dt}{\ln \frac{1}{x}}$;

b) $\lim\limits_{x \to \infty} \frac{\int\limits_0^x \sqrt{1 + t^4}\, dt}{x^3}$;

d) $\lim\limits_{x \to 0} x^{\alpha} \int\limits_x^1 \frac{f(t)}{t^{\alpha+1}} dt$

bu ýerde $\alpha > 0$ we $f(t)$ funksiýa $[0, 1]$ kesimde üznüksiz.

Integrallaryň ýygnanmagyny derňemeli:

185. $\int\limits_0^{+\infty} \frac{x^2 dx}{x^4 - x^2 + 1}$.

186. $\int\limits_1^{+\infty} \frac{dx}{x^3 \sqrt{x^2 + 1}}$.

187. $\int_0^2 \frac{dx}{\ln x}$.

188. $\int_0^{+\infty} x^{p-1}e^{-x}dx$.

189. $\int_0^1 x^p \ln^q \frac{1}{x}dx$.

190. $\int_0^{+\infty} \frac{x^m}{1+x^n}dx \quad (n \geq 0)$.

191. $\int_0^{+\infty} \frac{\operatorname{arctg} ax}{x^n}dx \quad (a \neq 0)$.

192. $\int_0^{+\infty} \frac{\ln(1+x)}{x^n}dx$.

193. $\int_0^{+\infty} \frac{x^m \operatorname{arctg} x}{2+x^n}dx \quad (n \geq 0)$.

194. $\int_0^{+\infty} \frac{\cos ax}{1+x^n} \quad (n \geq 0)$.

195. $\int_0^{+\infty} \frac{\sin^2 x}{x}dx$.

196. $\int_0^{\pi/2} \frac{dx}{\sin^p x \cos^q x}$.

197. $\int_0^1 \frac{x^n dx}{\sqrt{1-x^4}}$.

198. $\int_0^{+\infty} \frac{dx}{\sqrt{x^3+x}}$.

199. $\int_0^{+\infty} \frac{dx}{x^p + x^q}$.

200. $\int_0^1 \frac{\ln x}{1-x^2}dx$.

201. $\int_0^{\pi/2} \frac{\ln(\sin x)}{\sqrt{x}}dx$.

202. $\int_1^{+\infty} \frac{dx}{x^p \ln^q x}$.

203. $\int_e^{+\infty} \frac{dx}{x^p (\ln x)^q (\ln\ln x)^r}$.

204. $\int_{-\infty}^{+\infty} \frac{dx}{|x-a_1|^{p_1}|x-a_2|^{p_2}...|x-a_n|^{p_n}} \quad (a_1 < a_2 < ... < a_n)$.

205. $\int_0^{+\infty} x^\alpha |x-1|^\beta dx$.

206. $\int_0^{+\infty} \frac{P_m(x)}{P_n(x)}dx$, bu ýerde $P_m(x)$ we $P_n(x)$ degişlilikde m we n derejeli özara ýönekeý köpagzalar.

Aşakdaky integrallaryň absolýut we şertli ýygnanmaklaryny derňemeli:

207. $\int_0^{+\infty} \frac{\sin x}{x}dx$ (*Görkezme*: $|\sin x| \geq \sin^2 x$).

208. $\displaystyle\int_0^{+\infty} \frac{\sqrt{x}\cos x}{x+100}dx.$

209. $\displaystyle\int_0^{+\infty} x^p \sin(x^q)dx \quad (q \neq 0).$

210. $\displaystyle\int_0^{\pi/2} \sin(\sec x)dx.$

211. $\displaystyle\int_0^{+\infty} x^2 \cos(e^x)dx.$

212. $\displaystyle\int_0^{+\infty} \frac{x^p \sin x}{1+x^q}dx \quad (q \geq 0).$

213. $\displaystyle\int_0^{+\infty} \frac{\sin\left(x+\frac{1}{x}\right)}{x^n}dx.$

214. $\displaystyle\int_a^{+\infty} \frac{P_m(x)}{P_n(x)}\sin x dx,$ bu ýerde $P_m(x)$ we $P_n(x)$ – bitin köpagzalar we $x \geq a \geq 0$ bolanda $P_n(x) > 0$.

215. Eger $\displaystyle\int_a^{+\infty} f(x)dx$ ýygnanýan bolsa, onda $x \to +\infty$ bolanda $f(x) \to 0$ bolmagy hökmanmy:

a) $\displaystyle\int_0^{+\infty} \sin(x^2)dx;$ b) $\displaystyle\int_0^{+\infty} (-1)^{[x^2]}dx?$

216. Goý, $f(x) \in C^{(1)}[a, +\infty)$, $a \leq x < +\infty$ bolanda $|f'(x)| < C$ we $\displaystyle\int_a^{+\infty} |f(x)|dx$ integral ýygnanýan bolsun. $x \to +\infty$ bolanda $f(x) \to 0$ bolýandygyny subut etmeli.

(*Görkezme:* $\displaystyle\int_a^{+\infty} f(x)f'(x)dx$ *integrala seret*).

217. $[a, b]$ kesimde kesgitlenen çäksiz $f(x)$ funksiýanyň ýygnanýan hususy däl $\displaystyle\int_a^b f(x)dx$ integralyna şol funksiýanyň

$$\int_{i=1}^{n} f(t_i)\Delta x_i \quad (x_{i-1} \leq t_i \leq x_i, \quad \Delta x_i = x_i - x_{i-1})$$

integral jeminiň predeli hökmünde garamak bolarmy?

218. Goý, $\displaystyle\int_a^{+\infty} f(x)dx$ integral ýygnanýan we $\varphi(x)$ funksiýa çäkli bolsun. Onda

$$\int_a^{+\infty} f(x)\varphi(x)dx \tag{1}$$

integral hökman ýygnanýarmy? Degişli mysal getiriň.

Eger $\int\limits_{a}^{+\infty} f(x)dx$ integral absolýut ýygnanýan bolsa, onda (1) integralyň ýygnan-magy barada näme aýtmak bolar?

219. Eger $\int\limits_{a}^{+\infty} f(x)dx$ integral ýygnanýan we $f(x)$ monoton funksiýa bolsa, onda $f(x) = O\left(\frac{1}{x}\right)$ bolýandygyny subut etmeli.

220. Goý, $f(x)$ funksiýa $0 < x \leq 1$ aralykda monoton we $x = 0$ nokadyň golaý töwereginde çäksiz bolsun. Eger $\int\limits_{0}^{1} f(x)dx$ integral bar bolsa, onda

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int\limits_{0}^{1} f(x)dx$$

deňligi subut etmeli.

221. Eger $f(x)$ funksiýa $0 < x < a$ interwalda monoton we

$$\int\limits_{0}^{a} x^p f(x)dx$$

hususy däl integral bar bolsa, onda

$$\lim_{x\to+0} x^{p+1} f(x) = 0$$

deňligi subut etmeli.

222. Deňlikleri subut etmeli:

a) $V.P.\int\limits_{-1}^{1}\frac{dx}{x} = 0;$ b) $V.P.\int\limits_{0}^{+\infty}\frac{dx}{1-x^2} = 0;$ ç) $V.P.\int\limits_{-\infty}^{+\infty}\sin x dx = 0.$

223. $x \geq 0$ bolanda $lix = V.P.\int\limits_{0}^{x}\frac{d\xi}{\ln\xi}$ bardygyny subut etmeli.

Aşakdaky integrallary tapmaly:

224. $V.P.\int\limits_{0}^{+\infty}\frac{dx}{x^2-3x+2}.$

225. $V.P.\int\limits_{1/2}^{2}\frac{dx}{x\ln x}.$

226. $V.P.\int\limits_{-\infty}^{+\infty}\frac{1+x}{1+x^2}dx.$

227. $V.P.\int\limits_{-\infty}^{+\infty}\operatorname{arctg} x dx.$

§4. Kesgitli integrallaryň geometriýada ulanylyşlary

1. ***Tekiz figuranyň meýdany.*** Ýokarsyndan we aşagyndan degişlilikde $[a, b]$ kesimde üznüksiz $y = f_2(x)$ we $y = f_1(x)$ funksiýalaryň grafikleri, çepinden we sagyndan $x = a$ we $x = b$ $(a < b)$ göni çyzyklar bilen çäklenen tekiz figuranyň (*16-njy surat*) meýdany

$$S = \int_a^b [f_2(x) - f_1(x)]dx \qquad (1)$$

formula boýunça tapylýar.

Hususan-da, ýokarsyndan $[a, b]$ kesimde üznüksiz $y = f(x)$ funksiýanyň grafigi bilen çäklenen egri çyzykly trapesiýanyň meýdany (*17-nji surat*)

$$S = \int_a^b f(x)dx$$

formula boýunça tapylýar.

Eger egri çyzykly trapesiýany ýokarsyndan çäklendirýän çyzyk

$$x = \varphi(t), \quad y = \psi(t), \quad \alpha \le t \le \beta, \quad (a = \varphi(a) \le \varphi(t) \le \varphi(\beta) = b)$$

parametrik görnüşde berlen bolsa, onda onuň meýdany

$$S = \int_\alpha^\beta \psi(t)\varphi'(t)dt$$

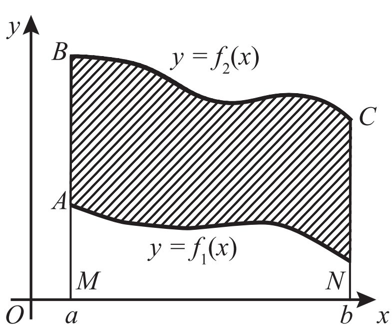
formula boýunça tapylýar.

Eger tekiz figura parametrik görnüşde berlen

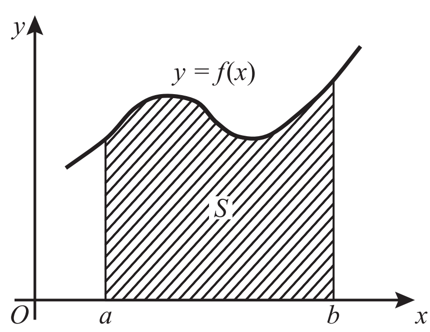
$$x = \varphi(t), \quad y = \psi(t), \quad (t_1 \le t \le t_2), \quad x(t_1) = x(t_2), \quad y(t_1) = y(t_2)$$

egri çyzyk bilen çäklenen bolsa, onda ol figuranyň meýdany

$$S = -\int_{t_1}^{t_2} \psi(t)\varphi'(t)dt, \quad S = \int_{t_1}^{t_2} \varphi(t)\psi'(t)dt$$



16-njy surat



17-nji surat

formulalar boýunça tapylýar. Olardan bolsa meýdan üçin

$$S = \frac{1}{2}\int_{t_1}^{t_2}[\varphi(t)\psi'(t) - \psi(t)\varphi'(t)]dt$$

formula alynýar.

2. *Egri çyzykly sektoryň meýdany.* Polýar koordinatalarynda $[\alpha, \beta]$ kesimde üznüksiz $\rho = \rho(\theta)$ funksiýanyň grafigi we polýar oky bilen α hem β burçlaryny emele getirýän şöhleler bilen çäklenen egri çyzykly sektoryň (*18-nji surat*) meýdany

$$S = \frac{1}{2}\int_{\alpha}^{\beta}\rho^2(\theta)d\theta \qquad (10)$$

formula boýunça tapylýar.

Gönükmeler

228. Göni parabolik segmentiň meýdanynyň

$$S = \frac{2}{3}bh$$

formula boýunça tapylýandygyny subut etmeli, bu ýerde b segmentiň esasy, h onuň beýikligi (*19-njy surat*).

Göni burçly dekart koordinatalarynda berlen çyzyklar bilen çäklenen figuralaryň meýdanlaryny tapmaly (parametrleri položiteldir):

229. $ax = y^2, \quad ay = x^2.$

230. $y = x^2, \quad x + y = 2.$

231. $y = 2x - x^2, \quad x + y = 0.$

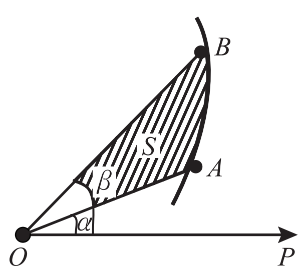
232. $y = |\lg x|, \quad y = 0, \quad x = 0{,}1, \quad x = 10.$

233. $y = 2^x, \quad y = 2, \quad x = 0.$

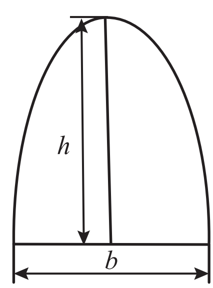
234. $y = (x + 1)^2, x = \sin\pi y, y = 0 \quad (0 \le y \le 1).$

235. $y = x, y = x + \sin^2 x \ (0 \le x \le \pi).$

236. $y = \dfrac{a^3}{a^2 + x^2}, \ y = 0.$



18-nji surat



19-njy surat

237. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

238. $y^2 = x^2(a^2 - x^2).$

239. $y^2 = 2px,\ 27py^2 = 8(x - p)^3.$

240. $Ax^2 + 2Bxy + Cy^2 = 1\quad (A > 1,\ AC - B^2 > 0).$

241. $y^2 = \dfrac{x^3}{2a - x},\ x = 2a$ (sissoida).

242. $x = a\ln\dfrac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2},\ y = 0$ (traktrisa).

243. $y^2 = \dfrac{x^n}{(1 + x^{n+2})^2}\quad (x > 0;\ n > -2).$

244. $y = e^{-x}|\sin x|,\ y = 0\quad (x \geq 0).$

245. $y^2 = 2x$ parabola $x^2 + y^2 = 8$ tegelegiň meýdanyny haýsy gatnaşykda bölýär?

246. $x^2 - y^2 = 1$ giperbolanyň $M(x, y)$ koordinatasyny giperbolanyň $M'M$ dugasy we iki OM we OM' şöhleler bilen çäklenen giperbolik $S = OM'M$ sektoryň meýdanynyň funksiýasy hökmünde aňlatmaly, bu ýerde $M'(x, -y)$ nokat Ox okuna görä M nokada simmetrik nokatdyr.

Parametrik görnüşde berlen çyzyklar bilen çäklenen figuralaryň meýdanlaryny tapmaly (parametrleriň hemmesi položiteldir):

247. $x = a(t - \sin t),\ y = a(1 - \cos t)\ (0 \leq t \leq 2\pi)$ we $y = 0.$

248. $x = 2t - t^2,\ y = 2t^2 - t^3.$

249. $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t)\ (0 \leq t \leq 2\pi)$ (tegelegiň ýazgyny) we $x = a,\ y \leq 0.$

250. $x = a(2\cos t - \cos 2t),\ y = a(2\sin t - \sin 2t).$

251. $x = \dfrac{c^2}{a}\cos^3 t,\ \ y = \dfrac{c^2}{b}\sin^3 t\ (c^2 = a^2 - b^2)$ (ellipsiň ewolýutasy).

252. $x = a\cos t,\ y = \dfrac{a\sin^2 t}{2 + \sin t}.$

Polýar koordinatalarynda berlen çyzyklar bilen çäklenen figuralaryň S meýdanlaryny tapmaly:

253. $r^2 = a^2\cos 2\varphi$ (lemniskata).

254. $r = a(1 + \cos\varphi)$ (kardioida).

255. $r = a\sin 3\varphi$ (üç ýapraklylyk).

256. $r = \dfrac{p}{1 - \cos\varphi}$ (parabola), $\varphi = \dfrac{\pi}{4}$, $\varphi = \dfrac{\pi}{2}$.

257. $r = \dfrac{p}{1 + \varepsilon\cos\varphi}$ $(0 < \varepsilon < 1)$ (ellips).

258. $r = 3 + 2\cos\varphi$.

259. $r = \dfrac{1}{\varphi}$, $r = \dfrac{1}{\sin\varphi}$ $\left(0 < \varphi \le \dfrac{\pi}{2}\right)$.

260. $r = a\cos\varphi$, $r = a(\cos\varphi + \sin\varphi)$ $(M(a/2, 0) \in S)$.

261. $\varphi = r\operatorname{arctg} r$ çyzyk we $\varphi = 0$ we $\varphi = \pi/\sqrt{3}$ iki şöhle bilen çäklenen sektoryň meýdanyny tapmaly.

262. $r^2 + \varphi^2 = 1$ çyzyk bilen çäklenen figuranyň meýdanyny tapmaly.

263. $\varphi = \sin(\pi r)$ $(0 \le r \le 1)$ çyzyk bilen çäklenen figuranyň meýdanyny tapmaly.

264. $\varphi = 4r - r^3$, $\varphi = 0$ çyzyklar bilen çäklenen figuranyň meýdanyny tapmaly.

265. $\varphi = r - \sin r$, $\varphi = \pi$ çyzyklar bilen çäklenen figuranyň meýdanyny tapmaly.

266. $r = \dfrac{2at}{1 + t^2}$, $\varphi = \dfrac{\pi t}{1 + t}$ ýapyk çyzyk bilen çäklenen figuranyň meýdanyny tapmaly.

Polýar koordinatalaryna geçip, berlen çyzyklar bilen çäklenen figuralaryň meýdanlaryny tapmaly:

267. $x^3 + y^3 = 3axy$ (Dekartyň ýapragy).

268. $x^4 + y^4 = a^2(x^2 + y^2)$.

269. $(x^2 + y^2)^2 = 2a^2xy$ (lemniskata).

Deňlemeleri parametrik görnüşe getirip, berlen çyzyklar bilen çäklenen figuralaryň meýdanlaryny tapmaly:

270. $x^{2/3} + y^{2/3} = a^{2/3}$ (astroida).

271. $x^4 + y^4 = ax^2y$ (*Görkezme*: $y = tx$ *almaly*).

3. *Çyzygyň dugasynyň uzynlygy*

Endigan (üznüksiz differensirlenýän) çyzygyň $y = y(x)$ $(a \le x \le b)$ dugasynyň uzynlygy

$$l = \int_a^b \sqrt{1 + y'^2(x)}\,dx$$

formula boýunça tapylýar.

Eger çyzyk parametrik görnüşde üznüksiz differensirlenýän

$$x = \varphi(t), \quad y = \psi(t) \quad (\alpha \le x \le \beta), \quad (a = \varphi(\alpha) \le \varphi(t) \le \varphi(\beta) = b)$$

funksiýalaryň grafigi hökmünde berlen bolsa, onda duganyň uzynlygy

$$l = \int_{\alpha}^{\beta} \sqrt{\varphi'^2(t) + \psi'^2(t)}\, dt$$

formula boýunça tapylýar.

Eger-de çyzyk polýar koordinatalarynda üznüksiz differensirlenýän $r = r(\varphi)$ $(\alpha \le \varphi \le \beta)$ funksiýanyň grafigi hökmünde berlen bolsa, onda duganyň uzynlygy

$$l = \int_{\alpha}^{\beta} \sqrt{r^2(\varphi) + r'^2(\varphi)}\, d\varphi$$

formula boýunça tapylýar.

Aşakdaky egri çyzyklaryň dugalarynyň uzynlyklaryny tapmaly:

272. $y = x^{3/2} \quad (0 \le x \le 4)$.

273. $y^2 = 2px \ (0 \le x \le x_0)$.

274. $y = a\,\mathrm{ch}\dfrac{x}{a}$ $A(0, a)$ nokatdan $B(b, h)$ nokada çenli.

275. $y = e^x \ (0 \le x \le x_0)$.

276. $x = \dfrac{1}{4}y^2 - \dfrac{1}{2}\ln y \ (1 \le y \le e)$.

277. $y = a\ln\dfrac{a^2}{a^2 - x^2} \ (0 \le x \le b < a)$.

278. $y = \ln\cos x \left(0 \le x \le a < \dfrac{\pi}{2}\right)$.

279. $x = a\ln\dfrac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2} \ (0 < b \le y \le a)$.

280. $y^2 = \dfrac{x^3}{2a - x} \left(0 \le x \le \dfrac{5}{3}a\right)$.

281. $x^{2/3} + y^{2/3} = a^{2/3}$ (astroida).

282. $x = \dfrac{c^2}{a}\cos^3 t, \ y = \dfrac{c^2}{b}\sin^3 t, \ c^2 = a^2 - b^2$ (ellipsiň ewolýutasy).

283. $x = \cos^4 t, \ y = \sin^4 t$.

284. $x = a(t - \sin t), \ y = a(1 - \cos t) \ (0 \le t \le 2\pi)$.

285. $x = a(\cos t + t\sin t), \ y = a(\sin t - t\cos t) \ 0 \le t \le 2\pi$ (töweregiň ýazgyny).

286. $x = a(\mathrm{sh}\, t - t), \ y = a(\mathrm{ch}\, t - 1) \ (0 \le t \le T)$.

287. $x = \mathrm{ch}^3 t, \ y = \mathrm{sh}^3 t, \ (0 \le t \le T)$.

288. $r = a\varphi$ $(0 \leq \varphi \leq 2\pi)$ (Arhimediň spiraly).

289. $r = ae^{m\varphi}$ $(m > 0)$ $(0 < r < a)$.

290. $r = a(1 + \cos\varphi)$.

291. $r = \dfrac{p}{1 + \cos\varphi}$ $\left(|\varphi| \leq \dfrac{\pi}{2}\right)$.

292. $r = a\sin^3 \dfrac{\varphi}{3}$.

293. $r = a\,\mathrm{th}\dfrac{\varphi}{2}$ $(0 \leq \varphi \leq 2\pi)$.

294. $\varphi = \dfrac{1}{2}\left(r + \dfrac{1}{r}\right)$ $(1 \leq r \leq 3)$.

295. $\varphi = \sqrt{r}$ $(0 \leq r \leq 5)$.

296. $\varphi = \displaystyle\int_0^r \frac{\mathrm{sh}\rho}{\rho} d\rho$ $(0 \leq r \leq R)$.

297. $r = 1 + \cos t$, $\varphi = t - \mathrm{tg}\dfrac{t}{2}$ $(0 \leq t \leq T < \pi)$.

298. $x = a\cos t$, $y = b\sin t$ ellipsiň dugasynyň uzynlygynyň $y = c\sin\dfrac{x}{b}$ sinusoidanyň bir tolkunynyň uzynlygyna deňdigini subut etmeli, bu ýerde $c = \sqrt{a^2 - b^2}$.

299. $4ay = x^2$ parabola Ox oky boýunça typylýar. Parabolanyň fokusynyň zynjyr çyzygyny emele getirýändigini subut etmeli.

300. $y = \pm\left(\dfrac{1}{3} - x\right)\sqrt{x}$ çyzyk bilen çäklenen meýdanyň töwereginiň uzynlygy şol çyzygyň konturynyň uzynlygyna deň bolan tegelegiň meýdanyna bolan gatnaşygyny tapmaly.

4. *Jisimiň göwrümi.* Eger Ox oky onuň x nokadynda kesýän perpendikulýar tekizligiň jisimiň kesigindäki meýdany üznüksiz $S = S(x)$ $(a \leq x \leq b)$ funksiýa bolsa, onda ol jisimiň göwrümi

$$V = \int_a^b S(x)dx$$

formula boýunça tapylýar.

Eger-de jisim $a \leq x \leq b$, $0 \leq y \leq f(x)$ egri çyzykly trapesiýanyň Ox okunyň daşyndan aýlanmagyndan alynýan bolsa, onda ol jisimiň göwrümi

$$V = \pi\int_a^b f^2(x)dx$$

formula boýunça tapylýar.

Eger-de jisim $[a, b]$ kesimde üznüksiz $y = f_1(x)$ we $y = f_2(x)$ funksiýalar üçin $a \leq x \leq b$, $f_1(x) \leq y \leq f_2(x)$ tekiz figuranyň Ox okunyň daşyndan aýlanmagyndan alynýan bolsa, onda ol jisimiň göwrümi

$$V=\pi\int_a^b\left(f_2^2(x)-f_1^2(x)\right)dx$$

formula boýunça tapylýar.

5. Aýlanma üstüň meýdany. Eger f funksiýa $[a, b]$ kesimde üznüksiz differensirlenip, otrisatel bolmasa, onda $y=f(x)$ $(a\le x\le b)$ çyzygyň Ox okunyň daşyndan aýlanmagyndan emele gelen üstüň meýdany

$$q=2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx$$

formula boýunça tapylýar.

301. Esaslary a we b taraply gönüburçluk, ýokarky gapyrgasy c we beýikligi h bolan üçegiň göwrümini tapmaly.

302. Parallel esaslarynyň taraplary A, B we a, b gönüburçluk bolan ýadygärligiň beýikligi h. Onuň göwrümini tapmaly.

303. Esaslarynyň ýarym oklary A, B we a, b ellips hem-de beýikligi h bolan kesik konusyň göwrümini tapmaly.

304. Esasy S, beýikligi H bolan aýlanma paraboloidiň göwrümini tapmaly.

305. Goý, jisimiň Ox okuna perpendikulýar kesiginiň $S=S(x)$ meýdany kwadratik $S(x)=Ax^2+Bx+C$ $[a\le x\le b]$ düzgün boýunça üýtgeýän bolsun, bu ýerde A, B we C – hemişelik ululyklar. Ol jisimiň göwrüminiň Simpsonyň

$$V=\frac{H}{6}\left[S(a)+4S\left(\frac{a+b}{2}\right)+S(b)\right]$$

formulasy boýunça tapylýandygyny subut etmeli, bu ýerde $H=b-a$.

306. Jisim $M(x, y, z)$ nokatlaryň z rasional bolanda $0\le z\le 1$; $0\le x\le 1$; $0\le y\le 1$ deňsizlikler bilen kesgitlenýän we irrasional bolanda $0\le z\le 1$; $-1\le x\le 0$; $-1\le y\le 0$ deňsizlikler bilen kesgitlenýän nokatlarynyň köplügidir. $\int_0^1 S(z)dz=1$ integral bolsa, ol jisimiň göwrüminiň ýokdugyny subut etmeli.

Berlen üstler bilen çäklenen jisimleriň göwrümlerini tapmaly:

307. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1,\ z=\dfrac{c}{a}x,\ z=0.$

308. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ (ellipsoid).

309. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1,\ z=\pm c.$

310. $x^2+z^2=a^2,\ y^2+z^2=a^2.$

311. $x^2 + y^2 + z^2 = a^2, x^2 + y^2 = ax.$

312. $z^2 = b(a - x), x^2 + y^2 = ax.$

313. $\frac{x^2}{a^2} + \frac{y^2}{z^2} = 1 \ (0 < z < a).$

314. $x + y + z^2 = 1, x = 0, y = 0, z = 0.$

315. $x^2 + y^2 + z^2 + xy + yz + zx = a^2.$

316. $a \le x \le b, 0 \le y \le y(x)$ meýdanyň Oy okunyň daşyndan aýlanmagyndan emele gelen jisimiň göwrüminiň

$$V_y = 2\pi \int_a^b xy(x)dx$$

formula boýunça tapylýandygyny subut etmeli, bu ýerde $y(x)$ birbahaly üznüksiz funksiýadyr.

Berlen çyzyklaryň görkezilen oklaryň daşyndan aýlanmagyndan alynýan üstler bilen çäklenen jisimleriň göwrümlerini tapmaly:

317. $y = b\left(\frac{x}{a}\right)^{2/3} \ (0 \le x \le a)$, Ox okunyň daşyndan (neýloid).

318. $y = 2x - x^2, y = 0$: a) Ox okunyň daşyndan; b) Oy okunyň daşyndan.

319. $y = \sin x, y = 0 \ (0 \le x \le \pi)$: a) Ox okunyň daşyndan; b) Oy okunyň daşyndan.

320. $y = b\left(\frac{x}{a}\right)^2, \ y = b\left|\frac{x}{a}\right|$: a) Ox okunyň daşyndan; b) Oy okunyň daşyndan.

321. $y = e^{-x}, y = 0 \ (0 \le x < +\infty)$: a) Ox okunyň daşyndan; b) Oy okunyň daşyndan.

322. $x^2 + (y - b)^2 = a^2 \ (0 < a \le b)$, Ox okunyň daşyndan.

323. $x^2 - xy + y^2 = a^2$, Ox okunyň daşyndan.

324. $y = e^{-x}\sqrt{\sin x} \ (0 \le x < +\infty)$, Ox okunyň daşyndan.

325. $x = a(t - \sin t), y = a(1 - \cos t) \ (0 \le t \le 2\pi), y = 0$: a) Ox okunyň daşyndan; b) Oy okunyň daşyndan; ç) $y = 2a$ göni çyzygyň daşyndan.

326. $x = a\sin^3 t, y = b\cos^3 t \ (0 \le t \le 2\pi)$: a) Ox okunyň daşyndan; b) Oy okunyň daşyndan.

327. $x = 2t - t^2, y = 4t - t^3$ çyzyklar bilen çäklenen meýdanyň a) Ox okunyň daşyndan; b) Oy okunyň daşyndan aýlanmagyndan alynýan jisimiň göwrümini tapmaly.

328. Polýar okunyň daşyndan $0 \le \alpha \le \varphi \le \beta \le \pi, 0 \le r \le r(\varphi)$ meýdanyň aýlanmagyndan alynýan jisimiň göwrüminiň

$$V = \frac{2\pi}{3}\int_{\alpha}^{\beta} r^3(\varphi)\sin\varphi d\varphi$$

formula boýunça tapylýandygyny subut etmeli, bu ýerde φ, r – polýar koordinatalary.

Polýar koordinatalarynda berlen meýdanlaryň görkezilen oklaryň daşyndan aýlanmagyndan alynýan jisimleriň göwrümlerini tapmaly:

329. $r = a(1 + \cos\varphi)$ $(0 \le \varphi \le 2\pi)$: a) polýar okunyň daşyndan; b) $r\cos\varphi = -\frac{a}{4}$ göni çyzygyň daşyndan.

330. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$: a) Ox okunyň daşyndan; b) Oy okunyň daşyndan; ç) $y = x$ göni çyzygyň daşyndan. (*Görkezme*: *Polýar koordinatalaryna geçmeli*).

331. $r = a\varphi$ $(a > 0;\ 0 \le \varphi \le \pi)$ Arhimediň spiralynyň ýarym aýlawy bilen çäklenen meýdanyň polýar okunyň daşyndan aýlanmagyndan alynýan jisimiň göwrümini tapmaly.

332. $\varphi = \pi r^3$, $\varphi = \pi$ çyzyklar bilen çäklenen meýdanyň polýar okunyň daşyndan aýlanmagyndan alynýan jisimiň göwrümini tapmaly.

333. $a \le r \le a\sqrt{2\sin 2\varphi}$ meýdanyň polýar okunyň daşyndan aýlanmagyndan alynýan jisimiň göwrümini tapmaly.

Berlen çyzyklaryň görkezilen oklaryň daşyndan aýlanmagyndan alynýan üstleriň meýdanlaryny tapmaly:

334. $y = x\sqrt{\frac{x}{a}}$ $(0 \le x \le a)$, Ox okunyň daşyndan.

335. $y = a\cos\frac{\pi x}{2b}$ $(|x| \le b)$, Ox okunyň daşyndan.

336. $y = \mathrm{tg}x$ $\left(0 \le x \le \frac{\pi}{4}\right)$, Ox okunyň daşyndan.

337. $y^2 = 2px$ $(0 \le x \le x_0)$: a) Ox okunyň daşyndan; b) Oy okunyň daşyndan.

338. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(0 < b \le a)$: a) Ox okunyň daşyndan; b) Oy okunyň daşyndan.

339. $x^2 + (y - b)^2 = a^2$ $(b \ge a)$, Ox okunyň daşyndan.

340. $x^{2/3} + y^{2/3} = a^{2/3}$, Ox okunyň daşyndan.

341. $y = a\,\mathrm{ch}\frac{x}{a}$ $(|x| \le b)$: a) Ox okunyň daşyndan; b) Oy okunyň daşyndan.

342. $\pm x = a\ln\frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}$, Ox okunyň daşyndan.

343. $x = a(t - \sin t)$, $y = a(1 - \cos t)$ $(0 \le t \le 2\pi)$: a) Ox okunyň daşyndan; b) Oy okunyň daşyndan; ç) $y = 2a$ göni çyzygyň daşyndan.

344. $x = a\cos^3 t$, $y = a\sin^3 t$, $y = x$ göni çyzygyň daşyndan.

345. $r = a(1 + \cos\varphi)$, polýar okunyň daşyndan.

346. $r^2 = a^2\cos 2\varphi$: a) polýar okunyň daşyndan; b) $\varphi = \pi/2$ okuň daşyndan; ç) $\varphi = \pi/4$ okuň daşyndan.

347. Jisim $ay = a^2 - x^2$ parabola we Ox oky bilen çäklenen figuranyň Ox okunyň daşyndan aýlanmagyndan alynýar. Aýlanma jisiminiň üstüniň deňululykly şaryň üstüne bolan gatnaşygyny tapmaly.

348. $y^2 = 2px$ parabola we $x = p/2$ göni çyzyk bilen çäklenen figura $y = p$ göni çyzygyň daşyndan aýlanýar. Aýlanma jisiminiň göwrümini we üstüni tapmaly.

§5. Kesgitli integrallaryň fizikada ulanylyşlary

Momentleriň we agyrlyk merkeziniň koordinatalarynyň hasaplanylyşy.

Birjynsly material $y = f(x)$ $(a \le x \le b)$ çyzygyň Ox we Oy oklaryna görä statiki momentleri

$$M_x = \int_a^b f(x)\sqrt{1 + f'^2(x)}\,dx, \qquad M_y = \int_a^b x\sqrt{1 + f'^2(x)}\,dx$$

formulalar boýunça, inersiýa momentleri bolsa

$$I_x = \int_a^b f^2(x)\sqrt{1 + f'^2(x)}\,dx, \qquad I_y = \int_a^b x^2\sqrt{1 + f'^2(x)}\,dx$$

formulalar boýunça tapylýar. Massasy we agyrlyk merkeziniň koordinatalary

$$m = \int_a^b \sqrt{1 + f'^2(x)}\,dx, \qquad x_c = \frac{M_y}{m} = \frac{\int_a^b x^2\sqrt{1 + f'^2(x)}\,dx}{\int_a^b \sqrt{1 + f'^2(x)}\,dx},$$

$$y_c = \frac{M_x}{m} = \frac{\int_a^b f(x)\sqrt{1 + f'^2(x)}\,dx}{\int_a^b \sqrt{1 + f'^2(x)}\,dx}$$

formulalar boýunça tapylýar.

Egri çyzykly trapesiýa görnüşindäki birjynsly tekiz material figuranyň Ox we Oy oklaryna görä statiki momentleri

$$M_x = \frac{1}{2}\int_a^b f^2(x)dx, \quad M_y = \int_a^b xf(x)dx$$

formulalar boýunça, massasy we agyrlyk merkeziniň koordinatalary

$$m = \int_a^b f(x)dx, \quad x_c = \frac{M_y}{m} = \frac{\int_a^b xf(x)dx}{\int_a^b f(x)dx}, \quad y_c = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b f^2(x)dx}{\int_a^b f(x)dx}$$

formulalar boýunça tapylýar.

Gönükmeler

349. Radiusy a bolan ýarym töweregiň dugasynyň şol duganyň uçlary arkaly geçýän diametre görä statiki we inersiýa momentlerini tapmaly.

350. $y^2 = 2px$ $(0 \le x \le p/2)$ parabolanyň dugasynyň $x = p/2$ göni çyzyga görä statiki momentini tapmaly.

351. Esasy b we beýikligi h bolan birjynsly üçburçly plastinanyň esasyna görä statiki we inersiýa momentlerini tapmaly ($\rho = 1$).

352. $ay = 2ax - x^2$ $(a > 0)$ we $y = 0$ çyzyklar bilen çäklenen parabolik segmentiň Ox we Oy oklaryna görä $I_x = M_2^{(x)}$ we $I_y = M_2^{(y)}$ inersiýa momentlerini tapmaly.

Inersiýanyň r_x we r_y radiuslary, ýagny

$$I_x = Sr_x^2, \quad I_y = Sr_y^2$$

deňlik boýunça kesgitlenýän ululyklar nämä deň? Bu ýerde S segmentiň meýdanydyr.

353. Ýarym oklary a we b bolan birjynsly elliptik plastinkanyň onuň esasy okuna görä inersiýa momentini tapmaly ($\rho = 1$).

354. Esasynyň radiusy r we beýikligi h bolan birjynsly tegelek konusyň onuň esasynyň tekizligine görä statiki we inersiýa momentlerini tapmaly ($\rho = 1$).

355. Radiusy R we massasy M bolan birjynsly şaryň diametrine görä inersiýa momentini tapmaly.

356. Guldeniň birinji teoremasyny subut etmeli: tekiz C duganyň şol duganyň tekizliginde ýatýan we ony kesmeýän okuň daşyndan aýlanmagyndan emele gelen üstüň meýdany şol duganyň uzynlygynyň duganyň agyrlyk merkeziniň çyzýan töwereginiň uzynlygyna köpeldilmegine deň.

357. Guldeniň ikinji teoremasyny subut etmeli: tekiz S figuranyň şol figuranyň tekizliginde ýatýan we ony kesmeýän okuň daşyndan aýlanmagyndan emele gelen jisimiň göwrümi şol figuranyň S meýdanynyň agyrlyk merkeziniň çyzýan töwereginiň uzynlygyna köpeldilmegine deň.

358. $x = a\cos\varphi$, $y = a\sin\varphi$ $(|\varphi| \le \alpha \le \pi)$ tegelek duganyň agyrlyk merkeziniň koordinatalaryny kesgitlemeli.

359. $ax = y^2$, $ay = x^2$ $(a > 0)$ parabolalar bilen çäklenen ýaýlanyň agyrlyk merkeziniň koordinatalaryny kesgitlemeli.

360. $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$ $(0 \le x \le a,\ 0 \le y \le b)$ ýaýlanyň agyrlyk merkeziniň koordinatalaryny kesgitlemeli.

361. Radiusy a bolan birjynsly ýarym şaryň agyrlyk merkezini kesgitlemeli.

362. $r = ae^{m\varphi}$ $(m > 0)$ logarifmik spiralyň $O(-\infty, 0)$ nokatdan $P(\varphi, r)$ nokada çenli OP duganyň $C(\varphi_0, r_0)$ agyrlyk merkeziniň koordinatalaryny kesgitlemeli.

363. $r = a(1 + \cos\varphi)$ çyzyk bilen çäklenen ýaýlanyň agyrlyk merkeziniň koordinatalaryny kesgitlemeli.

364. $x = a(t - \sin t)$, $y = a(1 - \cos t)$ $(0 \le t \le 2\pi)$ çyzygyň birinji arkasy bilen çäklenen ýaýlanyň agyrlyk merkeziniň koordinatalaryny kesgitlemeli.

365. $0 \le x \le a$; $y^2 \le 2px$ meýdanyň Ox okunyň daşyndan aýlanmagyndan emele gelen meýdanyň agyrlyk merkeziniň koordinatalaryny kesgitlemeli.

366. $x^2 + y^2 + z^2 = a^2$ $(z \ge 0)$ ýarym şaryň agyrlyk merkeziniň koordinatalaryny kesgitlemeli.

Degişli integral jemleri düzüp we olaryň predellerini tapyp, aşakdaky meseleleri çözmeli.

367. Eger uzynlygy $l = 10m$ bolan sterženiň çyzyk dykyzlygy $\delta = 6 + 0{,}3x$ kg/m düzgün boýunça üýtgeýän bolsa, onda şol sterženiň massasyny tapmaly.

368. Radiusy R bolan ýeriň üstünden m massaly jisimi h beýiklige galdyrmak üçin nähili iş sarp edilýär? Jisim tükeniksizlige daşlaşanda ol iş nämä deň bolar?

369. Eger 1 *kg* güýç pružini 1 *sm* dartýan bolsa, onda maýyşgak 10 *sm* pružini dartmak üçin näçe iş sarp ediler? (*Görkezme*: *Gukuň kanunyndan peýdalanmaly*).

370. Diametri 20 *sm* we uzynlygy 80 *sm* bolan silindr 10 kg/sm^2 basyş esasynda bug bilen doldurylan. Buguň temperaturasyny hemişelik hasap edip, buguň göwrümini iki esse azaltmak üçin nähili iş etmeli?

371. Diametri suwuň üstünde bolan we radiusy a bolan ýarym tegelek görnüşdäki dik diwarjyga suwuň basyş güýjüni kesgitlemeli.

372. Eger aşaky esasynyň suwa çümme derejesi $c = 20\ m$, aşaky esasy $a = 10\ m$, ýokarky esasy $b = 6\ m$ we beýikligi $h = 5\ m$ bolan trapesiýa görnüşdäki dik diwarjyga suwuň basyş güýjüni kesgitlemeli.

Differensial deňlemeleri düzüp, aşakdaky mysallary çözmeli:

373. Nokadyň tizligi

$$\vartheta = \vartheta_0 + at$$

düzgün boýunça üýtgeýär. $[0, T]$ wagt aralygynda ol nokat nähili ýol geçer?

374. Radiusy R we dykyzlygy δ bolan birjynsly şar ω burç tizligi bilen öz diametriniň daşyndan aýlanýar. Şaryň kinetik energiýasyny kesgitlemeli.

375. Hemişelik çyzyk dykyzlygy μ bolan tükeniksiz göni çyzyk şol çyzykdan a uzaklykdaky m massaly material nokady nähili güýç bilen dartar?

376. Hemişelik üst dykyzlygy δ_0 bolan a radiusly tegelek plastinkanyň şol plastinkanyň Q merkezi arkaly geçýän, plastinkanyň tekizligine perpendikulýar ýerleşýän, iň ýakyn PQ uzaklygy b deň m massaly material P nokady nähili güýç bilen dartýandygyny kesgitlemeli?

377. Toriçelliniň düzgüni boýunça suwuklygyň gapdan akyş tizligi

$$\vartheta = c\sqrt{2gh},$$

bu ýerde g – agyrlyk güýjüniň tizlenmesi, h – suwuklygyň deşikden ýokardaky derejesiniň beýikligi we $c = 0{,}6$ tejribe koeffisiýenti.

Ýokarsyna çenli doldurylan diametri $D = 1\ m$ we beýikligi $H = 2\ m$ bolan dik silindr aşaky düýbündäki diametri $d = 1\ sm$ tegelek deşik boýunça näçe wagtda boşar?

378. Aýlanma jisimi bolan gap nähili görnüşde bolanda ondan suwuklyk akanda peselme derejesi deňölçegli bolar?

379. Radiniň dargama tizligi her wagt pursadynda onuň mukdaryna proporsionaldyr. Eger başlangyç $t = 0$ pursatda radiniň mukdary Q_0 gram, $T = 1600$ ýyldan soň onuň mukdary iki esse azalan bolsa, onda radiniň dargama düzgünini tapmaly.

380. Ikinji tertipli prosesde A jisimi B jisime geçirmegiň himiki reaksiýasynyň tizligi ol jisimleriň konsentrasiýalarynyň köpeltmek hasylyna proporsional. Eger gapda $t = 0\ min$ B jisimiň 20%-i bar bolsa, $t = 15\ min$ soň onuň mukdary 80% bolan bolsa, onda $t = 1\ sag$ soň gapda B jisimiň näçe göterimi bolar?

381. Gukuň kanuny boýunça sterženiň ε uzalmasy degişli kese-kesikde σ güýjüň napryaženiýesine proporsional, ýagny

$$\varepsilon = \frac{\sigma}{E},$$

bu ýerde E – Ýunguň moduly.

Eger esasynyň radiusy R, konusyň beýikligi H we udel agramy γ bolan konus görnüşli steržen esasy boýunça berkidilen we depesi aşak ugrukdyrylan bolsa, onda ol agyr sterženiň uzalmasyny kesgitlemeli.

§6. Kesgitli integrallaryň takmyny hasaplanylyşy

Gönükmeler

382. Gönüburçluklar formulasyny ulanyp ($n = 12$),

$$\int_0^{2\pi} x \sin x dx$$

integralyň takmyny bahasyny hasaplamaly we ony takyk jogaby bilen deňeşdirmeli.

Trapesiýalar formulasyny ulanyp, integrallary hasaplamaly we olaryň ýalňyşlyklaryny bahalandyrmaly:

383. $\int_0^1 \frac{dx}{1+x}$ ($n = 8$).

384. $\int_0^1 \frac{dx}{1+x^3}$ ($n = 12$).

385. $\int_0^{\pi/2} \sqrt{1 - \frac{1}{4}\sin^2 x}\, dx$ ($n = 6$).

Simpsonyň formulasyny ulanyp, integrallary hasaplamaly:

386. $\int_1^9 \sqrt{x}\, dx$ ($n = 4$).

387. $\int_0^{\pi} \sqrt{3 + \cos x}\, dx$ ($n = 6$).

388. $\int_0^{\pi/2} \frac{\sin x}{x} dx$ ($n = 10$).

389. $\int_0^1 \frac{xdx}{\ln(1+x)}$ ($n = 6$).

390. $n = 10$ alyp, Katalanyň hemişeligini hasaplamaly:

$$G = \int_0^1 \frac{\text{arctg}x}{x} dx.$$

391. $\frac{\pi}{4} = \int\limits_0^1 \frac{dx}{1+x^2}$ formulany ulanyp, π sany 10^{-5}-e çenli takyklykda hasaplamaly.

392. $\int\limits_0^1 e^{x^2}dx$ integraly 0,001-e çenli takyklykda hasaplamaly.

393. $\int\limits_0^1 (e^x - 1)\ln\frac{1}{x}dx$ integraly 10^{-4}-e çenli takyklykda hasaplamaly.

394. $\int\limits_0^{+\infty} e^{-x^2}dx$ ähtimallyklar integralyny 0,001-e çenli takyklykda hasaplamaly.

395. Ýarym oklary $a = 10$ we $b = 6$ bolan ellipsiň uzynlygynyň takmyny bahasyny tapmaly.

396. $\Delta x = \pi/3$ alyp, nokatlar boýunça $y = \int\limits_0^x \frac{\sin t}{t}dt$ $(0 \le x \le 2\pi)$ funksiýanyň grafigini gurmaly.

VIII. KÖP ÜÝTGEÝÄNLI FUNKSIÝALAR

§1. Köp üýtgeýänli funksiýalaryň predeli we üznüksizligi

1. ***Köp üýtgeýänli funksiýa düşünjesi.*** Tertipleşdirilen hakyky $(x_1, ..., x_m)$ sanlaryň toplumyna m ölçegli nokat diýilýär we ol $x = (x_1, ..., x_m)$ bilen belgilenýär. Şeýle nokatlaryň köplügine bolsa m ölçegli giňişlik diýilýär. Nokatlarynyň arasyndaky uzaklyk

$$\rho(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_m - y_m)^2}$$

formula boýunça kesgitlenýän m ölçegli giňişlige m ölçegli Ýewklid giňişligi diýilýär we R^m bilen belgilenýär.

$M \subset R^m$ köplügiň her bir $x = (x_1, ..., x_m)$ nokadyna u hakyky sany degişli edýän f düzgüne M köplükde kesgitlenen köp üýtgeýänli funksiýa diýilýär.

$M \subset R^m$ köplükde kesgitlenen köp üýtgeýänli funksiýany belgilemek üçin

$$f: M \to R; \quad u = f(x), \quad x \in M; \quad u = f(x_1, ..., x_m)$$

ýazgylar ulanylýar. Şunlukda, funksiýanyň kesgitlenen M köplügine funksiýanyň kesgitleniş ýaýlasy diýilýär.

Eger funksiýa käbir aňlatmalar arkaly anyk görnüşde berlen bolsa, onda onuň kesgitleniş ýaýlasy diýlip şol aňlatmalaryň manyly nokatlarynyň köplügine düşünilýär.

1-nji mysal. Iki üýtgeýänli $u = \ln(1 - x^2 - y^2)$ funksiýa R^2 giňişligiň $M = \{(x,y): x^2 + y^2 < 1\}$ köplüginde kesgitlenendir. Onuň bahalar ýaýlasy bolsa $(-\infty, 0)$ interwaldyr.

2-nji mysal. Üç üýtgeýänli $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2 - 4}}$ funksiýa R^3 giňişligiň $M = \{(x,y,z): x^2 + y^2 + z^2 > 4\}$ köplüginde kesgitlenendir. Onuň bahalar ýaýlasy bolsa $(0; +\infty)$ interwaldyr.

Hemişelik c san üçin $M \subset R^m$ köplükde kesgitlenen f üçin

$$f(x_1, ..., x_m) = c$$

deňligi kanagatlandyrýan $(x_1, ..., x_m)$ nokatlar köplügine f funksiýanyň dereje köplügi diýilýär. Hususan-da, $m = 2$ we $m = 3$ bolanda oňa degişlilikde dereje çyzygy we dereje üsti diýilýär.

Mysal üçin, eger $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ bolsa, onda ol funksiýanyň dereje üsti merkezi koordinatalar başlangyjynda we radiusy $r = \sqrt{c}$ bolan $x_1^2 + x_2^2 + x_3^2 = c$ sferadyr.

2. ***Funksiýanyň nokatdaky predeli.*** Goý, f funksiýa $M \subset R^m$ köplükde kesgitlenen we $a \in R^m$ nokat M köplügiň predel nokady bolsun.

$\{x^{(n)}\} \subset R^m$ yzygiderligiň predeli we ýygnanmagy san yzygiderligiňki ýalydyr.

Geýnäniň kesgitlemesi. Eger a nokada ýygnanýan islendik $\{x^{(n)}\} \subset M$ ($x^{(n)} \neq a$) yzygiderlik üçin $\{f(x^{(n)})\}$ san yzygiderligi A sana ýygnanýan bolsa, onda A sana f funksiýanyň a nokatdaky ($x \to a$ bolandaky) predeli diýilýär.

f funksiýanyň a nokatdaky predeli ýazgyda şeýle aňladylýar:

$$\lim_{x \to a} f(x) = A, \qquad \lim_{\rho(x,a) \to 0} f(x) = A \qquad \text{ýa-da} \qquad \lim_{\substack{x_1 \to a_1 \\ \overline{x_m \to a_m}}} f(x_1, ..., x_m) = A.$$

Koşiniň kesgitlemesi. Eger $\forall \varepsilon > 0$ üçin $\delta > 0$ san tapylyp, $0 < \rho(x, a) < \delta$ şerti kanagatlandyrýan $\forall x \in M$ üçin $|f(x) - A| < \varepsilon$ deňsizlik ýerine ýetse, onda A sana f funksiýanyň a nokatdaky predeli diýilýär.

Bu kesgitlemeler bir üýtgeýänli funksiýanyň degişli kesgitlemeleriniň köp üýtgeýänli funksiýa üçin umumylaşdyrmasydyr.

Bellik. Funksiýanyň predeliniň $E \subset M$ köplük boýunça tapylýandygyny görkezmeklik zerur bolanda A sanyň f funksiýanyň E köplük boýunça a nokatdaky predeli bolýandygy

$$\lim_{x \to a, x \in E} f(x) = A$$

ýazgyda aňladylýar.

3-nji mysal. $f(x, y) = \dfrac{3x^2 y}{x^4 + 2y^2}$ funksiýanyň (0, 0) nokatda predeliniň ýokdugyny subut etmeli.

Ç.B. Geýnäniň kesgitlemesini ulanmak üçin, $(x^{(n)}, y^{(n)}) = \left(\frac{1}{n}, \frac{1}{n}\right)$, $(x_1^{(n)}, y_1^{(n)}) = \left(\frac{1}{n}, \frac{1}{n^2}\right)$ yzygiderliklere garalyň. Olar üçin $n \to \infty$ bolanda $(x^{(n)}, y^{(n)}) \to (0,0)$, $(x_1^{(n)}, y_1^{(n)}) \to (0, 0)$ bolýandygy aýdyňdyr. Ýöne

$$f(x^{(n)}, y^{(n)}) = \frac{3n}{1 + 2n^2}, \qquad f(x_1^{(n)}, y_1^{(n)}) = 1$$

deňlikleriň esasynda $\lim f(x^{(n)}, y^{(n)}) = 0$, $\lim f(x_1^{(n)}, y_1^{(n)}) = 1$. Şonuň üçin hem Geýnäniň kesgitlemesi esasynda funksiýanyň (0,0) nokatda predeli ýokdur. **Ç.S.**

Eger $\forall \varepsilon > 0$ üçin $\delta > 0$ san tapylyp, $\rho(x, a) > \delta$ şerti kanagatlandyrýan $\forall x \in M$ üçin $|f(x) - A| < \varepsilon$ deňsizlik ýerine ýetýän bolsa, onda A sana f funksiýanyň $x \to \infty$ bolandaky predeli diýilýär we

$$\lim_{x \to \infty} f(x) = A$$

diýlip, ýazgyda belgilenilýär.

4-nji mysal. $\lim\limits_{\substack{x\to+\infty \\ y\to+\infty}}(x^2+y^2)e^{-(x+y)}=0$ deňligi subut etmeli:

Ç.B. $\forall x>0$ we $\forall y>0$ üçin

$$0\le(x^2+y^2)e^{-(x+y)}\le(x+y)^2e^{-(x+y)}$$

we $\lim\limits_{x\to+\infty}t^2e^{-t}=0$ bolýandygy sebäpli, $\forall\varepsilon>0$ üçin $\delta>0$ tapylyp, $\forall t>\delta$ üçin $t^2e^{-t}<\varepsilon$ deňsizlik ýerine ýetýär. Ondan subut edilmeli deňlik gelip çykýar. **Ç.S.**

Bir üýtgeýänli funksiýalaryň predelleri üçin subut edilen häsiýetleriň, köp üýtgeýänli funksiýalar üçin hem ýerine ýetýändigi aňsatlyk bilen görkezilýär.

1-nji häsiýet. Goý, $M\subset R^m$ köplükde kesgitlenen f, g we φ funksiýalar üçin

$$f(x)\le g(x)\le\varphi(x),\quad x\in M,\quad \lim_{x\to a}f(x)=\lim_{x\to a}\varphi(x)=A$$

şertler ýerine ýetsin. Onda $\lim\limits_{x\to a}g(x)=A$.

2-nji häsiýet. Eger $M\subset R^m$ köplükde kesgitlenen f we g funksiýalaryň a nokatda predelleri bar bolsa, onda $f\pm g$, $f\times g$ we f/g ($\lim\limits_{x\to a}g(x)\neq 0$ bolanda) funksiýalaryň hem a nokatda predelleri bardyr.

Bir üýtgeýänli funksiýadan tapawutlylykda köp üýtgeýänli funksiýanyň $x\to a$ bolandaky predeli kesgitlenende x ululyk a nokada islendik ugur boýunça we islendik çyzyk boýunça ymtylyp biler. Şonda eger funksiýanyň a nokatda predeli bar bolsa, onda onuň şol nokatda islendik ugur boýunça we islendik çyzyk boýunça predelleri bardyr we olar deňdirler. Ýöne onuň tersi dogry däldir. Mysal üçin, 3-nji mysaldaky funksiýanyň (0,0) nokatda $y=x^2$ parabolanyň ugry boýunça predeli bardyr, çünki $f(x,x^2)=1$, ýöne onuň (0,0) nokatda predeli ýokdur (ony şol mysalda görkezipdik).

3. *Gaýtalanýan predel düşünjesi.* Köp üýtgeýänli funksiýanyň predeli baradaky kesgitlemelerde onuň m sany üýtgeýän ululyklarynyň ählisi bir wagtda käbir sanlara ymtylýar. Şonuň üçin ol predele m gat predel hem diýilýär ($m=2$ bolanda iki gat, $m=3$ bolanda üç gat). Ýöne köp üýtgeýänli funksiýalar üçin başga hili predel düşünjesi hem girizilýär. Ol predel düşünjesi funksiýanyň üýtgeýänleri boýunça yzygiderlikde, ýagny gaýtalap predele geçmek bilen baglanyşyklydyr. Şonuň üçin oňa gaýtalanýan predel hem diýilýär we ol predelleriň birisi

$$\lim_{x_1\to a_1}\lim_{x_2\to a_2}\dots\lim_{x_m\to a_m}f(x_1,x_2,\dots,x_m)$$

ýazgyda belgilenýär. Beýleki tertipdäki gaýtalanýan predeller hem şonuň ýaly kesgitlenilýär. Bu predel düşünjesiniň iki üýtgeýänli $u = f(x, y)$ funksiýa üçin kesgitlenişine giňişleýin garalyň.

Goý, $u = f(x, y)$ funksiýa $(a, b) \in R^2$ nokadyň, onuň özünden başga mümkin bolan, käbir $|x - a| < \delta_1$, $|y - b| < \delta_2$ göni burçly golaý töwereginde kesgitlenen bolsun.

Goý, $0 < |y - b| < \delta_2$ şerti kanagatlandyrýan her bir bellenen y üçin $u = f(x, y)$ funksiýanyň x ululyga görä a nokatda $\varphi(y) = \lim\limits_{x \to a} f(x, y)$ predeli we $\varphi(y)$ funksiýanyň b nokatda $B = \lim\limits_{y \to b} \varphi(y)$ predeli bar bolsun. Onda B sana $f(x, y)$ funksiýanyň (a, b) nokatdaky gaýtalanýan predeli diýilýär we ol

$$\lim_{y \to b} \lim_{x \to a} f(x, y) = B$$

ýazgyda belgilenilýär. Ýazgydan görnüşi ýaly, şeýle kesgitlenip gaýtalanýan predelde ilki $x \to a$ bolanda predele geçilip, soňra $y \to b$ bolanda predele geçilýär. Edil şuňa meňzeşlikde beýleki tertipdäki gaýtalanýan predel hem kesgitlenilýär:

$$\lim_{x \to a} \lim_{y \to b} f(x, y) = C.$$

Umuman aýdylanda, $B \neq C$. Ýokarda belleýşimiz ýaly, gaýtalanýan predellerden tapawutlandyrmak üçin

$$\lim_{(x,y) \to (a,b)} f(x, y) = \lim_{\substack{x \to a \\ y \to b}} f(x, y) = A$$

görnüşde kesgitlenen predele iki gat predel diýilýär.

Bu predelleriň biri-birleri bilen baglanyşygyny aşakda göreris.

5-nji mysal. $f(x, y) = x \sin \dfrac{1}{y} + y \sin \dfrac{1}{x}$ funksiýanyň $(0, 0)$ nokatdaky iki gat we gaýtalanýan predellerini hasaplamaly.

Ç.B. Predeliň 1-nji häsiýetini ulanyp,

$$0 \leq \left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| \leq |x| + |y|$$

deňsizligiň esasynda $\lim\limits_{(x,y) \to (0,0)} f(x, y) = 0$ deňligi alarys. Ýöne

$$\lim_{x \to 0} y \sin \frac{1}{x} \ (y \neq 0), \qquad \lim_{y \to 0} x \sin \frac{1}{y} \ (x \neq 0)$$

predelleriň ýokdugy sebäpli

$$\lim_{y \to 0} \left(\lim_{x \to 0} x \sin \frac{1}{y} + \lim_{x \to 0} y \sin \frac{1}{x} \right), \qquad \lim_{y \to 0} \left(\lim_{y \to 0} x \sin \frac{1}{y} + \lim_{y \to 0} y \sin \frac{1}{x} \right)$$

gaýtalanýan predeller ýokdur.

Diýmek, garalýan funksiýanyň (0, 0) nokatda iki gat predeli bolup, onuň gaýtalanýan predelleri ýokdur. **Ç.S.**

Bu mysal iki gat predeliň bar mahaly hem gaýtalanýan predelleriň bolmaýandygyny görkezýär. Tersine hem bolýandygyny aşakdaky mysal tassyklaýar.

6-njy mysal. $f(x,y) = \dfrac{xy}{x^2 + y^2}$ funksiýanyň (0, 0) nokatda iki gat we gaýtalanýan predellerini hasaplamaly.

Ç.B. Bu funksiýanyň (0, 0) nokatda gaýtalanýan predelleri bardyr, ýagny

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0} f(x,y) = 0,$$

ýöne iki gat predel ýokdur, çünki ol predel koordinata oklary boýunça nola deň, $y = x$ göni çyzyk boýunça bolsa 1/2 deňdir.

Diýmek, garalýan funksiýanyň (0, 0) nokatda nola deň gaýtalanýan predelleri bolup, ýöne onuň iki gat predeli ýokdur. **Ç.S.**

Şeýlelikde, funksiýanyň berlen nokatda diňe iki gat predeliniň bardygyndan gaýtalanýan predelleriň barlygy we tersine, gaýtalanýan predelleriň bardygyndan iki gat predelleriň barlygy gelip çykmaýar.

Gaýtalanýan predeller bar bolanda-da olar biri-birlerine hemişe deň däldirler, ýagny olaryň deň bolmaýan ýagdaýlary hem bardyr.

7-nji mysal. $f(x,y) = \dfrac{x^2 + y^2 + x - y}{x + y}$ funksiýanyň (0, 0) nokatdaky gaýtalanýan predellerini hasaplamaly.

Ç.B. Ilki bilen,

$$\varphi(y) = \lim_{x\to 0} f(x,y) = y - 1, \qquad \psi(x) = \lim_{y\to 0} f(x,y) = x + 1$$

bolýandygyny görýäris. Şonuň üçin hem

$$\lim_{y\to 0}\varphi(x,y) = \lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}(y - 1) = -1,$$

$$\lim_{y\to 0}\varphi(y) = \lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{x\to 0}(y - 1) = -1.$$

Bu ýerden gaýtalanýan predelleriň ikisiniň hem bardygyny, ýöne olaryň deň däldigini görýäris. **Ç.S.**

Gönükmeler

Funksiýalaryň kesgitleniş ýaýlasyny anyklamaly we grafigini gurmaly:

1. $u = x + \sqrt{y}$.

2. $u = \sqrt{1 - x^2} + \sqrt{y^2 - 1}$.

3. $u = \sqrt{1 - x^2 - y^2}$.

4. $u = \dfrac{1}{\sqrt{x^2 + y^2 - 1}}$.

5. $u = \sqrt{(x^2 + y^2 - 1)(4 - x^2 - y^2)}$.

6. $u = \sqrt{\dfrac{x^2 + y^2 - x}{2x - x^2 - y^2}}$.

7. $u = \sqrt{1 - (x^2 + y)^2}$.

8. $u = \ln(-x - y)$.

9. $u = \arcsin\dfrac{y}{x}$.

10. $u = \arccos\dfrac{x}{x + y}$.

11. $u = \arcsin\dfrac{x}{y^2} + \arcsin(1 - y)$.

12. $u = \sqrt{\sin(x^2 + y^2)}$.

13. $u = \arccos\dfrac{z}{\sqrt{x^2 + y^2}}$.

14. $u = \ln(xyz)$.

15. $u = \ln(-1 - x^2 - y^2 + z^2)$.

Funksiýalaryň dereje çyzyklaryny gurmaly:

16. $z = x + y$.

17. $z = x^2 + y^2$.

18. $z = x^2 - y^2$.

19. $z = (x + y)^2$.

20. $z = \dfrac{y}{x}$.

21. $z = \dfrac{1}{x^2 + 2y^2}$.

22. $z = \sqrt{xy}$.

23. $z = |x| + y$.

24. $z = |x| + |y| - |x + y|$.

25. $z = \min(x, y)$.

26. $z = \max(|x|, |y|)$.

27. $z = \min(x^2, y)$.

28. $z = e^{2x/(x^2 + y^2)}$.

29. $z = x^y$ $(x > 0)$.

30. $z = x^y e^{-x}$ $(x > 0)$.

31. $z = \ln\sqrt{\dfrac{(x - a)^2 + y^2}{(x + a)^2 + y^2}}$ $(a > 0)$.

32. $z = \operatorname{arctg}\dfrac{2ay}{x^2 + y^2 - a^2}$ $(a > 0)$.

33. $z = \operatorname{sgn}(\sin x \sin y)$.

Funksiýalaryň dereje üstlerini tapmaly:

34. $u = x + y + z$.

35. $u = x^2 + y^2 + z^2$.

36. $u = x^2 + y^2 - z^2$.

37. $u = (x + y)^2 + z^2$.

38. $u = \operatorname{sgn}\sin(x^2 + y^2 + z^2)$.

Deňlemeleri boýunça üstleriň häsiýetlerini derňemeli:

39. $z = f(y - ax)$.

40. $z = f(\sqrt{x^2 + y^2})$.

41. $z = xf\left(\frac{y}{x}\right).$ **42.** $z = f\left(\frac{y}{x}\right).$

43. $f(x,y) = \begin{cases} 1, & \text{eger } y \geq x, \\ 0, & \text{eger } y < x \end{cases}.$ bolsa, funksiýa üçin $F(t) = f(\cos t, \sin t)$ funksiýanyň grafigini gurmaly.

44. Berlen $f(x,y) = \dfrac{2xy}{x^2 + y^2}$ boýunça $f\left(1, \frac{y}{x}\right)$ tapmaly.

45. Berlen $f\left(\dfrac{y}{x}\right) = \dfrac{\sqrt{x^2 + y^2}}{x}$ $(x > 0)$ boýunça $f(x)$ funksiýany tapmaly.

46. Goý, $z = \sqrt{y} + f(\sqrt{x} - 1)$ bolsun. $y = 1$ bolanda $z = x$ bolýan f we z funksiýalary kesgitlemeli.

47. Goý, $z = x + y + f(x - y)$ bolsun. $y = 0$ bolanda $z = x^2$ bolýan f we z funksiýalary tapmaly.

48. Berlen $f\left(x + y, \dfrac{y}{x}\right) = x^2 - y^2$ boýunça $f(x, y)$ funksiýany tapmaly.

49. $f(x,y) = \dfrac{x - y}{x + y}$ funksiýa üçin

$$\lim_{x \to 0}\left\{\lim_{y \to 0} f(x,y)\right\} = 1; \qquad \lim_{y \to 0}\left\{\lim_{x \to 0} f(x,y)\right\} = -1$$

predelleriň bardygyny, ýöne oňa garamazdan $\lim\limits_{\substack{x \to 0 \\ y \to 0}} f(x,y)$ predeliň ýokdugyny subut etmeli.

50. $f(x,y) = \dfrac{x^2 y^2}{x^2 y^2 + (x - y)^2}$ funksiýa üçin

$$\lim_{x \to 0}\left\{\lim_{y \to 0} f(x,y)\right\} = \lim_{y \to 0}\left\{\lim_{x \to 0} f(x,y)\right\} = 0$$

predelleriň bardygyny, ýöne oňa garamazdan $\lim\limits_{\substack{x \to 0 \\ y \to 0}} f(x,y)$ predeliň ýokdugyny subut etmeli.

51. $f(x,y) = (x + y)\sin\dfrac{1}{x}\sin\dfrac{1}{y}$ funksiýa üçin

$$\lim_{x \to 0}\left\{\lim_{y \to 0} f(x,y)\right\} \quad \text{we} \quad \lim_{y \to 0}\left\{\lim_{x \to 0} f(x,y)\right\}.$$

Gaýtalanýan predelleriň ikisiniň hem ýokdugyny, ýöne oňa garamazdan $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0$ predeliň bardygyny subut etmeli.

52. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{2xy}{x^2+y^2}$ predel barmy?

53. $x = t\cos\alpha$, $y = t\sin\alpha$ $(0 \le t < +\infty)$ şöhle boýunça

$$f(x, y) = x^2 e^{-(x^2 - y)}$$

funksiýanyň $t \to +\infty$ bolandaky predeli nämä deň?

54. Berlen funksiýalar üçin

$$\lim_{x\to a}\left\{\lim_{y\to b} f(x,y)\right\} \quad \text{we} \quad \lim_{y\to b}\left\{\lim_{x\to a} f(x,y)\right\}$$

predelleri tapmaly:

a) $f(x,y) = \dfrac{x^2+y^2}{x^2+y^4}$, $a = \infty$, $b = \infty$;

b) $f(x,y) = \dfrac{x^y}{1+x^y}$, $a = \infty$, $b = +0$;

ç) $f(x,y) = \sin\dfrac{\pi x}{2x+y}$, $a = \infty$, $b = \infty$;

d) $f(x,y) = \dfrac{1}{xy}\operatorname{tg}\dfrac{xy}{1+xy}$, $a = 0$, $b = \infty$;

e) $f(x, y) = \log_x(x + y)$, $a = 1$, $b = 0$.

Ikigat predelleri tapmaly:

55. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{x+y}{x^2-xy+y^2}$.

56. $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{x^2+y^2}{x^4+y^4}$.

57. $\lim\limits_{\substack{x\to 0\\ y\to a}} \dfrac{\sin xy}{x}$.

58. $\lim\limits_{\substack{x\to +\infty\\ y\to +\infty}} (x^2+y^2)e^{-(x+y)}$.

59. $\lim\limits_{\substack{x\to +\infty\\ y\to +\infty}} \left(\dfrac{xy}{x^2+y^2}\right)^{x^2}$.

60. $\lim\limits_{\substack{x\to 0\\ y\to 0}} (x^2+y^2)^{x^2y^2}$.

61. $\lim\limits_{\substack{x\to\infty\\ y\to a}} \left(1+\dfrac{1}{x}\right)^{x^2/(x+y)}$.

62. $\lim\limits_{\substack{x\to 1\\ y\to 0}} \dfrac{\ln(x+e^y)}{\sqrt{x^2+y^2}}$.

63. Eger $x = \rho\cos\varphi$, $y = \rho\sin\varphi$ bolsa, onda haýsy φ ugur boýunça aşakdaky predeller bolmaly:

a) $\lim\limits_{\rho\to+0} e^{\frac{x}{x^2+y^2}}$; b) $\lim\limits_{\rho\to+\infty} e^{x^2-y^2}\cdot\sin 2xy$?

Funksiýalaryň üzülme nokatlaryny tapmaly:

64. $u = \dfrac{1}{\sqrt{x^2+y^2}}$.

65. $u = \dfrac{xy}{x+y}$.

66. $u = \dfrac{x+y}{x^3+y^3}$.

67. $u = \sin\dfrac{1}{xy}$.

68. $u = \dfrac{1}{\sin x \sin y}$.

69. $u = \ln(1 - x^2 - y^2)$.

70. $u = \dfrac{1}{xyz}$.

71. $u = \ln\dfrac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$.

72. $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{eger } x^2+y^2 \neq 0, \\ 0, & \text{eger } x^2+y^2 = 0 \end{cases}$

funksiýanyň üýtgeýän x we y ululyklaryň her biri boýunça (beýlekisiniň bellenen bahasynda) üznüksizdigini, ýöne üýtgeýän ululyklaryň toplumy boýunça üznüksiz däldigini subut etmeli.

73. $f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2}, & \text{eger } x^2+y^2 \neq 0, \\ 0, & \text{eger } x^2+y^2 = 0 \end{cases}$

bolsa, funksiýanyň (0, 0) nokatda şol nokat arkaly geçýän her bir

$$x = t\cos\alpha, \quad y = t\sin\alpha \quad (0 \le t < +\infty)$$

şöhläniň ugry boýunça üznüksizdigini, ýagny

$$\lim_{t\to 0} f(t\cos\alpha, t\sin\alpha) = f(0,0)$$

predeliň bardygyny, ýöne ol funksiýanyň (0, 0) nokatda üznüksiz däldigini subut etmeli.

74. Çyzykly $u = 2x - 3y + 5$ funksiýanyň tükeniksiz

$$E^2 = \{|x| < +\infty, |y| < +\infty\}$$

tekizlikde deňölçegli üznüksizligini derňemeli.

75. $u = \sqrt{x^2+y^2}$ funksiýanyň

$$E^2 = \{|x| < +\infty, |y| < +\infty\}$$

tekizlikde deňölçegli üznüksizligini derňemeli.

76. $f(x,y) = \sin\dfrac{\pi}{1 - x^2 - y^2}$ funksiýa $x^2 + y^2 < 1$ ýaýlada deňölçegli üznüksiz funksiýa bolarmy?

77. Berlen $u = \arcsin\dfrac{x}{y}$ funksiýa kesgitlenen B ýaýlasynda üznüksizmi? B ýaýlada u funksiýa deňölçegli üznüksiz funksiýa bolarmy?

78. Eger $y \neq 0$ we $f(x, 0) = 0$ bolsa, $f(x,y) = x\sin\dfrac{1}{y}$ funksiýanyň üzülme nokatlarynyň köplüginiň ýapyk köplük däldigini subut etmeli.

79. Eger $f(x, y)$ funksiýa käbir G ýaýlada üýtgeýän x ululyk boýunça üznüksiz we y boýunça x-e görä deňölçegli üznüksiz bolsa, onda ol funksiýanyň şol ýaýlada üznüksizdigini subut etmeli.

80. Eger $f(x, y)$ funksiýa käbir G ýaýlada üýtgeýän x ululyk boýunça üznüksiz we y boýunça Lipşisiň şertini kanagatlandyrýan bolsa, ýagny $\forall(x,y')\in G$, $\forall(x,y'')\in G$ bolup, hemişelik L üçin

$$|f(x, y') - f(x, y'')| \leq L|y' - y''|$$

deňsizlik ýerine ýetýän bolsa, onda ol funksiýanyň berlen ýaýlada üznüksizdigini subut etmeli.

81. Eger $f(x, y)$ funksiýa üýtgeýän x we y ululyklaryň her biri boýunça üznüksiz we olaryň biri boýunça monoton bolsa, onda ol funksiýanyň üýtgeýänleriň toplumy boýunça üznüksizdigini subut etmeli.

82. Goý, $f(x, y)$ funksiýa $a \leq x \leq A$, $b \leq y \leq B$ ýaýlada üznüksiz, $\varphi_n(x)$ $(n = 1, 2, ...)$ yzygiderlik $[a, A]$ kesimde deňölçegli ýygnanýan we $b \leq \varphi_n(x) \leq B$ şerti kanagatlandyrýan bolsun. Onda $F_n(x) = f(x, \varphi_n(x))$ $(n = 1, 2, ...)$ yzygiderligiň $[a, A]$ kesimde deňölçegli ýygnanýandygyny subut etmeli.

83. Goý, 1) $f(x, y)$ funksiýa $G(a < x < A;\ b < y < B)$ ýaýlada üznüksiz; 2) $\varphi(x)$ funksiýa (a, A) interwalda üznüksiz we onuň bahalary (b, B) interwala degişli bolsun. Onda $F(x) = f(x, \varphi(x))$ funksiýanyň (a, A) interwalda üznüksizdigini subut etmeli.

84. Goý, 1) $f(x, y)$ funksiýa $G(a < x < A;\ b < y < B)$ ýaýlada üznüksiz; 2) $x = \varphi(u, \upsilon)$ we $y = \psi(u, \upsilon)$ funksiýalar $G'(a' < u < A';\ b' < \upsilon < B')$ ýaýla üznüksiz bolup, olaryň bahalary degişlilikde (a, A) we (b, B) interwallara degişli bolsun. Onda $F(u, \upsilon) = f(\varphi(u, \upsilon), \psi(u, \upsilon))$ funksiýanyň G' ýaýlada üznüksizdigini subut etmeli.

§2. Köp üýtgeýänli funksiýalaryň hususy önümleri we differensiallary

Köp üýtgeýänli funksiýanyň üýtgeýän ululyklarynyň biri boýunça hususy önümi şol üýtgeýän ululykdan beýlekilerini hemişelik hasap edip, bir üýtgeýänli funksiýanyň önüminiň tapylyşy ýaly tapylýar.

Köp üýtgeýänli $u = f(x_1, x_2, ..., x_m)$ funksiýanyň birinji differensialy

$$du = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots + \frac{\partial f}{\partial x_m}dx_m$$

formula boýunça tapylýar.

Eger $u = f(x_1, x_2, ..., x_m)$ we $x_i = \varphi_i(t_1, t_2, ..., t_m)$ $(i = 1, ..., k)$ differensirlenýän bolsa, onda çylşyrymly funksiýanyň hususy önümleri

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \ldots + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_i} \quad (i = 1, ..., m)$$

formula boýunça tapylýar.

Differensirlenýän $u = f(x, y, z)$ funksiýanyň kosinus ugrukdyryjylary $\cos\alpha$, $\cos\beta$, $\cos\gamma$ bolan wektor bilen ugurdaş l ugur boýunça önümi

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta + \frac{\partial f}{\partial z}\cos\gamma$$

formula boýunça tapylýar.

Koordinatalary $u = f(x, y, z)$ funksiýanyň $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ hususy önümleri bolan wektora şol funksiýanyň gradiýenti diýilýär:

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

Gönükmeler

85. $f_x'(x,b) = \frac{d}{dx}[f(x,b)]$ deňligi subut etmeli.

86. $f(x,y) = x + (y-1)\arcsin\sqrt{\frac{x}{y}}$ funksiýa üçin $f_y'(x, 1)$ hususy önümi tapmaly.

87. $f(x,y) = \sqrt[3]{xy}$ funksiýa üçin $f_x'(0, 0)$ we $f_y'(0, 0)$ hususy önümleri tapmaly. Bu funksiýa $O(0, 0)$ nokatda differensirlenýärmi?

88. $f(x,y) = \sqrt[3]{x^3 + y^3}$ funksiýa $O(0, 0)$ nokatda differensirlenýärmi?

89. $x^2 + y^2 > 0$ bolanda $f(x,y) = e^{-\frac{1}{x^2+y^2}}$ we $f(0, 0) = 0$ deňlikler boýunça kesgitlenýän funksiýanyň $O(0, 0)$ nokatda differensirlenýändigini derňemeli.

Funksiýalaryň birinji we ikinji tertipli hususy önümlerini tapmaly:

90. $u = x^4 + y^4 - 4x^2y^2$.

91. $u = xy + \dfrac{x}{y}$.

92. $u = \dfrac{x}{y^2}$.

93. $u = \dfrac{x}{\sqrt{x^2 + y^2}}$.

94. $u = x\sin(x + y)$.

95. $u = \dfrac{\cos x^2}{y}$.

96. $u = \operatorname{tg}\dfrac{x^2}{y}$.

97. $u = x^y$.

98. $u = \ln(x + y^2)$.

99. $u = \operatorname{arctg}\dfrac{y}{x}$.

100. $u = \operatorname{arctg}\dfrac{x + y}{1 - xy}$.

101. $u = \arcsin\dfrac{x}{\sqrt{x^2 + y^2}}$.

102. $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$.

103. $u = \left(\dfrac{x}{y}\right)^z$.

104. $u = x^{y/z}$.

105. $u = x^{y^z}$.

106. Berlen funksiýalar üçin

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

deňligi barlamaly:

a) $u = x^2 - 2xy - 3y^2$; b) $u = x^{y^2}$; ç) $u = \arccos\sqrt{\dfrac{x}{y}}$.

107. $x^2 + y^2 \neq 0$ bolanda $f(x,y) = xy\dfrac{x^2 - y^2}{x^2 + y^2}$ we $f(0, 0) = 0$ deňlikler boýunça kesgitlenýän funksiýa üçin

$$f''_{xy}(0, 0) \neq f''_{yx}(0, 0)$$

deňsizligi subut etmeli.

108. $f(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & x^2 + y^2 > 0; \\ 0, & x = y = 0 \end{cases}$ bolsa, funksiýa üçin $f''_{xy}(0, 0)$ barmy?

109. n ölçegli birjynsly $u = f(x, y, z)$ funksiýa üçin, birjynsly funksiýalar üçin, Eýleriň teoremasyny barlamaly:

a) $u=(x-2y+3z)^2$; b) $u=\dfrac{x}{\sqrt{x^2+y^2+z^2}}$; ç) $u=\left(\dfrac{x}{y}\right)^{y/z}$.

110. Eger differensirlenýän $u=f(x, y, z)$ funksiýa

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu$$

deňlemäni kanagatlandyrýan bolsa, onda onuň n ölçegli, birjynsly funksiýadygyny subut etmeli. (*Görkezme*: *Kömekçi* $F(t)=\dfrac{f(tx,ty,tz)}{t^n}$ *funksiýadan peýdalanmaly*).

111. Eger $f(x, y, z)$ differensirlenýän n ölçegli birjynsly funksiýa bolsa, onda onuň $f_x'(x, y, z)$, $f_y'(x, y, z)$, $f_z'(x, y, z)$ hususy önümleriniň $(n-1)$ tertipli birjynsly funksiýalardygyny subut etmeli.

112. Eger $u=f(x, y, z)$ iki gezek differensirlenýän n ölçegli, birjynsly funksiýa bolsa, onda

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)^2u=n(n-1)u$$

deňligiň ýerine ýetýändigini subut etmeli.

Berlen funksiýalaryň birinji we ikinji tertipli differensiallaryny tapmaly (bu ýerde x, y, z baglanyşyksyz üýtgeýän ululyklar):

113. $u=x^my^n$.

114. $u=\dfrac{x}{y}$.

115. $u=\sqrt{x^2+y^2}$.

116. $u=\ln\sqrt{x^2+y^2}$.

117. $u=e^{xy}$.

118. $u=xy+yz+zx$.

119. $u=\dfrac{z}{x^2+y^2}$.

120. $f(x,y,z)=\sqrt[z]{\dfrac{x}{y}}$ funksiýanyň $df(1, 1, 1)$ we $d^2f(1, 1, 1)$ differensiallaryny tapmaly.

121. $u=\sqrt{x^2+y^2+z^2}$ funksiýa üçin $d^2u\geq 0$ deňsizligi derňemeli.

122. x we y ululyklary absolýut ululyklary boýunça kiçi hasap edip, berlen aňlatmalar üçin takmyny formulalary getirip çykarmaly:

a) $(1+x)^m\cdot(1+y)^n$; b) $\ln(1+x)\cdot\ln(1+y)$; ç) $\operatorname{arctg}\dfrac{x+y}{1+xy}$.

123. Funksiýanyň artymyny onuň differensialy bilen çalşyryp, aňlatmalary hasaplamaly:

a) $1{,}002 \cdot 2{,}003^2 \cdot 3{,}004^3$; ç) $\sqrt{1{,}02^3 + 1{,}97^3}$; e) $0{,}97^{1{,}05}$.

b) $\dfrac{1{,}03^2}{\sqrt[3]{0{,}98\sqrt[4]{1{,}05^3}}}$; d) $\sin 29^{\circ} \cdot \mathrm{tg}46^{\circ}$;

124. Eger taraplary $x = 6\ m$ we $y = 8\ m$ bolan gönüburçlugyň birinji tarapy $2\,mm$ uzaldylsa we ikinji tarapy $5\ mm$ gysgaldylsa, ol gönüburçlugyň diagonaly we meýdany näçe üýtgär?

125. Sektoryň merkezi $\alpha = 60^{\circ}$ burçy $\Delta\alpha = 1^{\circ}$ ulaldyldy. Sektoryň meýdanynyň üýtgemezligi üçin onuň $R = 20\ sm$ radiusyny näçe kiçeltmeli?

126. Köpeltmek hasylynyň otnositel ýalňyşlygynyň köpeldijileriň otnositel ýalňyşlyklarynyň jemine takmyny deňdigini subut etmeli.

127. Silindriň esasynyň R radiusy we H beýikligi ölçelende şeýle netijeler alyndy:

$$R = 2{,}5\ m \pm 0{,}1\ m; \quad H = 4{,}0\ m \pm 0{,}2\ m.$$

Silindriň göwrümi haýsy absolýut Δ we otnositel δ ýalňyşlyklar bilen hasaplanyp bilner?

128. Üçburçlugyň taraplary $a = 200\ m \pm 2\ m$, $b = 300\ m \pm 5\ m$ we olaryň arasyndaky burçy $C = 60^{\circ} \pm 1^{\circ}$. Üçburçlugyň üçünji c tarapy nähili absolýut ýalňyşlyk bilen hasaplanyp bilner?

129. $f(x,y) = \sqrt{|xy|}$ funksiýanyň (0, 0) nokatda üznüksizdigini, şol nokatda $f_x'(0, 0)$ we $f_y'(0, 0)$ hususy önümleriň ikisiniň hem bardygyny, ýöne (0, 0) nokatda differensirlenmeýändigini subut etmeli.

(0, 0) nokadyň golaý töwereginde $f_x'(x, y)$ we $f_y'(x, y)$ önümleriň üýtgeýşini anyklamaly.

130. $x^2 + y^2 \neq 0$ bolanda $f(x,y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$ we $f(0, 0) = 0$ deňlik bilen kesgitlenýän funksiýanyň (0, 0) nokadynyň golaý töwereginde üznüksiz we çäkli $f_x'(x, y)$ we $f_y'(x,y)$ hususy önümleriniň bardygyny, ýöne (0, 0) nokatda differensirlenmeýändigini subut etmeli.

131. $x^2 + y^2 \neq 0$ bolanda $f(x,y) = (x^2 + y^2)\sin\dfrac{1}{x^2 + y^2}$ we $f(0, 0) = 0$ deňlikler bilen kesgitlenýän funksiýanyň nokadyň golaý töwereginde $f_x'(x, y)$ we $f_y'(x,y)$ hususy önümleriniň bardygyny, (0, 0) nokatda olaryň üzülýändigini we şol nokadyň islendik golaý töwereginde çäksizdigini, oňa garamazdan, ol funksiýanyň (0, 0) nokatda differensirlenýändigini subut etmeli.

132. Käbir güberçek ýaýlada çäkli $f_x'(x, y)$ we $f_y'(x,y)$ hususy önümleri bar bolan $f(x, y)$ funksiýanyň şol ýaýlada deňölçegli üznüksizdigini subut etmeli.

133. Eger $f(x, y)$ funksiýa y-iň her bir bellenen bahasy üçin üýtgeýän x ululyk boýunça üznüksiz bolsa we üýtgeýän y ululyk boýunça çäkli $f_y'(x,y)$ hususy önümi bar bolsa, onda ol funksiýanyň üýtgeýän x we y ululyklaryň toplumy boýunça üznüksizdigini subut etmeli.

Berlen funksiýalaryň hususy önümlerini tapmaly:

134. $u = x - y + x^2 + 2xy + y^2 + x^3 - 3x^2y - y^3 + x^4 - 4x^2y^2 + y^4$ funksiýanyň $\dfrac{\partial^4 u}{\partial x^4}, \dfrac{\partial^4 u}{\partial x^3 \partial y}, \dfrac{\partial^4 u}{\partial x^2 \partial y^2}$ hususy önümlerini tapmaly.

135. $u = x\ln(xy)$ funksiýanyň $\dfrac{\partial^3 u}{\partial x^2 \partial y}$ hususy önümini tapmaly.

136. $u = x^3\sin y + y^3\sin x$ funksiýanyň $\dfrac{\partial^6 u}{\partial x^3 \partial y^3}$ hususy önümini tapmaly.

137. $u = \operatorname{arctg}\dfrac{x + y + z - xyz}{1 - xy - xz - yz}$ funksiýanyň $\dfrac{\partial^3 u}{\partial x \partial y \partial z}$ hususy önümini tapmaly.

138. $u = e^{xyz}$ funksiýanyň $\dfrac{\partial^3 u}{\partial x \partial y \partial z}$ hususy önümini tapmaly.

139. $u = \ln\dfrac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}$ funksiýanyň $\dfrac{\partial^4 u}{\partial x \partial y \partial \xi \partial \eta}$ hususy önümini tapmaly.

140. $u = (x - x_0)^p(y - y_0)^q$ funksiýanyň $\dfrac{\partial^{p+q} u}{\partial x^p \partial y^q}$ hususy önümini tapmaly.

141. $u = \dfrac{x + y}{x - y}$ funksiýanyň $\dfrac{\partial^{m+n} u}{\partial x^m \partial y^n}$ hususy önümini tapmaly.

142. $u = (x^2 + y^2)e^{x+y}$ funksiýanyň $\dfrac{\partial^{m+n} u}{\partial x^m \partial y^n}$ hususy önümini tapmaly.

143. $u = xyze^{x+y+z}$ funksiýanyň $\dfrac{\partial^{p+q+r} u}{\partial x^p \partial y^q \partial z^r}$ hususy önümini tapmaly.

144. $f(x, y) = e^x\sin y$ funksiýanyň $f^{(m+n)}_{x^m y^n}(0,0)$ hususy önümini tapmaly.

145. Eger $u = f(xyz)$ bolsa, onda $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = F(t)$, $t = xyz$ bolýandygyny subut etmeli we F funksiýany tapmaly.

146. $u = x^4 - 2x^3y - 2xy^3 + y^4 + x^3 - 3x^2y - 3xy^2 + y^3 + 2x^2 - xy + 2y^2 + x + y + 1$ funksiýanyň d^4u differensialyny tapmaly, $\frac{\partial^4 u}{\partial x^4}, \frac{\partial^4 u}{\partial x^3 \partial y}, \frac{\partial^4 u}{\partial x^2 \partial y^2}, \frac{\partial^4 u}{\partial x \partial y^3}$ we $\frac{\partial^4 u}{\partial y^4}$ hususy önümler nämä deň?

Berlen funksiýalaryň görkezilen tertipdäki doly differensialyny tapmaly:

147. $d^3u, \quad u = x^3 + y^3 - 3xy\,(x - y).$

148. $d^3u, \quad u = \sin(x^2 + y^2).$

149. $d^{10}u, \quad u = \ln(x + y).$

150. $d^6u, \quad u = \cos x \operatorname{ch} y.$

151. $d^3u, \quad u = xyz.$

152. $d^4u, \quad u = \ln(x^x y^y z^z).$

153. $d^n u, \quad u = e^{ax+by}.$

154. $d^n u, \quad u = X(x)Y(y).$

155. $d^n u, \quad u = f(x + y + z).$

156. $d^n u, \quad u = e^{ax+by+cz}.$

157. Birjynsly n derejeli $P_n(x, y, z)$ köpagza üçin

$$d^n P_n(x, y, z) = n!P_n(dx, dy, dz)$$

deňligi subut etmeli.

158. Goý, $Au = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ bolsun. Berlen funksiýalar üçin Au we $A^2u = A(Au)$ tapmaly:

a) $u = \frac{x}{x^2 + y^2}$;

b) $u = \ln\sqrt{x^2 + y^2}$.

159. Goý, $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ bolsun. Berlen funksiýalar üçin Δu tapmaly:

a) $u = \sin x \operatorname{ch} y$;

b) $u = \ln\sqrt{x^2 + y^2}$.

160. Goý, $\Delta_1 u = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2$ we $\Delta_2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ bolsun. Berlen funksiýalar üçin $\Delta_1 u$ we $\Delta_2 u$ tapmaly:

a) $u = x^3 + y^3 + z^3 - 3xyz$;

b) $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$.

Çylşyrymly funksiýalaryň birinji we ikinji tertipli önümlerini tapmaly:

161. $u = f(x^2 + y^2 + z^2).$

162. $u = f\left(x, \frac{x}{y}\right).$

163. $u = f(x, xy, xyz).$

164. $u = f(x + y, xy)$ funksiýanyň $\frac{\partial^2 u}{\partial x \partial y}$ önümini tapmaly.

165. $u = f(x + y + z, x^2 + y^2 + z^2)$ funksiýa üçin $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ tapmaly.

Çylşyrymly funksiýalaryň birinji we ikinji tertipli doly differensialyny tapmaly (x, y, z – baglanyşyksyz üýtgeýän ululyklar):

166. $u = f(t)$, $t = x + y$.

167. $u = f(t)$, $t = \frac{y}{x}$.

168. $u = f(\sqrt{x^2 + y^2})$.

169. $u = f(t)$, $t = xyz$.

170. $u = f(x^2 + y^2 + z^2)$.

171. $u = f(\xi, \eta)$, $\xi = ax$, $\eta = by$.

172. $u = f(\xi, \eta)$, $\xi = x + y$, $\eta = x - y$.

173. $u = f(\xi, \eta)$, $\xi = xy$, $\eta = \frac{x}{y}$.

174. $u = f(x + y, z)$.

175. $u = f(x + y + z, x^2 + y^2 + z^2)$.

176. $u = f\left(\frac{x}{y}, \frac{y}{z}\right)$.

177. $u = f(x, y, z)$, $x = t$, $y = t^2$, $z = t^3$.

178. $u = f(\xi, \eta, \zeta)$, $\xi = ax$, $\eta = by$, $\zeta = cz$.

179. $u = f(\xi, \eta, \zeta)$, $\xi = x^2 + y^2$, $\eta = x^2 - y^2$, $\zeta = 2xy$.

Berlen funksiýalar üçin $d^n u$ differensialy tapmaly:

180. $u = f(ax + by + cz)$.

181. $u = f(ax, by, cz)$.

182. $u = f(\xi, \eta, \zeta)$, $\xi = a_1x + b_1y + c_1z$; $\eta = a_2x + b_2y + c_2z$; $\zeta = a_3x + b_3y + c_3z$.

183. Goý, $u = f(r)$, $r = \sqrt{x^2 + y^2 + z^2}$ we f iki gezek differensirlenýän funksiýa bolsun.

$$\Delta u = F(r)$$

deňligi subut etmeli we F funksiýany tapmaly, bu ýerde

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Laplasyň operatory.

184. Goý, u we v iki gezek differensirlenýän funksiýa we Δ Laplasyň operatory bolsun.

$$\Delta(uv) = u\Delta v + v\Delta u + 2\Delta(u, v)$$

deňligi subut etmeli, bu ýerde

$$\Delta(u, v) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}.$$

185. $u = \ln\sqrt{(x-a)^2 + (y-b)^2}$ funksiýanyň hemişelik a we b üçin Laplasyň $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ deňlemesini kanagatlandyrýandygyny subut etmeli.

186. Eger $u = u(x, y)$ funksiýa Laplasyň deňlemesini kanagatlandyrýan bolsa, onda $v = u\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ funksiýanyň hem şol deňlemäni kanagatlandyrýandygyny subut etmeli.

187. $u = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-b)^2}{4a^2 t}}$ funksiýanyň hemişelik a we b üçin ýylylyk geçirijiligiň $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ deňlemesini kanagatlandyrýandygyny subut etmeli.

188. Eger $u = u(x, t)$ funksiýa ýylylyk geçirijiligiň deňlemesini kanagatlandyrýan bolsa, onda

$$v = \frac{1}{a\sqrt{t}} e^{-\frac{x^2}{4a^2 t}} u\left(\frac{x}{a^2 t}, -\frac{1}{a^4 t}\right) \quad (t > 0)$$

funksiýanyň hem şol deňlemäni kanagatlandyrýandygyny subut etmeli.

189. $u = \frac{1}{r}$, $r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$ funksiýanyň $r \neq 0$ bolanda Laplasyň

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

deňlemesini kanagatlandyrýandygyny subut etmeli.

190. Eger $u = u(x, y, z)$ funksiýa Laplasyň deňlemesini kanagatlandyrýan bolsa, onda hemişelik k we $r = \sqrt{x^2 + y^2 + z^2}$ üçin

$$v = \frac{1}{r} u\left(\frac{k^2 x}{r^2}, \frac{k^2 y}{r^2}, \frac{k^2 z}{r^2}\right)$$

funksiýanyň hem şol deňlemäni kanagatlandyrýandygyny subut etmeli.

191. $u = \frac{C_1 e^{-ar} + C_2 e^{ar}}{r}$, $r = \sqrt{x^2 + y^2 + z^2}$ funksiýanyň hemişelik C_1, C_2 üçin Gelmgolsyň

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = a^2 u$$

deňlemesini kanagatlandyrýandygyny subut etmeli.

192. Goý, $u_1 = u_1(x, y, z)$ we $u_2 = u_2(x, y, z)$ funksiýalar Laplasyň $\Delta u = 0$ deňlemesini kanagatlandyrýan bolsun.

$$\upsilon = u_1(x, y, z) + (x^2 + y^2 + z^2)u_2(x, y, z)$$

funksiýanyň bigarmonik $\Delta(\Delta v) = 0$ deňlemäni kanagatlandyrýandygyny subut etmeli.

193. Goý, $f(x, y, z)$ funksiýa m gezek differensirlenýän n derejeli, birjynsly funksiýa bolsun. Deňligi subut etmeli:

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)^m f(x,y,z) = n(n-1)...(n-m+1)f(x,y,z).$$

194. Differensirlenýän f funksiýa we $z = \sin y + f(\sin x - \sin y)$ üçin

$$\sec x\frac{\partial z}{\partial x} + \sec y\frac{\partial z}{\partial y}$$

aňlatmany ýönekeýleşdirmeli.

195. Erkin differensirlenýän f funksiýa üçin $z = x^n f\left(\frac{y}{x^2}\right)$ funksiýanyň

$$x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} = nz$$

deňlemäni kanagatlandyrýandygyny subut etmeli.

196. Erkin differensirlenýän f funksiýa üçin $z = yf(x^2 - y^2)$ funksiýanyň

$$y^2\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = xz$$

deňlemäni kanagatlandyrýandygyny subut etmeli.

197. Differensirlenýän f funksiýa we

$$u = \frac{1}{12}x^4 - \frac{1}{6}x^3(y+z) + \frac{1}{2}x^2yz + f(y-x, z-x)$$

üçin

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

aňlatmany ýönekeýleşdirmeli.

198. Goý, $x^2 = v\omega$, $y^2 = u\omega$, $z^2 = uv$ we

$$f(x, y, z) = F(u, v, \omega)$$

bolsun. Deňligi subut etmeli:

$$xf_x' + yf_y' + zf_z' = uF_u' + vF_v' + \omega F_\omega'.$$

Erkin φ we ψ funksiýalary ýeterlik tertipde differensirlenýän hasap edip, berlen funksiýalar üçin deňlikleri barlamaly:

199. $z = \varphi(x^2 + y^2),\ y\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = 0 \cdot$

200. $z = \dfrac{y^2}{3x} + \varphi(xy),\ x^2\dfrac{\partial z}{\partial x} - xy\dfrac{\partial z}{\partial y} + y^2 = 0 \cdot$

201. $z = e^y\varphi\left(ye^{\frac{x^2}{2y^2}}\right),\ (x^2 - y^2)\dfrac{\partial z}{\partial x} + xy\dfrac{\partial z}{\partial y} = xyz.$

202. $u = x^n\varphi\left(\dfrac{y}{x^\alpha}, \dfrac{z}{x^\beta}\right),\ x\dfrac{\partial u}{\partial x} + \alpha y\dfrac{\partial u}{\partial y} + \beta z\dfrac{\partial u}{\partial z} = nu.$

203. $u = \dfrac{xy}{z}\ln x + x\varphi\left(\dfrac{y}{x}, \dfrac{z}{x}\right),\ x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = u + \dfrac{xy}{z}.$

204. $u = \varphi(x - at) + \psi(x + at),\ \dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2}.$

205. $u = x\varphi(x + y) + y\psi(x + y),\ \dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

206. $u = \varphi\left(\dfrac{y}{x}\right) + x\psi\left(\dfrac{y}{x}\right),\ x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0.$

207. $u = x^n\varphi\left(\dfrac{y}{x}\right) + x^{1-n}\psi\left(\dfrac{y}{x}\right),\ x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u.$

208. $u = \varphi[x + \psi(y)],\ \dfrac{\partial u}{\partial x}\dfrac{\partial^2 u}{\partial x\partial y} = \dfrac{\partial u}{\partial y}\dfrac{\partial^2 u}{\partial x^2}.$

Yzygider differensirleme arkaly, erkin φ we ψ funksiýalary ýoklamaly:

209. $z = x + \varphi(xy).$

210. $z = x\varphi\left(\dfrac{x}{y^2}\right).$

211. $z = \varphi\left(\sqrt{x^2 + y^2}\right).$

212. $u = \varphi(x - y, y - z).$

213. $u = \varphi\left(\dfrac{x}{y}, \dfrac{y}{z}\right).$

214. $z = \varphi(x) + \psi(y).$

215. $z = \varphi(x)\psi(y).$

216. $z = \varphi(x + y) + \psi(x - y).$

217. $z = x\varphi\left(\dfrac{x}{y}\right) + y\psi\left(\dfrac{x}{y}\right).$

218. $z = \varphi(xy) + \psi\left(\dfrac{x}{y}\right).$

219. $z = x^2 - y^2$ funksiýanyň $M(1, 1)$ nokatdaky Ox okunyň položitel ugry bilen $\alpha = 60^\circ$ burçy emele getirýän l ugry boýunça önümini tapmaly.

220. $z = x^2 - xy + y^2$ funksiýanyň $M(1, 1)$ nokatdaky Ox okunyň položitel ugry bilen α burçy emele getirýän l ugry boýunça önümini tapmaly. Haýsy ugur boýunça bu önüm: a) iň uly bahany alar; b) iň kiçi bahany alar; ç) nola deň bolar?

221. $z = \ln(x^2 + y^2)$ funksiýanyň $M(x_0, y_0)$ nokatdaky şol nokat arkaly geçýän dereje çyzygyna perpendikulýar ugur boýunça önümini tapmaly.

222. $z = 1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$ funksiýanyň $M\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ nokatdaky şol nokatdan $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ çyzyga geçirilen içki normalyň ugry boýunça önümini tapmaly.

223. $u = xyz$ funksiýanyň $M(1, 1, 1)$ nokatdaky $l\{\cos\alpha, \cos\beta, \cos\gamma\}$ ugur boýunça önümini tapmaly.

Funksiýanyň şol nokatdaky gradiýentiniň ululygy nämä deň?

224. $u = \frac{1}{r}$, $r = \sqrt{x^2 + y^2 + z^2}$ funksiýanyň $M_0(x_0, y_0, z_0)$ nokatdaky gradiýentiniň ululygyny we ugruny tapmaly.

225. $u = x^2 + y^2 - z^2$ funksiýanyň $A(\varepsilon, 0, 0)$ we $B(0, \varepsilon, 0)$ nokatlardaky gradiýentleriniň arasyndaky burçy tapmaly.

226. $M(1, 2, 2)$ nokatda $u = x + y + z$ funksiýanyň gradiýentiniň ululygy $\upsilon = x + y + z + 0{,}001\sin\left(10^6\pi\sqrt{x^2 + y^2 + z^2}\right)$ funksiýanyň gradiýentiniň ululygyndan nähili tapawutlanýar?

227. $M_0(x_0, y_0, z_0)$ nokatdaky $u = ax^2 + by^2 + cz^2$ we $\upsilon = ax^2 + by^2 + cz^2 + 2mx + 2ny + 2pz$ (a, b, c, m, n, p – hemişelik we $a^2 + b^2 + c^2 \neq 0$) funksiýalaryň gradiýentleriniň arasyndaky burçunyň M_0 nokat tükeniksizlige daşlaşanda nola ymtylýandygyny subut etmeli.

228. Goý, $u = f(x, y, z)$ iki gezek differensirlenýän funksiýa bolsun. Eger $\cos\alpha$, $\cos\beta$, $\cos\gamma$ l ugruň kosinus ugrukdyryjylary bolsa, onda $\frac{\partial^2 u}{\partial l^2} = \frac{\partial}{\partial l}\left(\frac{\partial u}{\partial l}\right)$ ikinji önümi tapmaly.

229. Goý, $u = f(x, y, z)$ iki gezek differensirlenýän funksiýa bolsun we $l_1\{\cos\alpha_1, \cos\beta_1, \cos\gamma_1\}$, $l_2\{\cos\alpha_2, \cos\beta_2, \cos\gamma_2\}$, $l_3\{\cos\alpha_3, \cos\beta_3, \cos\gamma_3\}$ – özara perpendikulýar üç ugurlar bolsun. Deňlikleri subut etmeli:

a) $\left(\frac{\partial u}{\partial l_1}\right)^2 + \left(\frac{\partial u}{\partial l_2}\right)^2 + \left(\frac{\partial u}{\partial l_3}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2;$

b) $\dfrac{\partial^2 u}{\partial l_1^2}+\dfrac{\partial^2 u}{\partial l_2^2}+\dfrac{\partial^2 u}{\partial l_3^2}=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}$.

230. Goý, $u = u(x, y)$ differensirlenýän funksiýa bolsun we $y = x^2$ bolanda $u(x,y)=1$ we $\dfrac{\partial u}{\partial x} = x$ bolsun. $y = x^2$ bolanda $\dfrac{\partial u}{\partial y}$ hususy önümi tapmaly.

231. Goý, $u = u(x, y)$ funksiýa

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0$$

deňlemäni kanagatlandyrýan bolsun we ondan daşgary

$$u(x, 2x) = x, \quad u_x'(x, 2x) = x^2$$

şertler ýerine ýetsin. Önümleri tapmaly:

$$u_{xx}''(x, 2x), \quad u_{xy}''(x, 2x), \quad u_{yy}''(x, 2x).$$

$z = z(x, y)$ funksiýa üçin deňlemeleri çözmeli:

232. $\dfrac{\partial^2 z}{\partial x^2}=0$. **233.** $\dfrac{\partial^2 z}{\partial x \partial y}=0$. **234.** $\dfrac{\partial^n z}{\partial y^n}=0$.

235. $u = u(x, y, z)$ funksiýa üçin deňlemäni çözmeli:

$$\frac{\partial^3 u}{\partial x \partial y \partial z}=0.$$

236. $\dfrac{\partial z}{\partial y}=x^2+2y$ deňlemäniň $z(x, x^2) = 1$ şerti kanagatlandyrýan $z = z(x, y)$ çözüwini tapmaly.

237. $\dfrac{\partial^2 z}{\partial y^2}=2$ deňlemäniň $z(x, 0) = 1$, $z_y'(x, 0) = x$ şertleri kanagatlandyrýan $z=z(x, y)$ çözüwini tapmaly.

238. $\dfrac{\partial^2 z}{\partial x \partial y}=x+y$ deňlemäniň $z(x, 0) = x$, $z(0, y) = y^2$ şertleri kanagatlandyrýan $z=z(x, y)$ çözüwini tapmaly.

§3. Anyk däl funksiýalaryň barlygy we differensirlenmegi

1. ***Anyk däl funksiýalaryň barlygy.*** Goý, (x_0, y_0) nokadyň käbir golaý töweregi̇nde üznüksiz we F_y hususy önümi bar bolan $F(x, y)$ funksiýa üçin:

1) F_y önüm (x_0, y_0) nokatda üznüksiz we $F_y(x_0, y_0) \neq 0$;

2) $F(x_0, y_0) = 0$.

Onda

a) x_0 we y_0 nokatlaryň degişlilikde $U(x_0)$ we $U(y_0)$ golaý töwerekleri tapylyp, $\forall x \in U(x_0)$ üçin

$$F(x, y) = 0 \tag{1}$$

deňleme bilen ýeke-täk $y = f(x) \in U(y_0)$ funksiýa kesgitlenýär, ýagny ol deňlemäniň ýeke-täk $y = f(x)$ çözüwi bardyr;

b) $y = f(x)$ funksiýa $U(x_0)$ golaý töwerekde üznüksiz we $f(x_0) = y_0$.

2. ***Anyk däl funksiýalaryň differensirlenmegi.*** Eger-de goşmaça

3) (x_0, y_0) nokadyň käbir golaý töwereginde $F(x, y)$ funksiýanyň F_x we F_y üznüksiz hususy önümleri bar bolsa, onda

ç) $f(x)$ funksiýanyň x_0 nokadyň golaý töwereginde üznüksiz önümi bardyr we

$$f'(x) = -\frac{F_x(x,y)}{F_y(x,y)}.$$

Bellik. Eger (1) deňlemede $F(x, y) = F(x_1, ..., x_m, y)$ hasap etsek, onda ýokardakylar ýaly degişli şertlerde

$$F(x_1, ..., x_m, y) = 0$$

deňleme bilen ýeke-täk $y = f(x_1, ..., x_m)$ funksiýa kesgitlenýär.

3. ***Deňlemeler sistemasy bilen kesgitlenýän anyk däl funksiýalar.*** Eger $\left(x^0, y^0\right) = \left(x_1^0, ..., x_m^0;\ y_1^0, ..., y_n^0\right)$ nokadyň käbir golaý töwereginde üznüksiz differensirlenýän $F_i(x, y) = F_i(x_1, ..., x_m, y_1, ..., y_n)$ $(i = 1, ..., n)$ funksiýalar üçin:

1) (x^0, y^0) nokatda $\dfrac{\partial(F_1, ..., F_n)}{\partial(y_1, ..., y_n)} = \begin{vmatrix} \dfrac{\partial F_1}{\partial y_1} \cdots \dfrac{\partial F_1}{\partial y_n} \\ \dfrac{\partial F_n}{\partial y_1} \cdots \dfrac{\partial F_n}{\partial y_n} \end{vmatrix} \neq 0;$

2) $F_i(x^0, y^0) = 0$ $(i = 1, ..., n)$.

Onda $x^0 \in R^m$ we $y^0 \in R^n$ nokatlaryň degişlilikde $R^m \supset U(x^0)$ we $R^n \supset U(y^0)$ golaý töwerekleri tapylyp, $\forall x \in U(x^0)$ üçin

$$F_i(x, y) = 0 \ (i = 1, ..., n)$$

sistema bilen differensirlenýän ýeke-täk

$$y_i = f_i(x_1, ..., x_m) \in U(y^0) \ (i = 1, ..., n)$$

funksiýalar kesgitlenýär we $f_i\left(x_1^0, ..., x_m^0\right) = y_i^0$ $(i = 1, ..., n)$.

Ol funksiýalaryň hususy önümlerini tapmak üçin olary şol funksiýalaryň çözüwi bolan $F_i(x, y) = 0$ $(i = 1, ..., n)$ ulgamda ornuna goýalyň we alnan toždestwony x_i $(i = 1, ..., m)$ boýunça differensirläp alarys:

$$\sum_{j=1}^{n} \frac{\partial F_i}{\partial y_j}\frac{\partial y_j}{\partial x_l} + \frac{\partial F_i}{\partial x_l} \quad (i = 1,...,n).$$

Bu bolsa $\frac{\partial y_1}{\partial x_l},...,\frac{\partial y_n}{\partial x_l}$ hususy önümlere görä çyzykly deňlemeler sistemasydyr we onuň kesgitleýjisi şert boýunça noldan tapawutlydyr. Diýmek, ol sistemanyň ýeke-täk çözüwi bardyr we ony Krameriň usuly boýunça tapmak bolar.

Gönükmeler

239. Her bir nokatda üzülýän Dirihläniň

$$y = \begin{cases} 1, \text{ eger } x \text{ rasional,} \\ 0, \text{ eger } x \text{ irrasional} \end{cases}$$

bolsa, funksiýasynyň $y^2 - y = 0$ deňlemäni kanagatlandyrýandygyny subut etmeli.

240. Goý, $f(x)$ funksiýa (a, b) interwalda kesgitlenen bolsun. Haýsy ýagdaýda $a < x < b$ bolanda

$$f(x)y = 0$$

deňlemäniň ýeke-täk üznüksiz $y = 0$ çözüwi bar?

241. Goý, $f(x)$ we $g(x)$ funksiýalar (a, b) interwalda kesgitlenen we üznüksiz bolsun. Haýsy ýagdaýda

$$f(x)y = g(x)$$

deňlemäniň (a, b) interwalda ýeke-täk üznüksiz çözüwi bar?

242. Goý,

$$x^2 + y^2 = 1 \qquad (1)$$

deňleme berlen bolsun we

$$y = y(x) \quad (-1 \le x \le 1) \qquad (2)$$

şol deňlemäni kanagatlandyrýan birbahaly funksiýa bolsun.

1. Näçe birbahaly (2) funksiýa (1) deňlemäni kanagatlandyrýar?
2. Näçe birbahaly üznüksiz (2) funksiýa (1) deňlemäni kanagatlandyrýar?
3. Näçe birbahaly üznüksiz (2) funksiýa (1) deňlemäni:

a) $y(0) = 1$ bolanda; b) $y(1) = 0$ bolanda kanagatlandyrýar?

243. Goý,

$$x^2 = y^2 \qquad (1)$$

deňleme berlen bolsun we

$$y = y(x) \quad (-\infty \le x \le +\infty) \qquad (2)$$

şol deňlemäni kanagatlandyrýan birbahaly funksiýa bolsun.

1. Näçe birbahaly (2) funksiýa (1) deňlemäni kanagatlandyrýar?
2. Näçe birbahaly üznüksiz (2) funksiýa (1) deňlemäni kanagatlandyrýar?

3. Näçe birbahaly differensirlenýän (2) funksiýa (1) deňlemäni kanagatlandyrýar?

4. Näçe birbahaly üznüksiz (2) funksiýa (1) deňlemäni:

a) $y(1) = 1$ bolanda; b) $y(0) = 0$ bolanda kanagatlandyrýar?

5. Näçe birbahaly üznüksiz $y = y(x)$ $(1 - \delta < x < 1 + \delta)$ funksiýa $y(1) = 1$ we δ ýeterlik kiçi bolanda kanagatlandyrýar?

244. $x^2 + y^2 = x^4 + y^4$ deňleme bilen köpbahaly $y = y(x)$ funksiýa kesgitlenýär. Haýsy ýaýlalarda ol funksiýa 1) birbahaly, 2) ikibahaly, 3) üçbahaly, 4) dörtbahaly? Ol funksiýanyň şahalanýan nokatlaryny we onuň birbahaly şahalaryny kesgitlemeli.

245. $(x^2+y^2)^2=x^2-y^2$ deňleme bilen kesgitlenýän köpbahaly $y=y(x)$ $(-1\leq x\leq 1)$ funksiýanyň şahalanýan nokatlaryny we üznüksiz birbahaly şahalaryny kesgitlemeli.

246. Goý, $f(x)$ funksiýa $a < x < b$ bolanda üznüksiz we $\varphi(y)$ funksiýa $c<y<d$ kesimde artýan we üznüksiz bolsun. Haýsy ýagdaýda $\varphi(y)=f(x)$ deňleme birbahaly $y=\varphi^{-1}(f(x))$ funksiýany kesgitleýär? Bu şertlerde aşakdaky funksiýalary derňemeli:

a) $\sin y + \operatorname{sh} y = x$; b) $e^{-y} = -\sin^2 x$.

247. Goý,

$$x = y + \varphi(y) \tag{1}$$

bolsun, bu ýerde $-a < y < a$ bolanda $\varphi(0) = 0$ we $|\varphi'(y)| \leq k < 1$. $-\varepsilon < x < \varepsilon$ bolanda $y(0) = 0$ şerti we (1) deňlemäni kanagatlandyrýan ýeke-täk differensirlenýän $y=y(x)$ funksiýanyň bardygyny subut etmeli.

248. Goý, $y=y(x)$

$$x = ky + \varphi(y)$$

deňleme bilen anyk däl görnüşde kesgitlenýän funksiýa bolsun, bu ýerde hemişelik $k \neq 0$ we $\varphi(y)$ differensirlenýän ω-periodly periodik hem-de $|\varphi'(y)| < |k|$ şerti kanagatlandyrýan funksiýa.

$|k|\omega$-periodly periodik $\psi(x)$ funksiýa üçin

$$y = \frac{x}{k} + \psi(x)$$

deňligi subut etmeli.

Berlen deňlemeler bilen kesgitlenýän $y=y(x)$ funksiýanyň y' we y'' önümlerini tapmaly:

249. $x^2 + 2xy - y^2 = a^2$.

250. $\ln\sqrt{x^2 + y^2} = \operatorname{arctg}\frac{y}{x}$.

251. $y - \varepsilon\sin y = x$ $(0 < \varepsilon < 1)$.

252. $x^y = y^x$ $(x \neq y)$.

253. $y = 2x \operatorname{arctg}\dfrac{y}{x}$.

254. $1 + xy = k(x - y)$ deňligi kanagatlandyrýan hemişelik k üçin

$$\frac{dx}{1 + x^2} = \frac{dy}{1 + y^2}$$

deňligi subut etmeli.

255. Eger

$$x^2y^2 + x^2 + y^2 - 1 = 0$$

bolsa, onda $xy > 0$ bolanda

$$\frac{dx}{\sqrt{1 - x^4}} + \frac{dy}{\sqrt{1 - y^4}} = 0$$

deňligi subut etmeli.

256. $x = 0$, $y = 0$ nokadyň golaý töwereginde

$$(x^2 + y^2)^2 = a^2(x^2 - y^2) \quad (a \neq 0)$$

deňleme bilen iki differensirlenýän $y = y_1(x)$ we $y = y_2(x)$ funksiýalaryň kesgitlenýändigini subut etmeli. $y_1'(0)$ we $y_2'(0)$ önümleri tapmaly.

257. Eger $(x^2 + y^2)^2 = 3x^2y - y^3$ bolsa, onda $x = 0$, $y = 0$ bolanda y' önümi tapmaly.

258. Eger $x^2 + xy + y^2 = 3$ bolsa, onda y', y'' we y''' önümleri tapmaly.

259. Eger $x^2 - xy + 2y^2 + x - y - 1 = 0$ bolsa, onda $x = 0$, $y = 1$ bolanda y', y'' we y''' önümleri tapmaly.

260. Ikinji tertipli

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

deňlik üçin $\dfrac{d^3}{dx^3}[(y'')^{-2/3}] = 0$ deňligi subut etmeli.

$z = z(x, y)$ funksiýa üçin birinji we ikinji tertipli hususy önümleri tapmaly:

261. $x^2 + y^2 + z^2 = a^2$.

262. $z^3 - 3xyz = a^3$.

263. $x + y + z = e^z$.

264. $z = \sqrt{x^2 - y^2} \cdot \operatorname{tg}\dfrac{z}{\sqrt{x^2 - y^2}}$.

265. $x + y + z = e^{-(x+y+z)}$.

266. Goý,

$$x^2 + y^2 + z^2 - 3xyz = 0 \tag{1}$$

we

$$f(x, y, z) = xy^2z^3$$

bolsun. Tapmaly:

a) $z = z(x, y)$ funksiýa (1) deňleme bilen kesgitlenýän anyk däl funksiýa bolanda $f'_x(1, 1, 1)$ önümi;

b) $y = y(x, z)$ funksiýa (1) deňleme bilen kesgitlenýän anyk däl funksiýa bolanda $f'_x(1, 1, 1)$ önümi.

Ol önümleriň dürli bolýandygyny düşündirmeli.

267. Eger $x^2 + 2y^2 + 3z^2 + xy - z - 9 = 0$ bolsa, onda $x = 1, y = -2, z = 1$ bolanda

$$\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial x \partial y}, \quad \frac{\partial^2 z}{\partial y^2}$$

önümleri tapmaly.

Berlen deňlemelerden dz we d^2z differensiallary tapmaly:

268. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

269. $xyz = x + y + z.$

270. $\frac{x}{z} = \ln\frac{z}{y} + 1.$

271. $z = x + \mathrm{arctg}\frac{y}{z - x}.$

272. $u^3 - 3(x + y)u^2 + z^3 = 0$ deňlemeden du differensialy tapmaly.

Berlen deňlemelerden görkezilen hususy önümleri tapmaly:

273. $F(x + y + z, x^2 + y^2 + z^2) = 0, \ \frac{\partial^2 z}{\partial x \partial y}.$

274. $F(x - y, y - z, z - x) = 0, \ \frac{\partial z}{\partial x}$ we $\frac{\partial z}{\partial y}.$

275. $F(x, x + y, x + y + z) = 0, \ \frac{\partial z}{\partial x}, \ \frac{\partial z}{\partial y}$ we $\frac{\partial^2 z}{\partial x^2}.$

276. $F(xz, yz) = 0, \ \frac{\partial^2 z}{\partial x^2}.$

277. Berlen deňlemelerden d^2z differensialy tapmaly:

a) $F(x + z, y + z) = 0$; b) $F\left(\frac{x}{z}, \frac{y}{z}\right) = 0.$

278. $z^3 - xz + y = 0$ deňleme bilen kesgitlenýän $z = z(x, y)$ funksiýa $x = 3$, $y = -2$ bolanda $z = 2$ bahany alýar. $dz(3, -2)$ we $d^2z(3, -2)$ differensiallary tapmaly.

279. Goý, $x = x(y, z)$, $y = y(x, z)$, $z = z(x, y)$ funksiýalar $F(x, y, z) = 0$ deňleme bilen kesgitlenýän bolsun.

$$\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = -1$$

deňligi subut etmeli.

280. $x + y + z = 0$, $x^2 + y^2 + z^2 = 1$ deňlemelerden $\frac{\partial x}{\partial z}$ we $\frac{\partial y}{\partial z}$ hususy önümleri tapmaly.

281. $x^2 + y^2 = \frac{1}{2}z^2$, $x + y + z = 2$ deňlemelerden $x = 1$, $y = -1$, $z = 2$ bolanda

$\frac{dx}{dz}$, $\frac{dy}{dz}$, $\frac{d^2x}{dz^2}$ we $\frac{d^2y}{dz^2}$ hususy önümleri tapmaly.

282. $xu - y\upsilon = 0$, $yu + x\upsilon = 1$ deňlemelerden $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial \upsilon}{\partial x}$ we $\frac{\partial \upsilon}{\partial y}$ hususy önümleri tapmaly.

283. $\left.\begin{aligned} xe^{u+\upsilon} + 2u\upsilon &= 1, \\ ye^{u-\upsilon} - \frac{u}{1+\upsilon} &= 2x \end{aligned}\right\}$ sistema bilen $u(1, 2) = 0$ we $\upsilon(1, 2) = 0$ bahalary alýan, differensirlenýän $u = u(x, y)$ we $\upsilon = \upsilon(x, y)$ funksiýalar kesgitlenýär. $du(1, 2)$ we $d\upsilon(1, 2)$ differensiallary tapmaly.

284. $u + \upsilon = x + y$, $\frac{\sin u}{\sin \upsilon} = \frac{x}{y}$ deňlemelerden du, $d\upsilon$, d^2u we $d^2\upsilon$ differensiallary tapmaly.

285. $e^{u/x}\cos\frac{\upsilon}{y} = \frac{x}{\sqrt{2}}$, $e^{u/x}\sin\frac{\upsilon}{y} = \frac{y}{\sqrt{2}}$ deňliklerden $x = 1$, $y = 1$, $u = 0$, $\upsilon = \frac{\pi}{4}$ bolanda du, $d\upsilon$, d^2u we $d^2\upsilon$ differensiallary tapmaly.

286. Goý, $x = t + t^{-1}$, $y = t^2 + t^{-2}$, $z = t^3 + t^{-3}$ bolsun. $\frac{dy}{dx}$, $\frac{dz}{dx}$, $\frac{d^2y}{dx^2}$ we $\frac{d^2z}{dx^2}$ hususy önümleri tapmaly.

287. Oxy tekizligiň haýsy ýaýlasynda

$$x = u + \upsilon, \quad y = u^2 + \upsilon^2, \quad z = u^3 + \upsilon^3$$

deňlemeler sistemasy üýtgeýän z ululygy x we y-iň funksiýasy hökmünde kesgitleýär, bu ýerde u we υ hakyky bahalary alýan parametrler. $\frac{\partial z}{\partial x}$ we $\frac{\partial z}{\partial y}$ hususy önümleri tapmaly.

288. $\left.\begin{array}{l} x = u + \ln\upsilon, \\ y = \upsilon - \ln u, \\ z = 2u + \upsilon \end{array}\right\}$ sistemadan $u = 1$, $\upsilon = 1$ bolanda $\frac{\partial z}{\partial x}$ we $\frac{\partial z}{\partial y}$ hususy önümleri tapmaly.

289. $\left.\begin{array}{l} x = u + \upsilon^2, \\ y = u^2 - \upsilon^3, \\ z = 2u\upsilon \end{array}\right\}$ sistemadan $u = 2$, $\upsilon = 1$ bolanda $\frac{\partial^2 z}{\partial x \partial y}$ hususy önümi tapmaly.

290. $x = \cos\varphi\cos\psi$, $y = \cos\varphi\sin\varphi$, $z = \sin\varphi$ deňliklerden $\frac{\partial^2 z}{\partial x^2}$ hususy önümi tapmaly.

291. $x = u\cos\upsilon$, $y = u\sin\upsilon$, $z = \upsilon$ deňliklerden $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2}$ hususy önümleri tapmaly.

292. $x = e^{u+\upsilon}$, $y = e^{u-\upsilon}$, $z = u\upsilon$ (u, υ – parametrler) deňlemeler sistemasy bilen kesgitlenýän $z = z(x, y)$ funksiýanyň $u = 0$ we $\upsilon = 0$ bolanda dz we d^2z differensiallaryny tapmaly.

293. $z = x^2 + y^2$ bolup, $y = y(x)$ funksiýa $x^2 - xy + y^2 = 1$ deňleme bilen kesgitlenýän bolsun. $\frac{dz}{dx}$ we $\frac{d^2 z}{dx^2}$ hususy önümleri tapmaly.

294. $u = \frac{x + z}{y + z}$ bolup, z funksiýa $ze^z = xe^x + ye^y$ deňleme bilen kesgitlenýän bolsun. $\frac{\partial u}{\partial x}$ we $\frac{\partial u}{\partial y}$ hususy önümleri tapmaly.

295. Goý, $x = \varphi(u, \upsilon)$, $y = \psi(u, \upsilon)$, $z = \chi(u, \upsilon)$ funksiýalar üýtgeýän z ululygy x we y ululyklaryň funksiýasy hökmünde kesgitleýär. $\frac{\partial z}{\partial x}$ we $\frac{\partial z}{\partial y}$ hususy önümleri tapmaly.

296. Goý, $x = \varphi(u, \upsilon)$, $y = \psi(u, \upsilon)$ funksiýalar berlen bolsun. Ters $u = u(x, y)$, $\upsilon = \upsilon(x, y)$ funksiýalaryň birinji we ikinji tertipli hususy önümlerini tapmaly.

297. Eger a) $x = u\cos\frac{\upsilon}{u}$, $y = u\sin\frac{\upsilon}{u}$;

b) $x = e^u + u\sin\upsilon$, $y = e^u - u\cos\upsilon$

bolsa, onda $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial \upsilon}{\partial x}$ we $\frac{\partial \upsilon}{\partial y}$ hususy önümleri tapmaly.

298. $u = u(x)$ funksiýa $u = f(x, y, z)$, $g(x, y, z) = 0$, $h(x, y, z) = 0$ deňlemeler sistemasy bilen kesgitlenýär. $\frac{du}{dx}$ we $\frac{d^2u}{dx^2}$ hususy önümleri tapmaly.

299. $u = u(x, y)$ funksiýa $u = f(x, y, z, t)$, $g(y, z, t) = 0$, $h(z, t) = 0$ deňlemeler sistemasy bilen kesgitlenýär. $\frac{\partial u}{\partial x}$ we $\frac{\partial u}{\partial y}$ hususy önümleri tapmaly.

300. Goý, $x = f(u, \upsilon, \omega)$, $y = g(u, \upsilon, \omega)$, $z = h(u, \upsilon, \omega)$ bolsun. $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ we $\frac{\partial u}{\partial z}$ hususy önümleri tapmaly.

301. Goý, $z = z(x, y)$ funksiýa $f(x, y, z, t) = 0$, $g(x, y, z, t) = 0$ deňlemeler sistemasyny kanagatlandyrýan bolsun, bu ýerde t parametr. dz differensialy tapmaly.

302. Goý, $u = f(z)$ bolsun, bu ýerde z ululyk $z = x + y\varphi(z)$ deňleme bilen anyk däl görnüşde kesgitlenýän funksiýa. Lagranžyň

$$\frac{\partial^n u}{\partial y^n} = \frac{\partial^{n-1}}{\partial x^{n-1}}\left\{[\varphi(z)]^n \frac{\partial u}{\partial x}\right\}$$

formulasyny subut etmeli. (*Görkezme*: *Formulany $n = 1$ üçin subut edip, matematiki induksiýa usulyny ulanmaly*).

303. Erkin differensirlenýän $\phi(u, \upsilon)$ funksiýa üçin

$$\phi(x - az, y - bz) = 0 \tag{1}$$

deňleme bilen kesgitlenýän $z = z(x, y)$ funksiýanyň

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 1$$

deňlemäniň çözüwi bolýandygyny subut etmeli, bu ýerde a we b hemişelik sanlar.

(1) üstüň geometrik manysyny anyklamaly.

304. Erkin differensirlenýän $\varphi(u, \upsilon)$ funksiýa üçin

$$\varphi\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0 \tag{2}$$

deňleme bilen kesgitlenýän $z = z(x, y)$ funksiýanyň

$$(x - x_0)\frac{\partial z}{\partial x} + (y - y_0)\frac{\partial z}{\partial y} = z - z_0$$

deňlemäni kanagatlandyrýandygyny subut etmeli.

(2) üstüň geometrik manysyny anyklamaly.

305. Erkin differensirlenýän $\varphi(u)$ funksiýa üçin

$$ax + by + cz = \varphi(x^2 + y^2 + z^2) \qquad (3)$$

deňleme bilen kesgitlenýän $z = z(x, y)$ funksiýanyň

$$(cy - bz)\frac{\partial z}{\partial x} + (az - cx)\frac{\partial z}{\partial y} = bx - ay$$

deňlemäni kanagatlandyrýandygyny subut etmeli, bu ýerde a, b we c hemişelik sanlar.

(3) üstüň geometrik manysyny anyklamaly.

306. $z = z(x, y)$ funksiýa $x^2 + y^2 + z^2 = yf\left(\frac{z}{y}\right)$ deňleme bilen kesgitlenen. Deňligi subut etmeli:

$$(x^2 - y^2 - z^2)\frac{\partial z}{\partial x} + 2xy\frac{\partial z}{\partial y} = 2xz.$$

307. $z = z(x, y)$ funksiýa

$$F(x + zy^{-1}, y + zx^{-1}) = 0$$

deňleme bilen berlen. Deňligi subut etmeli:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z - xy.$$

308. Erkin differensirlenýän $f(\alpha)$ funksiýa üçin

$$\left.\begin{aligned} x\cos\alpha + y\sin\alpha + \ln z = f(\alpha), \\ -x\sin\alpha + y\cos\alpha = f'(\alpha) \end{aligned}\right\}$$

deňlemeler sistemasy bilen kesgitlenýän $z = z(x, y)$ funksiýanyň

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = z^2$$

deňlemäni kanagatlandyrýandygyny derňemeli, bu ýerde $\alpha = \alpha(x, y)$ üýtgeýän parametr.

309. $\left.\begin{aligned} z = \alpha x + \frac{y}{\alpha} + f(\alpha), \\ 0 = x - \frac{y}{\alpha^2} + f'(\alpha) \end{aligned}\right\}$ deňlemeler sistemasy bilen kesgitlenýän $z = z(x, y)$ funksiýanyň

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1$$

deňlemäni kanagatlandyrýandygyny subut etmeli.

310. $\left.\begin{array}{r}[z-f(\alpha)]^2 = x^2\left(y^2-\alpha^2\right),\\ [z-f(\alpha)]\,f'(\alpha) = \alpha x^2\end{array}\right\}$ deňlemeler sistemasy bilen kesgitlenýän $z=z(x,y)$ funksiýanyň

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = xy$$

deňlemäni kanagatlandyrýandygyny subut etmeli.

311. $\left.\begin{array}{l}z = \alpha x + y\varphi(\alpha) + \psi(\alpha),\\ 0 = x + y\varphi'(\alpha) + \psi'(\alpha)\end{array}\right\}$ deňlemeler sistemasy bilen kesgitlenýän $z=z(x,y)$ funksiýanyň

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$$

deňlemäni kanagatlandyrýandygyny subut etmeli.

312. $y = x\varphi(z) + \psi(z)$ deňleme bilen anyk däl görnüşde kesgitlenýän $z=z(x,y)$ funksiýanyň

$$\left(\frac{\partial z}{\partial y}\right)^2 \frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x \partial y} + \left(\frac{\partial z}{\partial x}\right)^2 \frac{\partial^2 z}{\partial y^2} = 0$$

deňlemäni kanagatlandyrýandygyny subut etmeli.

§4. Üýtgeýän ululyklary çalşyrmak

1. ***Ady önümleri özünde saklaýan***

$$P = F(x, y, y_x', y_{x^2}'', \ldots) \tag{1}$$

aňlatmada üýtgeýän ululyklary çalşyrmak

a) diňe x ululyk t bilen $x = \varphi(t)$ formula arkaly çalşyrylýar. Bu halda y hem t görä funksiýa bolar, ýagny $y = y(t)$. Şonuň üçin önümleri

$$y_x' = \frac{y_t'}{x_t'}, \qquad y_{x^2}'' = \frac{d}{dx}(y_x') = \frac{d}{dt}\left(\frac{y_t'}{x_t'}\right)\frac{1}{x_t'} = \frac{y_{t^2}''x_t' - x_{t^2}''y_t'}{x_{t^3}'}, \ldots \tag{2}$$

formula boýunça çalşyryp,

$$P = F_1(t, y, y_t', y_{t^2}'', \ldots)$$

aňlatmany alarys;

b) x we y ululyklaryň ikisi hem

$$x = \varphi(t, u), \qquad y = \psi(t, u) \tag{3}$$

formulalar arkaly t we u ululyklar bilen çalşyrylýar. Bu ýerde $u = u(t)$. Şonuň üçin x we y ululyklara t görä çylşyrymly funksiýa hökmünde garap, önümleri taparys:

$$
\begin{aligned}
&x_t' = \varphi_t' + \varphi_u' u_t', \quad y_t' = \psi_t' + \psi_u' u_t', \\
&x_{t^2}'' = \varphi_{t^2}'' + 2\varphi_{ut}'' u_t' + \varphi_{u^2}'' u_t'^2 + \varphi_u' u_{t^2}'', \\
&y_{t^2}'' = \psi_{t^2}'' + 2\psi_{ut}'' u_t' + \psi_{u^2}'' u_t'^2 + \psi_u' u_{t^2}''.
\end{aligned} \tag{4}
$$

Bu önümleri (2) formulada goýup, y funksiýanyň x görä önümlerini, u funksiýanyň t görä önümleri bilen çalşyrarys we şonuň esasynda alarys:

$$P = F_2(t, u, u_t', u_{t^2}'', \ldots).$$

Eger (1) aňlatmada x we y ululyklaryň orunlaryny çalşyrmaklyk talap edilse, onda $x = u$, $y = t$ çalşyrmany ulanyp, funksiýanyň önümleri üçin

$$y_x' = \frac{1}{x_y'}, \quad y_{x^2}'' = -\frac{x_{y^2}''}{(x_y')^3}, \quad y_{x^3}''' = \frac{3x_{y^2}''^{\,2} - x_y' x_{y^3}'''}{x_y'^5}, \ldots$$

formulalary alarys.

Bellik. Dekart koordinatalar sistemasyndan polýar koordinatalar sistemasyna geçmek üçin ulanylýan $x = r\cos\varphi$, $y = r\sin\varphi$ formulalar x we y ululyklary r we φ bilen çalşyrmaklygyň bir görnüşidir.

1-nji mysal. $P = \dfrac{y_{x^2}'' - y_x'(1 + y_x')^2}{(1 + y_x')^3}$ aňlatmany $x = t - y$ çalşyryp özgertmeli.

Ç.B. Bu halda (3) funksiýalary şeýle görnüşde ýazmak bolar:

$$x = t - u, \quad y = u.$$

Onda $x_t' = 1 - u_t'$, $y_t' = u_t'$; $x_{t^2}'' = -u_{t^2}''$, $y_{t^2}'' = u_{t^2}''$ deňlikleriň esasynda, (2) formulany ulanyp, önümleri taparys:

$$y_x' = \frac{u_t'}{1 - u_t'}, \quad y_{x^2}'' = \frac{u_{t^2}''(1 - u_t') + u_{t^2}'' u_t'}{(1 - u_t')^3} = \frac{u_{t^2}''}{(1 - u_t')^3}.$$

Şeýlelikde,

$$P = \frac{\dfrac{u_{t^2}''}{(1 - u_t')^3} - \dfrac{u_t'}{1 - u_t'}\left(1 + \dfrac{u_t'}{1 - u_t'}\right)^2}{\left(1 + \dfrac{u_t'}{1 - u_t'}\right)^3} = u_{t^2}'' - u_t'.$$ **Ç.S.**

2. ***Hususy önümleri özünde saklaýan***

$$P = F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \ldots\right)$$

aňlatmada üýtgeýän ululyklary çalşyrmak

a) diňe x we y ululyklar u we υ ululyklar bilen çalşyrylýar. Goý, ol ululyklar bir-birleri bilen

$$x = \varphi(u, \upsilon), \quad y = \psi(u, \upsilon) \tag{5}$$

formulalar arkaly baglanyşykda bolsun. Onda $z = z(x, y)$ funksiýa u, υ üýtgeýän ululyklara görä çylşyrymly funksiýa hökmünde garap, $\frac{\partial z}{\partial x}$ we $\frac{\partial z}{\partial y}$ hususy önümleri

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial \upsilon} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \upsilon} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \upsilon} \tag{6}$$

çyzykly deňlemeler sistemasyny çözüp taparys:

$$\frac{\partial z}{\partial x} = A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial \upsilon}, \quad \frac{\partial z}{\partial y} = C\frac{\partial z}{\partial u} + D\frac{\partial z}{\partial \upsilon}, \tag{7}$$

bu ýerde A, B, C we D koeffisiýentler üýtgeýän x, y ululyklara bagly, ýöne z-e bagly däl funksiýalardyr. Soňra olary ulanyp, ýokary tertipli önümleri taparys we netijede alarys:

$$P = F_1\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}, \ldots\right).$$

Bellik. x we y ululyklary u we υ bilen çalşyrmaklyk $u = \varphi(x, y)$, $\upsilon = \psi(x, y)$ formulalar arkaly amala aşyrylýan halda hususy önümler

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y}$$

formulalar boýunça tapylýar. Olardan peýdalanylyp, ýokary tertipli hususy önümler tapylýar.

b) goý, x, y we z ululyklaryň üçüsi hem

$$x = \varphi(u, \upsilon, w), \quad y = \psi(u, \upsilon, w), \quad z = g(u, \upsilon, w)$$

formulalar arkaly u, υ we w ululyklar bilen çalşyrylýan bolsun.

Bu halda $\frac{\partial z}{\partial x}$ we $\frac{\partial z}{\partial y}$ hususy önümler

$$\frac{\partial z}{\partial x}\left[\frac{\partial}{\partial u} + \frac{\partial}{\partial w}\frac{\partial w}{\partial u}\right] + \frac{\partial z}{\partial y}\left[\frac{\partial \psi}{\partial u} + \frac{\partial \psi}{\partial w}\frac{\partial w}{\partial u}\right] = \frac{\partial g}{\partial u} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial u},$$

$$\frac{\partial z}{\partial x}\left[\frac{\partial}{\partial \upsilon} + \frac{\partial}{\partial w}\frac{\partial w}{\partial \upsilon}\right] + \frac{\partial z}{\partial y}\left[\frac{\partial \psi}{\partial \upsilon} + \frac{\partial \psi}{\partial w}\frac{\partial w}{\partial \upsilon}\right] = \frac{\partial g}{\partial \upsilon} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial \upsilon}$$

çyzykly deňlemeler sistemasy çözülip tapylýar, soňra ikinji we ondan ýokary tertipli hususy önümleri kesgitlemek bolar we netijede alarys:

$$P = F_2\left(u, v, w, \frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}, \ldots\right).$$

2-nji mysal. $z_{xx} + z_{xy} + z_x = z$ deňlemede

$$x = u + \upsilon, \quad y = u - \upsilon, \quad z = we^{\upsilon - u} \quad (w = w(u, \upsilon))$$

formulalar boýunça u, υ, w ululyklara geçmeli.

Ç.B. Bu mysalda x, y, z ululyklary çalşyrmaklyk b) haldaky ýalydyr. Şonuň üçin $dw = w_u du + w_\upsilon d\upsilon$ deňligi ulanyp alarys:

$$\begin{cases} dx = du + d\upsilon \\ dy = du - d\upsilon \\ dz = e^{\upsilon - u}(w_u du + w_\upsilon d\upsilon) + we^{\upsilon - u}(d\upsilon - du). \end{cases}$$

Deňlikleriň ilki ikisinden $du = 0{,}5(dx + dy)$, $d\upsilon = 0{,}5(dx - dy)$ differensiallary tapyp, olary üçünji deňligiň sag böleginde goýarys, çep bölegini bolsa $dz = z_x dx + z_y dy$ bilen çalşyryp,

$$z_x dx + z_y dy = 0{,}5e^{\upsilon - u}[w_u(dx + dy) + w_\upsilon(dx - dy)] - we^{\upsilon - u}dy$$

deňligi alarys. Bu deňligiň iki bölegindäki dx differensialyň koeffisiýentlerini deňläp,

$$z_x = 0{,}5e^{\upsilon - u}(w_u + w_\upsilon)$$

deňligi alarys. $z(x, y)$ funksiýanyň ikinji tertipli hususy önümlerini täze girizilen funksiýalar arkaly aňlatmak üçin, ilki bilen, z_x önümiň differensialyny tapalyň: $dz_x = z_{xx}dx + z_{xy}dy$ deňligiň esasynda

$$z_{xx}dx + z_{xy}dy = 0{,}5e^{\upsilon - u}(d\upsilon - du)(w_u + w_\upsilon) + 0{,}5e^{\upsilon - u}(w_{uu}du + w_{u\upsilon}d\upsilon + w_{\upsilon u}du + w_{\upsilon\upsilon}d\upsilon).$$

Bu deňlikde du we $d\upsilon$ differensiallaryň bahalaryny goýup,

$$z_{xx}dx + z_{xy}dy = -0{,}5e^{\upsilon - u}(w_u + w_\upsilon)dy + 0{,}25e^{\upsilon - u}[w_{uu}(dx + dy) + 2w_{u\upsilon}dx + w_{\upsilon\upsilon}(dx - dy)]$$

deňligi alarys. Deňligiň iki bölegindäki dx we dy differensiallaryň koeffisiýentlerini deňläp, z_{xx} we $z_{xy} = z_{yx}$ önümleri taparys:

$$z_{xx} = 0{,}25e^{\upsilon - u}(w_{uu} + 2w_{u\upsilon} + w_{\upsilon\upsilon}),$$

$$z_{xy} = -0{,}5e^{\upsilon - u}(w_u + w_\upsilon) + 0{,}25e^{\upsilon - u}(w_{uu} - w_{\upsilon\upsilon}).$$

Indi tapylan önümleri berlen deňlemede goýup,

$$w_{uu} + w_{u\upsilon} = 2w$$

deňlemäni alarys. **Ç.S.**

Gönükmeler

y-i baglanyşyksyz üýtgeýän ululyk hasap edip, berlen deňlemeleri özgertmeli:

313. $y'y''' - 3y''^2 = x$.

314. $y'^2y^{IV} - 10y'y''y''' + 15y''^3 = 0$.

315. x-i funksiýa we $t = xy$ ululygy baglanyşyksyz üýtgeýän ululyk hasap edip, $y'' + \dfrac{2}{x}y' + y = 0$ deňlemäni özgertmeli.

Täze üýtgeýän ululyklary girizip, ady differensial deňlemeleri özgertmeli:

316. $x^2y'' + xy' + y = 0,\ x = e^t$.

317. $y''' = \dfrac{6y}{x^3},\ t = \ln|x|$.

318. $(1 - x^2)y'' - xy' + n^2y = 0,\ x = \cos t$.

319. $y'' + y'\operatorname{th}x + \dfrac{m^2}{\operatorname{ch}^2 x}y = 0,\ \ x = \ln \operatorname{tg}\dfrac{t}{2}$.

320. $y'' + p(x)y' + q(x)y = 0,\ \ y = ue^{-\frac{1}{2}\int\limits_{x_0}^{x} p(\xi)d\xi}$, bu ýerde $p(x) \in C(1)$

321. $x^4y'' + xyy' - 2y^2 = 0,\ x = e^t$ we $y = ue^{2t}$, bu ýerde $u = u(t)$.

322. $(1 + x^2)^2y'' = y,\ x = \operatorname{tg}t$ we $y = \dfrac{u}{\cos t}$, bu ýerde $u = u(t)$.

323. $(1 - x^2)^2y'' = -y,\ x = \operatorname{th}t$ we $y = \dfrac{u}{\operatorname{ch}t}$, bu ýerde $u = u(t)$.

324. $y'' + (x + y)(1 + y')^3 = 0,\ x = u + t$ we $y = u - t$, bu ýerde $u = u(t)$.

325. $y''' - x^3y'' + xy' - y = 0,\ x = \dfrac{l}{t}$ we $y = \dfrac{u}{t}$, bu ýerde $u = u(t)$.

326. Stoksuň $y'' = \dfrac{Ay}{(x-a)^2(x-b)^2}$ deňlemesini

$$u = \frac{y}{x - b}, \qquad t = \ln\left|\frac{x - a}{x - b}\right|$$

täze üýtgeýän ululyklary girizip we u ululygy t üýtgeýän ululygyň funksiýasy hasap edip özgertmeli.

327. Eger $\dfrac{d^2y}{dx^2} + p(x)\dfrac{dy}{dx} + q(x)y = 0$ deňleme $x = \varphi(\xi)$ çalşyrma girizilip,

$$\frac{d^2y}{d\xi^2} + P(\xi)\frac{dy}{d\xi} + Q(\xi)y = 0$$

deňleme özgerdilýän bolsa, onda

$$\left[2P(\xi)Q(\xi)+Q'(\xi)\right]\left[Q(\xi)\right]^{-\frac{3}{2}}=\left[2p(x)q(x)+q'(x)\right]\left[q(x)\right]^{-\frac{3}{2}}$$

deňligi subut etmeli.

328. y, y', y'' üýtgeýän ululyklara görä birjynsly bolan F funksiýa üçin $F(y,y',y'')=0$ deňlemäni $y=e^{\int_{x_0}^{x} u\,dx}$ alyp özgertmeli.

329. Argumentlerine görä birjynsly bolan F funksiýa üçin $F(x^2y'', xy', y)=0$ deňlemäni $u=x\dfrac{y'}{y}$ alyp özgertmeli.

330. $y'''(1+y'^2)-3y'y''^2=0$ deňlemäniň gomografik

$$x=\frac{a_1\xi+b_1\eta+c_1}{a\xi+b\eta+c}, \qquad y=\frac{a_2\xi+b_2\eta+c_2}{a\xi+b\eta+c}$$

özgertmede görnüşini üýtgetmeýändigini subut etmeli.

(Görkezme: Seredilýän özgertmäni ýönekeý

$$x=\alpha X+\beta Y+\gamma, \quad y=Y, \quad X=\frac{1}{X_1}, \qquad Y=\frac{Y_1}{X_1}, \quad X_1=a\xi+b\eta+c,$$

$$Y_1=a_2\xi+b_2\eta+c_2$$

özgertmeleriň kompozisiýasy hökmünde aňlatmaly).

331. $S[x(t)]=\dfrac{x'''(t)}{x'(t)}-\dfrac{3}{2}\left[\dfrac{x''(t)}{x'(t)}\right]^2$ şwarsianyň

$$y=\frac{ax(t)+b}{cx(t)+d}, \qquad (ad-bc\neq 0)$$

drob çyzygynyň çalşyrmada öz bahasyny üýtgetmeýändigini subut etmeli.

Dekart koordinatalaryny r we φ polýar koordinatalaryna özgerdýän $x=r\cos\varphi$ we $y=r\sin\varphi$ formulalary ulanyp, deňlemeleri özgertmeli:

332. $\dfrac{dy}{dx}=\dfrac{x+y}{x-y}$.

333. $(xy'-y)^2=2xy(1+y'^2)$.

334. $(x^2+y^2)^2y''=(x+yy')^3$.

335. $\dfrac{x+yy'}{xy'-y}$ aňlatmany polýar koordinatalaryna özgertmeli.

336. Tekiz çyzygyň $K = \dfrac{|y_{xx}''|}{(1 + y_x'^2)^{\frac{3}{2}}}$ egriligini r we φ polýar koordinatalarynda aňlatmaly.

337. $\dfrac{dx}{dt} = y + kx(x^2 + y^2)$, $\dfrac{dy}{dt} = -x + ky(x^2 + y^2)$ deňlemeler sistemasynda polýar koordinatalaryna geçmeli.

338. Täze $r = \sqrt{x^2 + y^2}$, $\varphi = \operatorname{arctg}\dfrac{y}{x}$ funksiýalary girizip,

$$W = x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}$$

aňlatmany özgertmeli.

339. Ležandryň özgertmesinde $y = y(x)$ çyzygyň her bir (x, y) nokadyna (X, Y) nokat degişli edilýär, bu ýerde

$$X = y', \qquad Y = xy' - y.$$

Y', Y'' we Y''' önümleri tapmaly.

Täze ξ we η üýtgeýän ululyklary girizip, deňlemeleri çözmeli:

340. $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y}$, $\xi = x + y$ we $\eta = x - y$.

341. $y\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = 0$, $\xi = x$ we $\eta = x^2 + y^2$.

342. $a\dfrac{\partial z}{\partial x} + b\dfrac{\partial z}{\partial y} = 1$, $(a \neq 0)$, $\xi = x$ we $\eta = y - bz$.

343. $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$, $\xi = x$ we $\eta = \dfrac{y}{x}$.

u we v täze baglanyşyksyz üýtgeýän ululyklary girizip, deňlemeleri özgertmeli:

344. $x\dfrac{\partial z}{\partial x} + \sqrt{1 + y^2}\,\dfrac{\partial z}{\partial y} = xy$, $u = \ln x$ we $v = \ln\left(y + \sqrt{1 + y^2}\right)$.

345. $(x + y)\dfrac{\partial z}{\partial x} - (x - y)\dfrac{\partial z}{\partial y} = 0$, $u = \ln\sqrt{x^2 + y^2}$ we $v = \operatorname{arctg}\dfrac{y}{x}$.

346. $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z + \sqrt{x^2 + y^2 + z^2}$, $u = \dfrac{y}{x}$ we $v = z + \sqrt{x^2 + y^2 + z^2}$.

347. $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \dfrac{x}{z}$, $u = 2x - z^2$ we $v = \dfrac{y}{z}$.

348. $(x+z)\dfrac{\partial z}{\partial x}+(y+z)\dfrac{\partial z}{\partial y}=x+y+z$, $u=x+z$ we $v=y+z$.

349. Täze baglanyşyksyz üýtgeýän $\xi = y + ze^{-x}$ we $\eta = x + ze^{(-y)}$ ululyklary girizip,

$$(z+e^{x})\frac{\partial z}{\partial x}+(z+e^{y})\frac{\partial z}{\partial y}-(z^{2}-e^{x+y})$$

aňlatmany özgertmeli.

350. $x=uv$, $y=\dfrac{1}{2}(u^{2}-v^{2})$ alyp, $\left(\dfrac{\partial z}{\partial x}\right)^{2}+\left(\dfrac{\partial z}{\partial y}\right)^{2}$ aňlatmany özgertmeli.

351. $\xi=x$, $\eta=y-x$, $\zeta=z-x$ alyp, $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0$ deňlemäni özgertmeli.

352. x-i funksiýa, y we z-i baglanyşyksyz üýtgeýän ululyk hasap edip, $(x-z)\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=0$ deňlemäni özgertmeli.

353. x-i funksiýa, $u=y-z$, $v=y+z$ ululyklary bolsa baglanyşyksyz üýtgeýän ululyk hasap edip, $(y-z)\dfrac{\partial z}{\partial x}+(y+z)\dfrac{\partial z}{\partial y}=0$ deňlemäni özgertmeli.

354. x-i funksiýa, $u=xz$, $v=yz$ ululyklary bolsa baglanyşyksyz üýtgeýän ululyk hasap edip,

$$A=\left(\frac{\partial z}{\partial x}\right)^{2}+\left(\frac{\partial z}{\partial y}\right)^{2}$$

aňlatmany özgertmeli.

355. $(y+z+u)\dfrac{\partial u}{\partial x}+(x+z+u)\dfrac{\partial u}{\partial y}+(x+y+u)\dfrac{\partial u}{\partial z}=x+y+z$

deňlemede $e^{\xi}=x-u$, $e^{\eta}=y-u$, $e^{\zeta}=z-u$ orun çalşyrma girizmeli.

Aşakdaky deňlemelerde täze üýtgeýän u, v, w ululyklara geçmeli, bu ýerde $w=w(u, v)$:

356. $y\dfrac{\partial z}{\partial x}-x\dfrac{\partial z}{\partial y}=(y-x)z$, $u=x^{2}+y^{2}$, $v=\dfrac{1}{x}+\dfrac{1}{y}$, $w=\ln z-(x+y)$.

357. $x^{2}\dfrac{\partial z}{\partial x}+y^{2}\dfrac{\partial z}{\partial y}=z^{2}$, $u=x$, $v=\dfrac{1}{y}-\dfrac{1}{x}$, $w=\dfrac{1}{z}-\dfrac{1}{x}$.

358. $(xy+z)\dfrac{\partial z}{\partial x}+(1-y^{2})\dfrac{\partial z}{\partial y}=x+yz$, $u=yz-x$, $v=xz-y$, $w=xy-z$.

359. $\left(x\dfrac{\partial z}{\partial x}\right)^{2}+\left(y\dfrac{\partial z}{\partial y}\right)^{2}=z^{2}\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial z}{\partial y}$, $x=ue^{w}$, $y=ve^{w}$, $z=we^{w}$.

360. $u = \ln\sqrt{x^2 + y^2}$, $v = \operatorname{arctg} z$, $w = x + y + z$ alyp,

$$(x - y):\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

aňlatmany özgertmeli, bu ýerde $w = w(u, v)$.

361. $u = xe^z$, $v = ye^z$, $w = ze^z$, alyp, $A = \frac{\partial z}{\partial x} : \frac{\partial z}{\partial y}$ aňlatmany özgertmeli, bu ýerde $w = w(u, v)$.

362. $\xi = \frac{x}{z}$, $\eta = \frac{y}{z}$, $\zeta = z$, $w = \frac{u}{z}$ orun çalşyrmalary girizip,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = u + \frac{xy}{z}$$

aňlatmany özgertmeli, bu ýerde $w = w(\xi, \eta, \zeta)$.

Dekart koordinatalaryny r we φ polýar koordinatalaryna özgerdýän $x = r\cos\varphi$ we $y = r\sin\varphi$ formulalary ulanyp, aňlatmalary özgertmeli:

363. $w = x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x}$.

364. $w = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

365. $w = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$.

366. $w = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

367. $w = x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

368. $w = y^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + x^2\frac{\partial^2 z}{\partial y^2} - \left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$.

369. $I = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$ aňlatmada $x = r\cos\varphi$ we $y = r\sin\varphi$ ornuna goýmaly.

370. Täze $\xi = x - at$, $\eta = x + at$ üýtgeýän ululyklary girizip,

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$$

deňlemäni çözmeli.

u we v üýtgeýän ululyklary täze baglanyşyksyz üýtgeýän ululyklar hasap edip, deňlemeleri özgertmeli:

371. $2\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$, $u = x + 2y + 2$ we $v = x - y - 1$.

372. $(1+x^2)\dfrac{\partial^2 z}{\partial x^2}+(1+y^2)\dfrac{\partial^2 z}{\partial y^2}+x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=0,\ u=\ln\left(x+\sqrt{1+x^2}\right)$ we $v=\ln\left(y+\sqrt{1+y^2}\right)$.

373. $ax^2\dfrac{\partial^2 z}{\partial x^2}+2bxy\dfrac{\partial^2 z}{\partial x\partial y}+cy^2\dfrac{\partial^2 z}{\partial y^2}=0,\ u=\ln x$ we $v=\ln y$ (a, b, c – hemişelik sanlar).

374. $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0,\ u=\dfrac{x}{x^2+y^2}$ we $v=-\dfrac{y}{x^2+y^2}$.

375. $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}+m^2z=0,\ x=e^u\cos v$ we $y=e^u\sin v$.

376. $\dfrac{\partial^2 z}{\partial x^2}-y\dfrac{\partial^2 z}{\partial y^2}=\dfrac{1}{2}\dfrac{\partial z}{\partial y}$ $(y>0)$, $u=x-2\sqrt{y}$ we $v=x+2\sqrt{y}$.

377. $x^2\dfrac{\partial^2 z}{\partial x^2}-y^2\dfrac{\partial^2 z}{\partial y^2}=0,\ u=xy$ we $v=\dfrac{x}{y}$.

378. $x^2\dfrac{\partial^2 z}{\partial x^2}-(x^2+y^2)\dfrac{\partial^2 z}{\partial x\partial y}+y^2\dfrac{\partial^2 z}{\partial y^2}=0,\ u=x+y$ we $v=\dfrac{1}{x}+\dfrac{1}{y}$.

379. $xy\dfrac{\partial^2 z}{\partial x^2}-(x^2+y^2)\dfrac{\partial^2 z}{\partial x\partial y}+xy\dfrac{\partial^2 z}{\partial y^2}+y\dfrac{\partial z}{\partial x}+x\dfrac{\partial z}{\partial y}=0,\ u=\dfrac{1}{2}(x^2+y^2)$ we $v=xy$.

380. $x^2\dfrac{\partial^2 z}{\partial x^2}-2x\sin y\dfrac{\partial^2 z}{\partial x\partial y}+\sin^2 y\dfrac{\partial^2 z}{\partial y^2}=0,\ u=x\,\mathrm{tg}\dfrac{y}{2}$, we $v=x$.

381. $x\dfrac{\partial^2 z}{\partial x^2}-y\dfrac{\partial^2 z}{\partial y^2}=0,\ (x>0, y>0),\ x=(u+v)^2$ we $y=(u-v)^2$.

382. $\dfrac{\partial^2 z}{\partial x\partial y}=\left(1+\dfrac{\partial z}{\partial y}\right)^3,\ u=x$ we $v=y+z$.

383. Çyzykly $\xi=x+\lambda_1 y$, $\eta=x+\lambda_2 y$ orun çalşyrmanyň kömegi bilen

$$A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}=0 \qquad (1)$$

deňlemäni $\dfrac{\partial^2 u}{\partial\xi\partial\eta}=0$ deňlemä özgertmeli, bu ýerde A, B we C – hemişelik sanlar we $AC-B^2<0$.

(1) deňlemäni kanagatlandyrýan funksiýanyň umumy görnüşini tapmaly.

384. Laplasyň $\Delta z \equiv \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ deňlemesiniň

$$\frac{\partial \varphi}{\partial u} = \frac{\partial \psi}{\partial v}, \quad \frac{\partial \varphi}{\partial v} = -\frac{\partial \psi}{\partial u}$$

şerti kanagatlandyrýan $x = \varphi(u, v)$, $y = \psi(u, v)$ orun çalşyrmada üýtgemeýändigini subut etmeli.

385. $u = f(r)$ bu ýerde $r = \sqrt{x^2 + y^2}$ orun çalşyrmany ulanyp, deňlemäni özgertmeli: a) $\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$; *b*) $\Delta(\Delta u) = 0$.

386. $\omega = f(u)$ bu ýerde $u = (x - x_0)(y - y_0)$ çalşyrmada

$$\frac{\partial^2 \omega}{\partial x \partial y} + c\omega = 0$$

deňleme haýsy görnüşi alar?

387. $A = x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x}$ aňlatmany $x + y = X, y = XY$ orun çalşyrmany ulanyp özgertmeli.

388. $\frac{\partial^2 z}{\partial x^2} + 2xy^2\frac{\partial z}{\partial x} + 2(y - y^3)\frac{\partial z}{\partial y} + x^2y^2z = 0$ deňlemäniň

$$x = uv \quad \text{we} \quad y = \frac{1}{v}$$

orun çalşyrmada görnüşiniň üýtgemeýändigini subut etmeli.

389. $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ deňlemäniň

$$u = x + z, \quad v = y + z$$

orun çalşyrmada görnüşiniň üýtgemeýändigini subut etmeli.

390. $xy\frac{\partial^2 u}{\partial x \partial y} + yz\frac{\partial^2 u}{\partial y \partial z} + xz\frac{\partial^2 u}{\partial x \partial z} = 0$ deňlemäni

$$x = \eta\zeta, \quad y = \xi\zeta, \quad z = \xi\eta$$

orun çalşyrmany ulanyp özgertmeli.

391. $\frac{\partial^2 z}{\partial x_1^2} + \frac{\partial^2 z}{\partial x_2^2} + \frac{\partial^2 z}{\partial x_3^2} + \frac{\partial^2 z}{\partial x_1 \partial x_2} + \frac{\partial^2 z}{\partial x_1 \partial x_3} + \frac{\partial^2 z}{\partial x_2 \partial x_3} = 0$ deňlemäni

$$y_1 = x_2 + x_3 - x_1, \quad y_2 = x_1 + x_3 - x_2, \quad y_3 = x_1 + x_2 - x_3$$

orun çalşyrmany ulanyp özgertmeli.

392. $x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+2xz\frac{\partial^2 u}{\partial x\partial z}+2yz\frac{\partial^2 u}{\partial y\partial z}=0$ deňlemäni

$$\xi=\frac{y}{x},\qquad \eta=\frac{z}{x},\qquad \zeta=y-z$$

orun çalşyrmany ulanyp özgertmeli. (*Görkezme*: *Deňlemäni $A^2u-Au=0$ görnüşde ýazmaly, bu ýerde* $A=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$).

393. $\Delta_1 u=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2$, $\Delta_2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$ deňlemeleri

$$x=r\sin\theta\cos\varphi,\qquad y=r\sin\theta\sin\varphi,\qquad z=r\cos\theta$$

çalşyrmalary ulanyp, sferik koordinatalaryna geçmeli. (*Görkezme*: *Çalşyrmany*

$$x=R\cos\varphi,\qquad y=R\sin\varphi,\qquad z=z$$

we

$$R=r\sin\theta,\qquad \varphi=\varphi,\qquad z=r\cos\theta$$

iki çalşyrmalaryň kompozisiýasy hökmünde aňlatmaly).

394. $z\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$ deňlemede $\omega=z^2$ çalşyrmany ulanyp, täze ω funksiýany girizmeli.

Täze üýtgeýän u we v ululyklary we $\omega=\omega(u,v)$ funksiýany täze funksiýa hökmünde alyp, aşakdaky deňlemeleri özgertmeli:

395. $y\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial y}=\frac{2}{x}$, $u=\frac{x}{y}$, $v=x$, $\omega=xz-y$.

396. $\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0$, $u=x+y$, $v=\frac{y}{x}$, $\omega=\frac{z}{x}$.

397. $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0$, $u=x+y$, $v=x-y$, $\omega=xy-z$.

398. $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial x}=z$, $u=\frac{x+y}{2}$, $v=\frac{x-y}{2}$, $\omega=ze^y$.

399. $\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\left(1+\frac{y}{x}\right)\frac{\partial^2 z}{\partial y^2}=0$, $u=x$, $v=x+y$, $\omega=x+y+z$.

400. $(1-x^2)\frac{\partial^2 z}{\partial x^2}+(1-y^2)\frac{\partial^2 z}{\partial y^2}=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$, $x=\sin u$, $y=\sin v$, $z=e^{\omega}$.

401. $(1-x^2)\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} - 2x\dfrac{\partial z}{\partial x} - \dfrac{1}{4}z = 0,\ u = \dfrac{1}{2}(y + \arccos x),$

$v = \dfrac{1}{2}(y - \arccos x),\ \omega = z\sqrt[4]{1-x^2}\ \ (|x| < 1).$

402. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 2\dfrac{x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y}}{x^2 - y^2} - \dfrac{3(x^2+y^2)z}{(x^2-y^2)^2}$ $(|x| > |y|), u = x + y, v = x - y,$

$\omega = \dfrac{z}{\sqrt{x^2 - y^2}}.$

403. $\dfrac{\partial^2 z}{\partial x \partial y} + a\dfrac{\partial z}{\partial x} + b\dfrac{\partial z}{\partial y} + cz = 0$, ($a$, b, c – hemişelik sanlar) görnüşdäki her bir deňlemäniň

$$z = ue^{\alpha x + \beta y},\ \alpha, \beta \text{ – hemişelik ululyklar we } u = u(x, y)$$

çalşyrma arkaly

$$\frac{\partial^2 u}{\partial x \partial y} + c_1 = 0 \qquad (c_1 = \text{const})$$

görnüşdäki deňlemä getirilýändigini subut etmeli.

404. $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y}$ deňlemäniň $x' = \dfrac{x}{y},\ y' = -\dfrac{1}{y},\ u' = \dfrac{u}{\sqrt{y}}e^{-\frac{x^2}{4y}}$ çalşyrmada görnüşini üýtgetmeýändigini subut etmeli, bu ýerde u' ululyk x' we y' üýtgeýän ululyklaryň funksiýasydyr.

405. $q(1+q)\dfrac{\partial^2 z}{\partial x^2} - (1 + p + q + 2pq)\dfrac{\partial^2 z}{\partial x \partial y} + p(1+p)\dfrac{\partial^2 z}{\partial y^2} = 0$ deňlemede

$p = \dfrac{\partial z}{\partial x}$ we $q = \dfrac{\partial z}{\partial y}$. $u = x + z,\ v = y + z,\ \omega = x + y + z$ orun çalşyrma girizmeli we $\omega = \omega(u, v)$ hasap etmeli.

406. $x^2\dfrac{\partial^2 u}{\partial x^2} + y^2\dfrac{\partial^2 u}{\partial y^2} + z^2\dfrac{\partial^2 u}{\partial z^2} = \left(x\dfrac{\partial u}{\partial x}\right)^2 + \left(y\dfrac{\partial u}{\partial y}\right)^2 + \left(z\dfrac{\partial u}{\partial z}\right)^2$ deňlemede

$x = e^{\xi}, y = e^{\eta}, z = e^{\zeta}, u = e^{\omega}$ çalşyrmany girizmeli, bu ýerde $\omega = \omega(\xi, \eta, \zeta)$.

407. $\dfrac{\partial^2 z}{\partial x^2}\dfrac{\partial^2 z}{\partial y^2} - \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$ deňlemede x, y we z üýtgeýän ululyklaryň arasyndaky wezipeler nähili paýlananda-da onuň görnüşiniň üýtgemeýändigini subut etmeli.

408. x-i y-iň we z-iň funksiýasy hökmünde alyp,

$$\left(\frac{\partial z}{\partial y}\right)^2 \frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x \partial y} + \left(\frac{\partial z}{\partial x}\right)^2 \frac{\partial^2 z}{\partial y^2} = 0$$

deňlemäni çözmeli.

409. Ležandryň $X = \frac{\partial z}{\partial x}$, $Y = \frac{\partial z}{\partial y}$, $Z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z$ özgertmesini ulanyp we $Z = Z(X, Y)$ hasap edip,

$$A\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)\frac{\partial^2 z}{\partial x^2} + 2B\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)\frac{\partial^2 z}{\partial x \partial y} + C\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)\frac{\partial^2 z}{\partial y^2} = 0$$

deňlemäni özgertmeli.

§5. Geometrik goşundylar

1. *Galtaşýan göni çyzyk we normal tekizlik.* $M(x, y, z)$ nokatda

$$x = \varphi(t), \quad y = \psi(t), \quad z = g(t)$$

çyzyga galtaşýan göni çyzygyň deňlemesi

$$\frac{X - x}{\frac{dx}{dt}} = \frac{Y - y}{\frac{dy}{dt}} = \frac{Z - z}{\frac{dz}{dt}}$$

görnüşde bolýar.

Şol nokatda normal tekizligiň deňlemesi:

$$\frac{dx}{dt}(X - x) + \frac{dy}{dt}(Y - y) + \frac{dz}{dt}(Z - z) = 0.$$

2. *Galtaşýan tekizlik we normal tekizlik.* $z = f(x, y)$ üstüň $M(x, y, z)$ nokadynda şol üste galtaşýan tekizligiň deňlemesi:

$$(Z - z) = \frac{\partial z}{\partial x}(X - x) + \frac{\partial z}{\partial y}(Y - y),$$

M nokatdaky normalyň deňlemesi:

$$\frac{X - x}{\frac{\partial z}{\partial x}} = \frac{Y - y}{\frac{\partial z}{\partial y}} = \frac{Z - z}{-1}.$$

Eger üstüň deňlemesi anyk däl, ýagny $F(x, y, z) = 0$ görnüşde berlen bolsa, onda degişlilikde galtaşýan tekizligiň deňlemesi

$$\frac{\partial F}{\partial x}(X - x) + \frac{\partial F}{\partial y}(Y - y) + \frac{\partial F}{\partial z}(Z - z) = 0$$

we normalyň deňlemesi

$$\frac{X-x}{\frac{\partial F}{\partial x}}=\frac{Y-y}{\frac{\partial F}{\partial y}}=\frac{Z-z}{\frac{\partial F}{\partial z}}$$

görnüşde bolar.

3. ***Tekiz çyzyklaryň egreldiji çyzygy.*** Eger çyzyk her bir nokadynda bir parametrli $f(x, y, \alpha)=0$ (α – parametr) çyzyklaryň diňe birine galtaşýan we dürli nokatlarynda çyzyklaryň dürli çyzyklaryna galtaşýan bolsa, onda oňa şol çyzyklaryň egreldijisi diýilýär. Bir parametrli çyzyklaryň egreldiji çyzygy

$$f(x, y, \alpha)=0, \quad f_\alpha'(x, y, \alpha)=0$$

deňlemeler sistemasyny kanagatlandyrýar.

4. ***Üstleriň egreldiji üsti.*** Eger üst bir parametrli $F(x, y, z, \alpha)=0$ üstleriň ählisine galtaşýan bolsa, onda oňa şol üstleriň egreldijisi diýilýär. Bir parametrli üstleriň egreldiji üsti

$$F(x, y, z, \alpha)=0, \quad F_\alpha'(x, y, z, \alpha)=0$$

deňlemeler sistemasyny kanagatlandyrýar.

Iki parametrli $\Phi(x, y, z, \alpha, \beta)=0$ üstler üçin egreldiji üst

$$\Phi(x, y, z, \alpha, \beta)=0, \quad \Phi_\alpha'(x, y, z, \alpha, \beta)=0, \quad \Phi_\beta'(x, y, z, \alpha, \beta)=0$$

deňlemeler sistemasyny kanagatlandyrýar.

Gönükmeler

Görkezilen egri çyzyklara berlen nokatlarda galtaşýan göni çyzyklaryň we normal tekizlikleriň deňlemelerini ýazmaly:

410. $x=a\cos\alpha\cos t$, $y=a\sin\alpha\cos t$, $z=a\sin t$; $t=t_0$ nokatda.

411. $x=a\sin^2 t$, $y=b\sin t\cos t$, $z=c\cos^2 t$; $t=\pi/4$ nokatda.

412. $y=x$, $z=x^2$; $M(1, 1, 1)$ nokatda.

413. $x^2+z^2=10$, $y^2+z^2=10$; $M(1, 1, 3)$ nokatda.

414. $x^2+y^2+z^2=6$, $x+y+z=0$; $M(1, -2, 1)$ nokatda.

415. $x=t$, $y=t^2$, $z=t^3$ egri çyzygyň haýsy nokadyna geçirilen galtaşýan $x+2y+z=4$ tekizlige parallel bolar, şol nokady tapmaly.

416. $x=a\cos t$, $y=a\sin t$, $z=bt$ aýlawly çyzyga galtaşýan çyzygyň Oz oky bilen hemişelik burçy emele getirýändigini subut etmeli.

417. $x=ae^t\cos t$, $y=ae^t\sin t$, $z=ae^t$ egri çyzygyň $x^2+y^2=z^2$ konusyň ähli emele getirijilerini şol bir burç boýunça kesýändigini subut etmeli.

418. $\operatorname{tg}\left(\frac{\pi}{4}+\frac{\psi}{2}\right)=e^{k\varphi}$ $(k = \text{const})$ loksodromanyň sferanyň ähli meridianalaryny hemişelik burç boýunça kesýändigini subut etmeli, bu ýerde: φ – sferanyň nokadynyň uzaklygy, ψ – onuň giňligi.

419. $z = f(x, y)$, $\frac{x-x_0}{\cos\alpha}=\frac{y-y_0}{\sin\alpha}$ egri çyzyga $M_0(x_0, y_0)$ nokatda geçirilen galtaşýanyň Oxy tekizlik bilen emele getirýän burçunyň tangensini tapmaly, bu ýerde f – differensirlenýän funksiýa.

420. $u=\frac{x}{\sqrt{x^2+y^2+z^2}}$ funksiýanyň $M(1, 2, –2)$ nokatdaky şol nokada $x = t$, $y = 2t^2$, $z = –2t^4$ egri çyzyga geçirilen galtaşýanyň ugry boýunça önümini tapmaly.

Berlen üstlere degişli nokatlarda galtaşýan tekizligiň we normalyň deňlemesini ýazmaly:

421. $z=x^2+y^2$; $M_0(1, 2, 5)$ nokatda.

422. $x^2+y^2+z^2=169$; $M_0(3, 4, 12)$ nokatda.

423. $z = \operatorname{arctg}\frac{y}{x}$; $M_0(1, 1, \pi/4)$ nokatda.

424. $ax^2 + by^2 + cz^2 = 1$; $M_0(x_0, y_0, z_0)$ nokatda.

425. $z = y + \ln\frac{x}{z}$; $M_0(1, 1, 1)$ nokatda.

426. $2^{x/z} + 2^{y/z} = 8$; $M_0(2, 2, 1)$ nokatda.

427. $x = a\cos\psi\cos\varphi$, $y = b\cos\psi\sin\varphi$, $z = c\sin\psi$; $M_0(\varphi_0, \psi_0)$ nokatda.

428. $x = r\cos\varphi$, $y = r\sin\varphi$, $z = r\operatorname{ctg}\alpha$; $M_0(\varphi_0, r_0)$ nokatda.

429. $x = u\cos\upsilon$, $y = u\sin\upsilon$, $z = a\upsilon$; $M_0(u_0, \upsilon_0)$ nokatda.

430. $M(u, \upsilon)$ $(u \neq \upsilon)$ galtaşma nokadyň üstüň $u = \upsilon$ gyraky çyzygynyň $M_0(u_0, \upsilon_0)$ nokadyna çäksiz ýakynlaşýan ýagdaýynda $x = u + \upsilon$, $y = u^2 + \upsilon^2$, $z = u^3 + \upsilon^3$ üste galtaşýan tekizligiň predel ýagdaýyny tapmaly.

431. $x^2 + 2y^2 + 3z^2 + 2xy + 2xz + 4yz = 8$ üstüň haýsy nokatlarynda geçirilen galtaşýan tekizlikler koordinatalar tekizliklerine parallel bolar?

432. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ellipsoidiň haýsy nokadynda geçirilen normal koordinatalar oklary bilen deň burçlary emele getirýär?

433. $x^2 + 2y^2 + 3z^2 = 21$ üste $x + 4y + 6z = 0$ tekizlige parallel bolan galtaşýan tekizlikleri geçirmeli.

434. $xyz = a^3$ $(a > 0)$ üste galtaşýan tekizlikleriň koordinatalar tekizlikleri bilen hemişelik göwrümli tetraedri emele getirýändigini subut etmeli.

435. $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ $(a > 0)$ üste galtaşýan tekizlikleriň koordinatalar oklaryndan kesip alýan kesimleriniň jeminiň hemişelikdigini subut etmeli.

436. $z = xf(y/x)$ konusa galtaşýan tekizlikleriň onuň depesinden geçýändigini subut etmeli.

437. $z = f\left(\sqrt{x^2 + y^2}\right)$ $(f' \neq 0)$ aýlanma üstüň normallarynyň aýlanma okuny kesip geçýändigini subut etmeli.

438. $x^2 + y^2 + z^2 - xy = 1$ ellipsoidiň koordinatalar tekizliklerine proýeksiýalaryny tapmaly.

439. $\{0 \leq x \leq 1, 0 \leq y \leq 1\}$ kwadrat diametrleri $\leq \delta$ bolan σ sany çäkli böleklere bölünen. Eger $z = 1 - x^2 - y^2$ üstüniň şol bir σ bölegine degişli $P(x, y)$ we $P_1(x_1, y_1)$ islendik nokatlarynda geçirilen normallaryň ugurlarynyň arasyndaky tapawudy 1°-dan az bolan ýagdaýynda, δ sany ýokardan çäklendirmeli.

440. Goý,

$$z = f(x, y), \quad (x, y) \in D \tag{1}$$

üstüň deňlemesi we $\varphi = (P_1, P)$ bolsa $P(x, y) \in D$ we $P_1(x_1, y_1) \in D$ nokatlarda (1) üste geçirilen normallaryň arasyndaky burç bolsun.

Eger D ýaýla çäkli we ýapyk bolup, $f(x, y)$ funksiýanyň D ýaýlada ikinji tertipli çäkli önümleri bar bolsa, onda Lýapunowyň

$$\varphi(P_1, P) < C\rho(P_1, P) \tag{2}$$

deňsizliginiň ýerine ýetýändigini subut etmeli. Bu ýerde C – hemişelik we $\rho(P_1, P)$ san P we P_1 nokatlaryň arasyndaky uzaklyk.

441. $x^2 + y^2 = a^2$ silindr $bz = xy$ üst bilen umumy $M_0(x_0, y_0, z_0)$ nokatda haýsy burç boýunça kesişýändigini tapmaly.

442. $x^2 + y^2 + z^2 = r^2$, $y = x\,\mathrm{tg}\varphi$, $x^2 + y^2 = z^2\mathrm{tg}^2\theta$ sferik koordinatalaryň koordinatalar üstleriniň jübütleýin ortogonaldygyny subut etmeli.

443. $x^2 + y^2 + z^2 = 2ax$, $x^2 + y^2 + z^2 = 2by$, $x^2 + y^2 + z^2 = 2cz$ sferalaryň üç ortogonal sistemany emele getirýändigini subut etmeli.

444. Her bir $M(x, y, z)$ nokat arkaly $\lambda = \lambda_1$, $\lambda = \lambda_2$, $\lambda = \lambda_3$ bolanda ikinji tertipli üç sany

$$\frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} + \frac{z^2}{c^2 - \lambda^2} = -1 \qquad (a > b > c > 0)$$

üst geçýär. Olaryň ortogonaldygyny subut etmeli.

445. $u = x + y + z$ funksiýasynyň $x^2 + y^2 + z^2 = 1$ sferanyň $M_0(x_0, y_0, z_0)$ nokatdaky şol nokatda geçirilen daşky normalyň ugry boýunça önümini tapmaly.

Sferanyň haýsy nokatlarynda u funksiýanyň normal önümi a) iň uly baha; b) iň kiçi baha; ç) nola deň bolar?

446. $u = x^2 + y^2 + z^2$ funksiýasynyň $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ellipsoidiň $M_0(x_0, y_0, z_0)$ nokatdaky şol nokatda geçirilen daşky normalyň ugry boýunça önümini tapmaly.

447. Goý, $\frac{\partial u}{\partial n}$ we $\frac{\partial \upsilon}{\partial n}$, u we υ funksiýalaryň $F(x, y, z) = 0$ üstüň nokadyndaky normal önümleri bolsun. $\frac{\partial}{\partial n}(u\upsilon) = u\frac{\partial \upsilon}{\partial n} + \upsilon\frac{\partial u}{\partial n}$ deňligiň dogrudygyny subut etmeli.

Bir parametrli tekiz çyzyklaryň egreldijilerini tapmaly:

448. $x\cos\alpha + y\sin\alpha = p$ (p = const).

449. $(x - a)^2 + y^2 = a^2/2$.

450. $y = kx + a/k$ (a = const).

451. $y^2 = 2px + p^2$.

452. Uzynlygy l, uçlary koordinatalar oklary boýunça süýşýän kesim bilen egreldilýän egri çyzygy tapmaly.

453. Hemişelik S meýdany bolan $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipsleriň egreldijisini tapmaly.

454. Howasyz giňişlikde ϑ_0 başlangyç tizlik bilen atylan okuň atylyş burçunyň wertikal tekizlikde α burç boýunça üýtgemegindäki traýektoriýasynyň egreldijisini tapmaly.

455. Tekiz çyzygyň normallarynyň egreldijisiniň şol çyzygyň ewolýutasy bolýandygyny subut etmeli.

456. Bir parametrli $f(x, y, \alpha) = 0$ çyzyklaryň

$$f(x, y, \alpha) = 0, \quad f'_\alpha(x, y, \alpha) = 0$$

deňlemeler ulgamyny kanagatlandyrýan nokadyna ol çyzyklaryň *häsiýetlendiriji nokady*, ol nokatlaryň köplügine bolsa *diskriminant çyzygy* diýilýär.

Aşakda görkezilen çyzyklaryň diskriminant çyzyklarynyň häsiýetlerini derňemeli (c – üýtgeýän parametr):

a) $y = (x - c)^3$ kubiki parabolalaryň;

b) $y^2 = (x - c)^3$ ýarym kubiki parabolalaryň;

ç) $y^3 = (x - c)^2$ Neýliň parabolalarynyň;

d) $(y - c)^2 = x^2\frac{a - x}{a + x}$ strofoidleriň.

457. Merkezleri $x = R\cos t$, $y = R\sin t$, $z = 0$ töweregiň üstünde ýerleşýän, r radiusly şarlaryň egreldijisini kesgitlemeli, bu ýerde t – parametr, $R > r$.

458. $(x - t\cos\alpha)^2 + (y - t\cos\beta)^2 + (z - t\cos\gamma)^2 = 1$ şarlaryň egreldijisini tapmaly, bu ýerde t – üýtgeýän parametr we $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

459. Göwrümi hemişelik V bolan $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ellipsoidleriň egreldijisini kesgitlemeli.

460. Merkezleri $x^2 + y^2 = z^2$ konusyň üstünde ýerleşýän, ρ radiusly sferalaryň egreldijisini tapmaly.

461. Ýalpyldaýan nokat koordinatalar başlangyjynda ýerleşýär. $x_0^2 + y_0^2 + z_0^2 > R^2$ bolanda

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \le R^2$$

şaryň berýän konus görnüşli kölegesini kesgitlemeli.

462. $p^2 + q^2 = 1$ deňleme arkaly baglanyşykda bolan p we q parametrler üçin

$$z - z_0 = p(x - x_0) + q(y - y_0)$$

tekizlikleriň egreldijisini tapmaly.

§6. Teýloryň formulasy

1. *Teýloryň formulasy.* Eger $(x_1^0, \ldots, x_m^0)$ nokadyň käbir δ golaý töwereginde $u = f(x_1, \ldots, x_m)$ funksiýanyň $(n + 1)$ tertipli üznüksiz hususy önümleri bar bolsa, onda $|\Delta x| = \sqrt{\Delta x_1^2 + \ldots + \Delta x_m^2} < \delta$ şerti kanagatlandyrýan $\forall \Delta x = (\Delta x_1, \ldots, \Delta x_m)$ üçin şeýle $\theta (0 < \theta < 1)$ san tapylyp,

$$f(x^0 + \Delta x) = f(x^0) + \sum_{k=1}^{n} \frac{d^k f(x^0)}{k!} + r_n(x) \tag{1}$$

Teýloryň formulasy dogrudyr, bu ýerde

$$r_n(x) = \frac{1}{(n+1)!} d^{n+1} f(x^0 + \theta\Delta x) = \frac{1}{(n+1)!} \left(\sum_{i=1}^{m} \Delta x_i \frac{\partial}{\partial x_i} \right)^{n=1} f(x^0 + \theta\Delta x) \tag{2}$$

funksiýa Lagranžyň galyndy agzasy,

$$r_n(x) = \frac{1}{n!} d^n f(x^0) + o(|\Delta x|^n), \quad |\Delta x| \to 0$$

bolsa Peanonyň galyndy agzasy diýilýär.

Teýloryň formulasyndan we onuň galyndy agzalaryndan $x^0 = 0$ bolanda alynýan formula galyndy agzalary Lagranžyň we Peanonyň görnüşindäki Makloreniň formulasy diýilýär.

Teýloryň formulasyndan hususy hal hökmünde alynýan

$$f(x^0+\Delta x)-f(x^0)=\sum_{i=1}^{m}\frac{\partial f(x^0+\theta\Delta x)}{\partial x_i}\Delta x_i$$

formula Lagranžyň formulasy diýilýär.

2. *Teýloryň hatary.* Eger $(x_1^0,...,x_m^0)$ nokadyň käbir δ golaý töwereginde $u=f(x_1,...,x_m)$ funksiýanyň tükeniksiz tertipdäki hususy önümleri bar we $\lim\limits_{n\to\infty} r_n(x)=0$ bolsa, onda ol funksiýa

$$f(x^0+\Delta x)=f(x^0)+\sum_{k=1}^{\infty}\frac{d^k f(x^0)}{k!}$$

Teýloryň hatary görnüşinde aňladylýar. Hususy halda alynýan

$$f(x)=f(0)+\sum_{k=1}^{\infty}\frac{d^k f(0)}{k!}$$

hatara Makloreniň hatary diýilýär.

3. *Tekiz çyzyklaryň aýratyn nokatlary.* Goý, iki gezek differensirlenýän $f(x,y)=0$ çyzygyň käbir $M(a,b)$ nokadynda

$$f(a,b)=0,\quad f_x'(a,b)=0,\quad f_y'(a,b)=0$$

şertler ýerine ýetýän bolup,

$$A=f_{xx}''(a,b),\quad B=f_{xy}''(a,b),\quad C=f_{yy}''(a,b)$$

sanlaryň hemmesi nola deň bolmasyn. Onda

1) $AC-B^2>0$ bolanda M – üzňe nokat;
2) $AC-B^2<0$ bolanda M – ikigat nokat;
3) $AC-B^2=0$ bolanda M – gaýdyş nokady ýa-da üzňe nokat.

Mysal. $f(x,y)=2x^2-xy-y^2-6x-3y+5$ funksiýa üçin $(1;-2)$ nokadyň golaý töwereginde Teýloryň formulasyny ýazmaly.

Ç.B. Funksiýanyň ikiden uly tertipdäki hususy önümleriniň nola deň bolýandygy üçin, bu halda Teýloryň formulasy

$$f(x,y)=f(1;-2)+\frac{\partial f(1;-2)}{\partial x}(x-1)+\frac{\partial f(1;-2)}{\partial y}(y+2)+$$

$$+\frac{1}{2!}\frac{\partial^2 f(1;-2)}{\partial x^2}(x-1)^2+\frac{2}{2!}\frac{\partial^2 f(1;-2)}{\partial x\partial y}(x-1)(y+2)+\frac{1}{2!}\frac{\partial^2 f(1;-2)}{\partial y^2}(y+2)^2$$

görnüşi alar. Indi hususy önümleri tapalyň:

$$\frac{\partial f}{\partial x}\bigg|_{\substack{x=1\\y=-2}}=(4x-y-6)\bigg|_{\substack{x=1\\y=-2}}=0,\qquad \frac{\partial f}{\partial y}\bigg|_{\substack{x=1\\y=-2}}=(-x-2y-3)\bigg|_{\substack{x=1\\y=-2}}=0,$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{\substack{x=1\\y=-2}} = 4, \quad \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{\substack{x=1\\y=-2}} = -1, \quad \left.\frac{\partial^2 f}{\partial y^2}\right|_{\substack{x=1\\y=-2}} = -2.$$

Şeýlelikde,

$$f(x, y) = 5 + 2(x-1)^2 - (x-1)(y+2) - (y+2)^2.$$ **Ç.S.**

Gönükmeler

463. $f(x, y) = 2x^2 - xy - y^2 - 6x - 3y + 5$ funksiýany $A(1, -2)$ nokadyň golaý töwereginde Teýloryň formulasy boýunça dagytmaly.

464. $f(x, y, z) = x^3 + y^3 + z^3 - 3xyz$ funksiýany $A(1, 1, 1)$ nokadyň golaý töwereginde Teýloryň formulasy boýunça dagytmaly.

465. $f(x, y) = x^2y + xy^2 - 2xy$ funksiýanyň $x = 1$, $y = -1$ bahalardan $x_1 = 1 + h$, $y_1 = -1 + k$ bahalara geçendäki artymyny tapmaly.

466. Eger $f(x, y, z) = Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz$ bolsa, onda $f(x+h, y+k, z+l)$ funksiýany h, k we l ululyklaryň bitin položitel derejeleri boýunça dagytmaly.

467. $f(x, y) = x^y$ funksiýanyň $A(1, 1)$ nokadyň golaý töweregindäki dagytmasynyň ikinji derejä çenli agzalaryny (hasaba alyp) ýazmaly.

468. $f(x,y) = \sqrt{1 - x^2 - y^2}$ funksiýany dördünji derejeli agzalaryna çenli (hasaba alyp) Makloreniň formulasy boýunça dagytmaly.

469. $|x|$ we $|y|$ ululyklaryň 1 bilen deňeşdirilende kiçi bolan hallarynda

a) $\dfrac{\cos x}{\cos y}$; b) $\operatorname{arctg}\dfrac{1+x+y}{1-x+y}$

aňlatmalar üçin ikinji derejeli agzalaryna çenli takyklyk bilen takmyny formulalary getirip çykarmaly.

470. x, y, z ululyklary absolýut ululyklary boýunça kiçi ululyk hasap edip,

$$\cos(x + y + z) - \cos x \cos y \cos z$$

aňlatmany ýönekeýleşdirmeli.

471. $F(x,y) = \frac{1}{4}[f(x+h,y) + f(x,y+h) + f(x-h,y) + f(x,y-h)] - f(x,y)$

funksiýany h-yň derejeleri boýunça h^4-e çenli takyklykda dagytmaly.

472. Goý, $f(P) = f(x, y)$ we $P_i(x_i, y_i)$ $(i = 1, 2, 3)$ merkezi $P(x, y)$ nokatda ρ radiusly töweregiň içinden çyzylan dogry üçburçlugyň depeleri bolsun, şeýle-de, $x_1 = x + \rho$, $y_1 = y$.

$$F(\rho) = \frac{1}{3}[f(P_1) + f(P_2) + f(P_3)]$$

funksiýany ρ-nyň bitin položitel derejeleri boýunça ρ^2-a çenli takyklyk bilen dagytmaly.

473. $\Delta_{xy} f(x, y) = f(x + h, y + k) - f(x + h, y) - f(x, y + k) + f(x, y)$ funksiýany h-yň we k-nyň derejeleri boýunça dagytmaly.

474. ρ-nyň derejeleri boýunça

$$F(\rho) = \frac{1}{2\pi}\int_0^{2\pi} f(x + \rho\cos\phi, y + \rho\sin\phi)d\phi$$

funksiýany dagytmaly.

Aşakdaky funksiýalary Makloreniň hatary boýunça dagytmaly:

475. $f(x, y) = (1 + x)^m(1 + y)^n$.

476. $f(x, y) = \ln(1 + x + y)$.

477. $f(x, y) = e^x\sin y$.

478. $f(x, y) = e^x\cos y$.

479. $f(x, y) = \sin x \operatorname{sh} y$.

480. $f(x, y) = \cos x \operatorname{ch} y$.

481. $f(x, y) = \sin(x^2 + y^2)$.

482. $f(x, y) = \ln(1 + x)\ln(1 + y)$.

483. $f(x,y) = \int_0^1 (1 + x)^{t^2 y} dt$ funksiýanyň Makloreniň hataryna dagydylmasynyň üç agzasyny ýazmaly.

484. e^{x+y} funksiýany $x - 1$ we $y + 1$ binomlaryň bitin položitel derejeleri boýunça derejeli hatara dagytmaly.

485. $f(x,y) = \frac{x}{y}$ funksiýanyň $M(1, 1)$ nokadyň golaý töwereginde Teýloryň hataryna dagydylyşyny ýazmaly.

486. Goý, z funksiýa x we y ululyklaryň anyk däl funksiýasy hökmünde $z^3 - 2xz + y = 0$ deňleme bilen kesgitlenýän bolsun we $x = 1$ we $y = 1$ bolanda onuň bahasy $z = 1$ bolsun. $x - 1$ we $y - 1$ binomlaryň artýan derejeleri boýunça z funksiýanyň dagydylyşynyň birnäçe agzalaryny ýazmaly.

Aşakdaky egri çyzyklaryň aýratyn nokatlarynyň görnüşlerini öwrenip, ol çyzyklary takmyny şekillendirmeli:

487. $y^2 = ax^2 + x^3$.

488. $x^3 + y^3 - 3xy = 0$.

489. $x^2 + y^2 = x^4 + y^4$.

490. $x^2 + y^4 = x^6$.

491. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

492. $(y - x^2)^2 = x^5$.

493. $(a + x)y^2 = (a - x)x^2$.

494. $y^2 = (x - a)(x - b)(x - c)$ egri çyzygyň a, b, c ($a \le b \le c$) parametrleriň bahalaryna baglylykda görnüşini derňemeli.

Transsendent egri çyzyklaryň aýratyn nokatlaryny derňemeli:

495. $y^2 = 1 - e^{-x^2}$.

496. $y^2 = 1 - e^{-x^3}$.

497. $y = x\ln x$.

498. $y = \dfrac{x}{1 + e^{1/x}}$.

499. $y = \operatorname{arctg}\left(\dfrac{1}{\sin x}\right)$.

500. $y^2 = \sin\dfrac{\pi}{x}$.

501. $y^2 = \sin x^2$.

502. $y^2 = \sin^3 x$.

§7. Köp üýtgeýänli funksiýanyň ekstremumy

1. ***Funksiýanyň ekstremumynyň kesgitlenişi we zerur şerti.*** Goý, köp üýtgeýänli $f(x) = f(x_1, ..., x_m)$ funksiýa $x^0 = (x_1^0, \ldots, x_m^0)$ nokadyň käbir $U(x^0, \delta)$ golaý töwereginde kesgitlenen bolsun. Eger $\forall x \in U(x^0, \delta)$ üçin $f(x) \ge f(x^0)$ ($f(x) \le f(x^0)$) deňsizlik ýerine ýetse, onda x^0 nokada f funksiýanyň minimum (maksimum) nokady diýilýär.

Ekstremumyň zerur şerti. Differensirlenýän f funksiýanyň x^0 ekstremum nokadynda

$$\frac{\partial f}{\partial x_i}(x^0) = 0 \qquad (i = 1, ..., m) \tag{1}$$

$$\text{ýa-da} \qquad df(x^0) = 0. \tag{2}$$

Funksiýanyň hususy önümleriniň nol we hususy önümleriniň ýok nokatlaryna onuň ekstremumynyň bolup biljek nokatlary diýilýär.

2. ***Silwestriň ölçegleri.*** f funksiýanyň x^0 nokatdaky

$$d^2 f(x^0) = \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial^2 f(x^0)}{\partial x_i \partial x_j} dx_i dx_i \tag{3}$$

ikinji differensialy dx_1, ..., dx_m ululyklara görä kwadrat formadyr we $a_{ij} = \dfrac{\partial^2 f(x^0)}{\partial x_i \partial x_j}$ ($i, j = 1, ..., m$) onuň koeffisiýentleridir. Olardan düzülen

$$A_1 = a_{11}, \qquad A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, ..., \qquad A_m = \begin{vmatrix} a_{11}\, a_{12} \ldots a_{1m} \\ a_{21}\, a_{22} \ldots a_{2m} \\ \ldots\ldots\ldots\ldots\ldots \\ a_{m1}\, a_{m2} \ldots a_{mm} \end{vmatrix}$$

kesgitleýjiler üçin Silwestriň ölçegleri şeýle okalýar:

Kwadrat formanyň položitel kesgitlenmegi üçin $A_1>0,\ A_2>0, ..., A_m>0$ şertleriň ýerine ýetmegi we otrisatel kesgitlenmegi üçin $(-1)^k A_k>0$ $(k=1, ..., m)$ şertleriň ýerine ýetmegi zerur we ýeterlikdir.

3. *Ekstremumyň ýeterlik şertleri.* Eger x^0 nokadyň käbir golaý töwereginde f funksiýanyň ikinji tertipli üznüksiz hususy önümleri bar bolup, $df(x^0)=0$ we $d^2f(x^0)$ položitel kesgitlenen (otrisatel kesgitlenen) kwadrat forma bolsa, onda x^0 nokat f funksiýanyň minimum (maksimum) nokadydyr. Eger-de kwadrat formanyň alamaty üýtgeýän bolsa, onda onuň x^0 nokatda ekstremumy ýokdur.

Iki üýtgeýänli $f(x, y)$ funksiýanyň ekstremumynyň ýeterlik şerti $a_{11}=f''_{xx}(a,b)$, $a_{12}=f''_{xy}(a,b)$, $a_{22}=f''_{yy}(a,b)$ üçin şeýle okalýar. Eger: 1) $a_{11}a_{22}-a_{12}^2>0$ we $a_{11}>0$ bolsa, onda (a, b) nokat $f(x, y)$ funksiýanyň minimum nokady; 2) $a_{11}a_{22}-a_{12}^2>0$ we $a_{11}<0$ bolsa, onda (a, b) nokat $f(x, y)$ funksiýanyň maksimum nokady; 3) $a_{11}a_{22}-a_{12}^2<0$ bolsa, onda (a, b) nokat $f(x, y)$ funksiýanyň ekstremum nokady däldir; 4) $a_{11}a_{22}-a_{12}^2=0$ bolsa, onda (a, b) nokat $f(x, y)$ funksiýanyň ekstremum nokady bolup hem, bolman hem biler.

4. *Şertli ekstremum.* Köp üýtgeýänli $f(x_1, ..., x_m)$ funksiýanyň $f_k(x_1, ..., x_m)=0$ $(k = 1, ..., n,\ n < m)$ baglanyşyk şertleri kanagatlandyrýan ekstremumyny tapmaklyga şertli ekstremum diýilýär. Funksiýanyň şertli ekstremumyny tapmaklyk: 1) baglanyşyk şertlerdäki $x_1, ..., x_m$ üýtgeýän ululyklaryň birnäçesini (mysal üçin, n sanysyny) beýlekileri arkaly aňladyp, üýtgeýän $m-n$ ululykly funksiýanyň ady ekstremumyny tapmaklyga getirilýär; 2) Lagranžyň

$$L(x_1, ..., x_m) = f(x_1, ..., x_m) + \sum_{i=1}^{n} \lambda_i f_i(x_1, ..., x_m)$$

funksiýasynyň ady ekstremumyny tapmaklyga getirilýär.

5. *Funksiýanyň iň uly we iň kiçi bahasy.* Eger f funksiýa ýapyk çäkli G ýaýlada differensirlenýän bolsa, onda ol funksiýa iň uly (iň kiçi) bahasyny ýa hususy önümleriň nol nokatlarynda, ýa-da ýaýlanyň araçäk nokatlarynda alýar.

1-nji mysal. $f(x, y) = x^3 + 3xy^2 - 15x - 12y$ funksiýanyň ekstremum nokatlaryny kesgitlemeli.

Ç.B. Ilki bilen $\left.\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2+3y^2-15=0, \\ \frac{\partial f}{\partial y} &= 6xy-12=0 \end{aligned}\right\}$ sistemany çözüp, $M_1(1;2)$, $M_2(2;1)$, $M_3(-1;-2)$, $M_4(-2;-1)$ nokatlaryň f funksiýanyň hususy önümleriniň nol nokatlarydygyny görýäris.

$$\frac{\partial^2 f}{\partial x^2}=6x, \qquad \frac{\partial^2 f}{\partial x \partial y}=6y, \qquad \frac{\partial^2 f}{\partial y^2}=6x\cdot$$

Ikinji tertipli hususy önümleriň esasynda:

1. M_1 nokatda $a_{11}=6$, $a_{12}=a_{21}=12$, $a_{22}=6$, $a_{11}a_{22}-a_{12}^2=36-144<0$. Diýmek, M_1 nokatda ekstremum ýok.

2. M_2 nokatda $a_{11}=12$, $a_{12}=a_{21}=6$, $a_{22}=12$, $a_{11}a_{22}-a_{12}^2=144-36>0$. Diýmek, M_2 minimum nokadydyr.

3. M_3 nokatda $a_{11}=-6$, $a_{12}=a_{21}=-12$, $a_{22}=-6$, $a_{11}a_{22}-a_{12}^2=144-36<0$. Diýmek, M_3 nokatda ekstremum ýok.

4. M_4 nokatda $a_{11}=-12$, $a_{12}=a_{21}=-6$, $a_{22}=-12$, $a_{11}a_{22}-a_{12}^2=144-36>0$. Diýmek, M_4 maksimum nokadydyr.

8-nji mysal. $f(x, y)=x^2-y^2$ funksiýanyň $2x-y-3=0$ göni çyzykdaky ekstremum nokadyny tapmaly.

Ç.B. Ilki bilen Lagranžyň funksiýasyny düzeliň:

$$L(x, y, \lambda)=x^2-y^2+\lambda(2x-y-3).$$

Indi bu funksiýanyň hususy önümleriniň nol nokatlaryny kesgitläliň:

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 2x+2\lambda=0, \\ \frac{\partial L}{\partial y} &= -2y-\lambda=0, \\ \frac{\partial L}{\partial \lambda} &= 2x-y-3=0. \end{aligned}\right\}$$

Bu sistemanyň çözüwi: $x=2$, $y=1$, $\lambda=-2$, ýagny $(2, 1, -2)$ nokat Lagranžyň funksiýasynyň hususy önümleriniň noludyr. Şol nokatda ikinji tertipli differensialy tapalyň:

$$d^2L=2dx^2-2dy^2.$$

$2x-y-3=0$ baglanyşyk şertiň esasynda $2dx=dy$, şonuň üçin hem bu şert ýerine ýetende Lagranžyň funksiýasy garalýan funksiýa bilen gabat gelýär we $d^2L=-6dx^2<0$. Şonuň üçin hem ekstremumyň ýeterlik şerti boýunça $(2, 1)$ nokat $f(x, y)=x^2-y^2$ funksiýanyň şertli maksimum nokady, ýagny $2x-y-3=0$ göni çyzykdaky maksimum nokadydyr. **Ç.S.**

3-nji mysal. $f(x, y) = x^2 + y^2 - xy + x + y$ funksiýanyň

$$x \leq 0, \quad y \leq 0, \quad x + y \geq -3$$

ýaýladaky iň kiçi we iň uly bahalaryny kesgitlemeli.

Ç.B. Seredilýän ýaýla AOB üçburçlukdyr: $A(-3, 0)$, $O(0, 0)$, $B(0, -3)$. Ilki bilen funksiýanyň hususy önümleriniň nollaryny tapalyň:

$$\begin{cases} f_x' = 2x - y + 1 = 0, \\ f_y' = 2y - x + 1 = 0. \end{cases}$$

Bu ýerden görnüşi ýaly, hususy önümleriň noly: $x=-1$, $y=-1$, ýagny $M(-1, -1)$ nokatdyr. Funksiýanyň şol nokatdaky bahasy $f(M) = -1$ bolar. Indi funksiýany ýaýlanyň araçäklerinde derňäliň. OB kesimde, ýagny $x = 0$ bolanda $g(y) = f(0, y) = y^2 + y$ bolar we mesele bir üýtgeýänli g funksiýanyň $-3 \leq y \leq 0$ kesimdäki iň uly we iň kiçi bahalaryny tapmaklyga getirilýär. Ol funksiýanyň önüminiň noly $y = -1/2$ bolar. Şonuň üçin g funksiýanyň $y = -3$, $y = -1/2$, $y = 0$ nokatlardaky bahalaryny deňeşdirip, berlen funksiýanyň OB kesimdäki iň uly we iň kiçi bahalaryny taparys:

$$f_{i.u.}(OB) = f(0, -3) = 6, \quad f_{i.k.}(OB) = f(0, -1/2) = -1/4.$$

Edil şoňa meňzeşlikde, AO kesimde, ýagny $y=0$ bolanda alynýan $p(x)=f(x, 0)=x^2 + x$ funksiýanyň $-3 \leq x \leq 0$ kesimdäki we şonuň esasynda berlen funksiýanyň AO kesimdäki iň uly we iň kiçi bahalaryny taparys:

$$f_{i.u.}(AO) = f(-3, 0) = 6, \quad f_{i.k.}(OB) = f(-1/2, 0) = -1/4.$$

AB kesimde, ýagny $x + y = -3$ göni çyzykda $y = -3 - x$ bolar we şonda bir üýtgeýänli $q(x) = f(x, -3 - x) = 3x^2 + 9x + 6$ funksiýany alarys. Ol funksiýany derňäp, berlen funksiýanyň AB kesimdäki iň uly we iň kiçi bahalaryny taparys:

$$f_{i.u.}(AB) = f(0, -3) = f(-3, 0) = 6, \quad f_{i.k.}(AB) = f(-3/2, -3/2) = -3/4.$$

Berlen f funksiýalaryň bahalaryny deňeşdirip, onuň iň uly bahany $A(-3, 0)$ we $B(0, -3)$ nokatlarda, iň kiçi bahany bolsa $M(-1, -1)$ nokatda alýandygyny görýäris.
Ç.S.

Gönükmeler

Köp üýtgeýänli funksiýalaryň ekstremumlaryny derňemeli:

503. $z = x^2 + (y - 1)^2$.

504. $z = x^2 - (y - 1)^2$.

505. $z = (x - y + 1)^2$.

506. $z = x^2 - xy + y^2 - 2x + y$.

507. $z = x^2y^3(6 - x - y)$.

508. $z = x^3 + y^3 - 3xy$.

509. $z = x^4 + y^4 - x^2 - 2xy - y^2$.

510. $z = 2x^4 + y^4 - x^2 - 2y^2$.

511. $z = xy + \frac{50}{x} + \frac{20}{y}$ $(x>0, y>0)$. **512.** $z = xy\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ $(a>0, b>0)$.

513. $z = \frac{ax + by + c}{\sqrt{x^2 + y^2 + 1}}$ $(a^2+b^2+c^2\neq 0)$. **514.** $z = 1 - \sqrt{x^2 + y^2}$.

515. $z = e^{2x+3y}(8x^2 - 6xy + 3y^2)$. **516.** $z = e^{x^2-y}(5 - 2x + y)$.

517. $z = (5x + 7y - 25)e^{-(x^2+xy+y^2)}$. **518.** $z = x^2 + xy + y^2 - 4\ln x - 10\ln y$.

519. $z = \sin x + \cos y + \cos(x - y)$ $\left(0 \le x \le \frac{\pi}{2};\ 0 \le y \le \frac{\pi}{2}\right)$.

520. $z = \sin x \sin y \sin(x + y)$ $(0 \le x \le \pi;\ 0 \le y \le \pi)$.

521. $z = x - 2y + \ln\sqrt{x^2 + y^2} + 3\operatorname{arctg}\frac{y}{x}$.

522. $z = xy\ln(x^2 + y^2)$. **523.** $z = x + y + 4\sin x \sin y$.

524. $z = (x^2 + y^2)e^{-(x^2+y^2)}$. **525.** $u = x^2 + y^2 + z^2 + 2x + 4y - 6z$.

526. $u = x^3 + y^2 + z^2 + 12xy + 2z$.

527. $u = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$ $(x > 0, y > 0, z > 0)$.

528. $u = xy^2z^3(a - x - 2y - 3z)$ $(a > 0)$.

529. $u = \frac{a^2}{x} + \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{b}$ $(x > 0, y > 0, z > 0, a > 0, b > 0)$.

530. $u = \sin x + \sin y + \sin z - \sin(x + y + z)$ $(0 \le x \le \pi;\ 0 \le y \le \pi;\ 0 \le z \le \pi)$.

531. $u = x_1x_2^2\dots x_n^n\,(1 - x_1 - 2x_2 - \dots - nx_n)$ $(x_1 > 0, x_2 > 0, \dots, x_n > 0)$.

532. $u = x_1 + \frac{x_2}{x_1} + \frac{x_3}{x_2} + \dots + \frac{x_n}{x_{n-1}} + \frac{2}{x_n}$ $(x_i > 0, i = 1, 2, \dots, n)$.

533. *Gýuýgensiň meselesi.* $u = \frac{x_1x_2\dots x_n}{(a + x_1)(x_1 + x_2)\dots(x_n + b)}$ drobuň ululygy iň uly bolar ýaly, a we b položitel sanlaryň arasynda n sany $x_1, x_2, \dots, x_n$ sanlary ýerleşdirmeli.

Üýtgeýän x we y ululyklara baglylygy anyk däl görnüşde berlen z funksiýanyň ekstremal bahalaryny tapmaly:

534. $x^2 + y^2 + z^2 - 2x + 2y - 4z - 10 = 0$.

535. $x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2 = 0$.

536. $(x^2 + y^2 + z^2)^2 = a^2(x^2 + y^2 - z^2)$.

Aşakdaky berlen funksiýalaryň görkezilen şertleri kanagatlandyrýan şertli ekstremum nokatlaryny tapmaly:

537. $z = xy,\ x + y = 1.$

538. $z = \frac{x}{a} + \frac{y}{b},\ x^2 + y^2 = 1.$

539. $z = x^2 + y^2,\ \frac{x}{a} + \frac{y}{b} = 1.$

540. $z = Ax^2 + 2Bxy + Cy^2,\ x^2 + y^2 = 1.$

541. $z = x^2 + 12xy + 2y^2,\ 4x^2 + y^2 = 25.$

542. $z = \cos^2 x + \cos^2 y,\ x - y = \frac{\pi}{4}.$

543. $u = x - 2y + 2z,\ x^2 + y^2 + z^2 = 1.$

544. $u = x^m y^n z^p,\ x + y + z = a\ (m > 0,\ n > 0,\ p > 0,\ a > 0).$

545. $u = x^2 + y^2 + z^2,\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\ (a > b > c > 0).$

546. $u = xy^2z^3,\ x + 2y + 3z = a\ (x > 0,\ y > 0,\ z > 0,\ a > 0).$

547. $u = xyz,\ x^2 + y^2 + z^2 = 1,\ x + y + z = 0.$

548. $u = xy + yz,\ x^2 + y^2 = 2,\ y + z = 2\ (x > 0,\ y > 0,\ z > 0).$

549. $u = \sin x \sin y \sin z,\ x + y + z = \frac{\pi}{2}\ (x > 0,\ y > 0,\ z > 0).$

550. $u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2},\ x^2 + y^2 + z^2 = 1,\ x\cos\alpha + y\cos\beta + z\cos\gamma = 0\ (a > b > c > 0,$ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1).$

551. $u = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2,\ Ax + By + Cz = 0,\ x^2 + y^2 + z^2 = R^2,$ $\frac{\xi}{\cos\alpha} = \frac{\eta}{\cos\beta} = \frac{\zeta}{\cos\gamma},\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$

552. $u = x_1^2 + x_2^2 + \ldots + x_n^2,\ \frac{x_1}{a_1} + \frac{x_2}{a_2} + \ldots + \frac{x_n}{a_n} = 1\ (a_i > 0;\ i = 1, 2, \ldots, n).$

553. $u = x_1^p + x_2^p + \ldots + x_n^p\ (p > 1),\ x_1 + x_2 + \ldots + x_n = a\ (a > 0).$

554. $u = \frac{\alpha_1}{x_1} + \frac{\alpha_2}{x_2} + \ldots + \frac{\alpha_n}{x_n},\ \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n = 1,\ \alpha_i > 0,\ \beta_i > 0,\ x_i > 0,$ $(i = 1, 2, \ldots, n).$

555. $u = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n},\ x_1 + x_2 + \ldots + x_n = a,\ a > 0,\ \alpha_i > 1,\ (i = 1, 2, \ldots, n).$

556. $\sum_{i=1}^{n} x_i^2 = 1$ şertde $u = \sum_{i,j}^{n} a_{ij} x_i x_j\ (a_{ij} = a_{ji})$ kwadrat formanyň ekstremumyny tapmaly.

557. $n \geq 1$ we $x \geq 0, y \geq 0$ bolanda $\dfrac{x^n + y^n}{2} \geq \left(\dfrac{x+y}{2}\right)^n$ deňsizligi subut etmeli. (*Görkezme*: $x + y = s$ *şertde* $z = \frac{1}{2}(x^n + y^n)$ *funksiýanyň minimumyny tapmaly*).

558. Gýolderiň $\sum_{i=1}^{n} a_i x_i \leq \left(\sum_{i=1}^{n} a_i^k\right)^{\frac{1}{k}} \left(\sum_{i=1}^{n} x_i^{k'}\right)^{\frac{1}{k'}}$ deňsizligini subut etmeli $\left(a_i \geq 0,\ x_i \geq 0,\ i = 1, 2, ..., n;\ k > 1,\ \frac{1}{k} + \frac{1}{k'} = 1\right)$. (*Görkezme*: $\sum_{i=1}^{n} a_i x_i = A$ *şertde* $u = \left(\sum_{i=1}^{n} a_i^k\right)^{1/k} \left(\sum_{i=1}^{n} x_i^{k'}\right)^{1/k'}$ *funksiýanyň minimumyny tapmaly*).

559. n tertipli $A = |a_{ij}|$ kesgitleýji üçin Adamaryň deňsizligini subut etmeli: $A^2 \leq \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}^2\right)$. (*Görkezme*: $A = |a_{ij}|$ *kesgitleýjiniň ekstremumyna* $\sum_{j=1}^{n} a_{ij}^2 = S_i$ $(i = 1, 2, ..., n)$ *şertde garamaly*).

Aşakdaky funksiýalaryň görkezilen ýaýlalarda iň uly (sup) we iň kiçi (inf) bahalaryny kesgitlemeli:

560. $z = x - 2y - 3,\ 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq x + y \leq 1.$

561. $z = x^2 + y^2 - 12x + 16y,\ x^2 + y^2 \leq 25.$

562. $z = x^2 - xy + y^2,\ |x| + |y| \leq 1.$

563. $u = x^2 + 2y^2 + 3z^2,\ x^2 + y^2 + z^2 \leq 100.$

564. $u = x + y + z,\ x^2 + y^2 \leq z \leq 1.$

565. $x > 0, y > 0, z > 0$ ýaýlada $u = (x + y + z)e^{-(x+2y+3z)}$ funksiýanyň aşaky (inf) we ýokarky (sup) takyk çägini tapmaly.

566. $z = (1 + e^y)\cos x - ye^y$ funksiýanyň tükeniksiz köp maksimumynyň bardygyny we hiç bir minimumynyň ýokdugyny subut etmeli.

567. $f(x, y)$ funksiýanyň $M_0(x_0, y_0)$ nokatda minimumynyň bolmagy üçin ol funksiýanyň M_0 nokatdan geçýän her bir göni çyzyk boýunça minimumynyň bolmagy ýeterlikmi? Bu şertde şu funksiýany derňemeli: $f(x,y) = (x - y^2)(2x - y^2)$.

568. Berlen položitel a sany ters ululyklarynyň jemi, iň uly bolar ýaly, n položitel köpeldijilere dagytmaly.

569. Berlen položitel a sany kwadratlarynyň jemi, iň kiçi bolar ýaly, n goşulyjylara dagytmaly.

570. Berlen položitel a sany berlen položitel derejeleriň jemi, iň kiçi bolar ýaly, n položitel köpeldijilere dagytmaly.

571. Tekizlikde massalary $m_1, m_2, \ldots, m_n$ bolan n sany $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, ..., $P_n(x_n, y_n)$ material nokatlar berlen. $P(x, y)$ nokat nähili ýerleşende material nokatlar sistemasynyň şol nokada görä inersiýa momenti iň kiçi bolar?

572. Sygymy V bolan göni burçly açyk wannanyň ölçegleri nähili bolanda onuň üsti iň kiçi bolar?

573. Üsti S we kese-kesigi ýarym tegelek bolan açyk silindr şekilli wannanyň ölçegleri nähili bolanda ol iň uly sygymly bolar?

574. $x^2 + y^2 + z^2 = 1$ sferada berlen $M_i(x_i, y_i, z_i)$ $(i = 1, 2, ..., n)$ nokatlardan uzaklyklarynyň kwadratlarynyň jemi iň kiçi bolýan nokadyny tapmaly.

575. Jisim konus şekilli göni tegelek silindrden ybarat. Doly üsti belli we Q-a deň bolan ol jisimiň ölçegleri nähili bolanda onuň göwrümi iň uly bolar?

576. Göwrümi V bolan jisim dogry göni burçly parallelepiped bolup, onuň aşaky we ýokarky esaslary deň dogry dörtburçly piramidadyr. Piramidanyň gapdal granlary onuň esasyna haýsy burç boýunça gyşaranda jisimiň doly üsti iň uly bolar?

577. Perimetri $2p$ we taraplarynyň biriniň daşyndan aýlananda göwrümi iň uly bolýan gönüburçlugy tapmaly.

578. Perimetri $2p$ we taraplarynyň biriniň daşyndan aýlananda göwrümi iň uly bolýan üçburçlugy tapmaly.

579. R radiusly ýarym şaryň içinden göwrümi iň uly bolan göni burçly parallelepipedi çyzmaly.

580. Berlen göni tegelek konusyň içinden göwrümi iň uly bolan göni burçly parallelepipedi çyzmaly.

581. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ ellipsoidiň içinden göwrümi iň uly bolan göni burçly parallelepipedi çyzmaly.

582. l emele getirijisi esasynyň tekizligi bilen α burç emele getirýän göni tegelek konusyň içinden doly üsti iň uly bolan göni burçly parallelepipedi çyzmaly.

583. $\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$, $z = c$ elliptik paraboloidiň segmentiniň içinden göwrümi iň uly bolan göni burçly parallelepipedi çyzmaly.

584. $M_0(x_0, y_0, z_0)$ nokadyň $Ax + By + Cz + D = 0$ tekizlikden iň kiçi uzaklygyny tapmaly.

585. Giňişlikdäki

$$\frac{x-x_1}{m_1}=\frac{y-y_1}{n_1}=\frac{z-z_1}{p_1} \quad \text{we} \quad \frac{x-x_2}{m_2}=\frac{y-y_2}{n_2}=\frac{z-z_2}{p_2}$$

iki göni çyzyklaryň arasyndaky iň ýakyn d uzaklygy tapmaly.

586. $y = x^2$ parabola bilen $x - y - 2 = 0$ göni çyzygyň arasyndaky iň ýakyn uzaklygy tapmaly.

587. Ikinji tertipli $Ax^2 + 2Bxy + Cy^2 = 1$ merkezi çyzygyň ýarym oklaryny tapmaly.

588. Ikinji tertipli $Ax^2 + By^2 + Cz^2 + 2Dxy + 2Eyz + 2Fxz = 1$ merkezi üstüň ýarym oklaryny tapmaly.

589. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ silindr bilen $Ax + By + Cz = 0$ tekizligiň kesişmeginden alynýan ellipsiň meýdanyny tapmaly.

590. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ellipsoid bilen $x\cos\alpha + y\cos\beta + z\cos\gamma = 0$ tekizligiň kesiginiň meýdanyny tapmaly, bu ýerde

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

591. Fermanyň prinsipi boýunça A nokatdan çykýan we B nokada düşýän ýagtylyk geçmek üçin iň az wagt talap edýän çyzyk boýunça ýaýraýar.

A we B nokatlar tekizlik bilen bölünen dürli optiki gurşawda ýerleşýär diýip hasap edip we ýagtylygyň birinji we ikinji gurşawlardaky ýaýrama tizliklerini degişlilikde υ_1 we υ_2 alyp, ýagtylygyň döwülme kanunyny getirip çykarmaly.

592. Döwülme burçy α we döwülme görkezijisi n bolan prizmadan geçýän ýagtylyk şöhlesi haýsy burç boýunça düşende ýagtylyk şöhlesiniň gyşarmasy (ýagny düşýän we çykýan şöhleleriň arasyndaky burç) iň kiçi bolar? Şol iň kiçi gyşarmany tapmaly.

593. Üýtgeýän x we y ululyklar koeffisiýentlerini kesgitlemek talap edilýän $y = ax + b$ çyzykly deňlemeleri kanagatlandyrýar. Birnäçe deň-takyk ölçegleriň netijesinde x we y ululyklar üçin x_i, y_i $(i = \overline{1, n})$ bahalar alyndy.

Iň kiçi kwadratlar usulyndan peýdalanyp, a we b koeffisiýentleriň iň ähtimal bahalaryny tapmaly. (*Görkezme*: *Iň kiçi kwadratlar usuly boýunça a we b koeffisiýentleriň iň ähtimal bahalary ýalňyşlyklarynyň kwadratlarynyň jemi*

$$\sum_{i=1}^{n}\Delta_i^2 = \sum_{i=1}^{n}(ax_i + b - y_i)^2$$

iň kiçi bolýan bahalardyr).

594. Tekizlikde n nokatlaryň sistemasy berlen. $M_i(x_i, y_i)$ $(i = 1, 2, ..., n)$. $x\cos\alpha + y\sin\alpha - p = 0$ göni çyzyk nähili ýerleşende ol nokatlaryň şol göni çyzykdan gyşarmalarynyň kwadratlarynyň jemi iň kiçi bolar?

595. x^2 funksiýany (1, 3) interwalda $\Delta = \sup|x^2 - (ax + b)|$ $(1 \leq x \leq 3)$ absolýut gyşarma iň kiçi bolar ýaly $ax + b$ çyzykly funksiýa bilen takmyny çalşyrmaly.

JOGAPLAR

I. §1. Köplükler we olar bilen geçirilýän amallar

1. $A \cap B$ – topardaky başlikçileriň köplügi; $A \backslash B$ – topardaky başlikçi däl talyplaryň köplügi; $B \backslash A$ – fakultetiň beýleki toparlaryndaky başlikçileriň köplügi. **2.** $A \cup B = \{x: 0<x \le \le 3\}$, $A \cap B = \{x: 1 \le x < 2\}$; $A \backslash B = \{x: 0<x<1\}$; $B \backslash A = \{x: 2 \le x \le 3\}$; $A \Delta B = \{x: (0<x<1) \vee (2< <x<3)\}$. **4.** 0; 1. **5.** $-\sqrt{2}, \sqrt{2}$. **10.** $-1{,}01<x<-0{,}99$. **11.** $x \le -8$, $x \ge 12$. **12.** $x<-1/2$. **13.** $0<x<2/3$. **14.** $|x| \le 6$. **15.** $x>-1/2$. **16.** $-1/2<x<1/2$. **17.** $(5-\sqrt{30})/10 < x < < (5-\sqrt{20})/10$; $(5+\sqrt{20})/10 < x < (5+\sqrt{30})/10$. **19.** Ikinji. **20.** $9{,}9102\ sm^2 \le S \le \le 10{,}0902\ sm^2$; $\Delta \le 0{,}0902\ sm^2$; $\delta \le 0{,}91\%$. **21.** $3{,}93\ g/sm^3 \pm 0{,}27\ g/sm^3$; $\delta \le 7{,}3\%$. **22.** $\delta \le 3{,}05\%$. **23.** $172{,}480\ m^3 \le v \le 213{,}642\ m^3$; $v = 192{,}660\ m^3 \pm 20{,}982\ m^3$, $\delta \approx 12\%$. **24.** $\Delta \le 0{,}17\ mm$. **25.** $\Delta < 0{,}0005\ m$.

II. §1. San yzygiderlikleri we olaryň häsiýetleri

1. a) $n_0 \ge 1/\varepsilon$; b) $n_0 \ge 1/\varepsilon$; ç) $n_0 \ge \sqrt{2/\varepsilon}$; d) $n_0 \ge 1+\lg(1/\varepsilon)/\lg 2$; e) $n_0 \ge \lg\varepsilon/\lg 0{,}999 \approx \approx 2330\lg(1/\varepsilon)$. **2.** a) $n_0 > K$; b) $n_0 > (\lg K/\lg 2)^2$; ç) $n_0 > 10^{10}$. **6.** 0. **7.** 0. **8.** 0. **9.** 1/3. **10.** $(1-b)/(1-a)$. **11.** 1/2. **12.** 1/2. **13.** 1/3. **14.** 4/3. **15.** 3. **16.** 1. **17.** 2. **27.** a) ikinji; b) birinji; ç) ikinji. **32.** $e=2{,}71828...$. **52.** $a \ne 0$ bolanda 1-e deň we $[-1,\ 1]$ kesime degişli ýa-da $a=0$ bolanda predeli ýok. **56.** $x_3=9/8$. **57.** $x_{100}=1/20$. **58.** $x_{1000}=1000^{1000}/1000! \approx 2{,}49 \cdot 10^{452}$. **59.** $x_4=x_5=-120$. **60.** $x_{10}=20$. **61.** 0; 1; 1; 1. **62.** –7/2; 5; –2; 2. **63.** –1; 3/2; 0; 1. **64.** 0; 2; 0; 2. **65.** –4; 6; –4; 6. **66.** –1/2; 1; –1/2; 1. **67.** $-\infty$; $+\infty$; $-\infty$; $+\infty$. **68.** $-\infty$; –1; $-\infty$; $-\infty$. **69.** 0; $+\infty$; 0; $+\infty$. **70.** $-\infty$; $+\infty$; $-\infty$; $+\infty$. **71.** –5; 1,25; 0; 0. **72.** –1/2; 1. **73.** $-(e+1/\sqrt{2})$; $e+1$. **74.** 0; 1. **75.** 1; 2. **76.** 0; 1. **77.** 0; 1. **78.** 1; 1/2; 1/3; …; 0. **79.** 0 we 1-iň arasyndaky ähli hakyky sanlar, şolary hem girizip. **80.** 1; 5. **81.** a; b. **88.** a) dargaýar; b) ýygnanýan hem bolup biler, dargaýan hem. **89.** a) ýok; b) ýok. **90.** Ýok. **91.** Ýok. **105.** a) 0; b) 0. **108.** ln2. **109.** $(a+2b)/3$.

III. §1. Funksiýa we onuň grafigi

1. $-\infty<x<+\infty$, $x \ne -1$. **2.** $-\infty < x \le -\sqrt{3}$ we $0 \le x \le \sqrt{3}$. **3.** $-1 \le x < 1$. **4.** a) $|x|>2$; b) $x>2$. **5.** $4k^2\pi^2 \le x \le (2k+1)^2\pi^2$ ($k=0,\ 1,\ 2,\ ...$). **6.** $|x| \le \sqrt{\pi/2}$ we $\sqrt{\pi(4k-1)/2} \le \le |x| \le \sqrt{\pi(4k+1)/2}$ ($k=1, 2,\ ...$). **7.** $1/(2k+1)<x<1/2k$ we $-1/(2k+1)<x<-1/(2k+2)$ ($k=0,\ 1,\ 2,\ ...$). **8.** $x>0$, $x \ne n$ ($n=1,\ 2,\ ...$). **9.** $-1/3 \le x \le 1$. **10.** $|x-k\pi| \le \pi/6$ ($k=0,\ \pm 1,$

$\pm2, \ldots)$. **11.** $10^{(2k-1/2)\pi} < x < 10^{(2k+1/2)\pi}$ $(k=0, \pm1, \pm2, \ldots)$. **12.** $x=-1, -2, -3, \ldots$ we $x\geq0$. **13.** $x<0, x\neq-n$ $(n=1, 2, \ldots)$. **14.** $1<x\leq2$. **15.** $x=1/2, 1, 3/2, 2, \ldots$. **16.** $x>4$. **17.** $k\pi+\pi/4\leq x<k\pi+$ $+\pi/2$ $(k=0, \pm1, \ldots)$. **18.** $0\leq x\leq\pi/3$ we $4\pi/3\leq x\leq 3\pi/2$. **19.** $-1\leq x\leq2$; $0\leq y\leq3/2$. **20.** $2k\pi+\pi/3<x<2k\pi+5\pi/3$ $(k=0, \pm1, \pm2, \ldots)$; $-\infty<y\leq\lg3$. **21.** $-\infty<x<+\infty$; $0\leq y\leq\pi$. **22.** $1\leq x\leq100$; $-\pi/2\leq y\leq\pi/2$. **23.** $x=p/(2q+1)$, bu ýerde p we q bitin sanlar; $y=\pm1$. **24.** $P=2b+2(1-b/h)x$ $(0<x<h)$; $S=bx(1-x/h)$ $(0<x<h)$. **25.** $a=\sqrt{100-96\cos x}$ $(0<x<\pi)$, $S=24\sin x\,(0<x<\pi)$. **26.** $S=hx^2/(a-b)$, $0\leq x\leq(a-b)/2$ bolanda; $S=h(x-(a-b)/4)$, $(a-b)/2<x<(a+b)/2$ bolanda; $S=h[(a+b)/2-(a-x)^2/(a-b)]$, $(a+b)/2\leq x\leq a$ bolanda. **27.** $m(x)=0$, $-\infty<x\leq0$ bolanda; $m(x)=2x$, $0<x\leq1$ bolanda; $m(x)=2$, $1<x\leq2$ bolanda; $m(x)=3$, $2<x\leq3$ bolanda; $m(x)=4$, $3<x<+\infty$ bolanda. **31.** $E_y=\{0\leq y\leq4\}$. **32.** $E_y=\{1<y<3\}$. **33.** $E_y=\{0<y<1\}$. **34.** $E_y=\{1\leq|y|<+\infty\}$. **35.** $E_y=\{1\leq y\leq2\}$. **36.** $a<y<b$, $a<b$ bolanda we $b<y<a$, $a>b$, bolanda. **37.** $1<y<+\infty$. **38.** $0>y>-\infty$ we $+\infty>y>1$. **39.** $0<y\leq1/2$. **40.** $+\infty>y>-\infty$. **41.** $0<y<1/2$ we $3/2\leq y<2$. **42.** 0; 0; 0; 0; 24. **43.** 0; –6; 4. **44.** 1; 1; 1; 2. **45.** –1; 0; 1; 2; 4. **46.** $1, \dfrac{1+x}{1-x}, \dfrac{-x}{2+x}, \dfrac{2}{1+x}, \dfrac{x-1}{x+1}, \dfrac{1+x}{1-x}$. **47.** a) $f(x)=0$, $x=-1$, $x=0$ we $x=1$ bolanda; $f(x)>0$, $-\infty<x<-1$ we $0<x<1$ bolanda; $f(x)<0$, $-1<x<0$ we $1<x<+\infty$ bolanda; b) $f(x)=0$, $x=\pm1/k$ bolanda; $f(x)>0$, $\dfrac{1}{2k+1}<x<\dfrac{1}{2k}$ we $-\dfrac{1}{2k+1}<x<-\dfrac{1}{2k+2}$ $(k=0, 1, 2, \ldots)$ bolanda; $f(x)<0$, $\dfrac{1}{2k+2}<x<\dfrac{1}{2k+1}$ we $-\dfrac{1}{2k}<x<-\dfrac{1}{2k+1}$ $(k=0, 1, 2, \ldots)$ bolanda; ç) $f(x)=0$, $x\leq0$ we $x=1$ bolanda; $f(x)>0$, $0<x<1$ bolanda; $f(x)<0, 1<x<+\infty$ bolanda. **48.** a) a; b) $2x+h$; ç) $a^x(a^h-1)/h$. **50.** $f(x)=7x/3-2$; $f(1)=1/3$; $f(2)=8/3$. **51.** $f(x)=7x^2/6+17x/6+1$; $f(-1)=-2/3$; $f(0{,}5)=65/24$. **52.** $f(x)=10x^3/3-7x^2/2-$ $-29x/6+2$. **53.** $f(x)=10+5\cdot2^x$. **56.** a) $2k\pi<x<\pi+2k\pi$ $(k=0, \pm1, \pm2, \ldots)$; b) $1<x<e$; ç) $x>0$, $x\neq k$ $(k=0, 1, 2, \ldots)$. **58.** a) $z=x+y$; b) $z=\dfrac{xy}{x+y}$; ç) $z=\dfrac{x+y}{1-xy}$; d) $z=\dfrac{x+y}{1+xy}$. **59.** $\varphi(\varphi(x))=x^4$; $\psi(\psi(x))=2^{2x}$; $\varphi(\psi(x))=2^{2x}$; $\psi(\varphi(x))=2^{x^2}$. **60.** $\varphi(\varphi(x))=\operatorname{sgn}x$; $\psi(\psi(x))=x$ $(x\neq0)$; $\varphi(\psi(x))=\psi(\varphi(x))=\operatorname{sgn}x$ $(x\neq0)$. **61.** $\varphi(\varphi(x))=\varphi(x)$; $\psi(\varphi(x))=\psi(x)$; $\psi(\psi(x))=\varphi(\psi(x))=0$. **62.** $-(1-x)/x$; x $(x\neq0, x\neq1)$. **63.** $f_n(x)=\dfrac{x}{\sqrt{1+nx^2}}$. **64.** x^2-5x+6. **65.** $x^2-2\left(|x|\geq2\dfrac{1}{2}\right)$. **66.** $\dfrac{1+\sqrt{1+x^2}}{x}$. **67.** $f(x)=(x/(1-x))^2$. **75.** a) $a>0$ bolanda artýar we $a<0$ bolanda kemelýär; b) $a>0$ bolanda $(-\infty, -b/2a)$ interwalda kemelýär we $(-b/2a, +\infty)$ interwalda artýar; ç) artýar; d) $ad-bc>0$ bolanda $(-\infty, -d/c)$ we $(-d/c, +\infty)$ interwallarda artýar; e) $a>1$ bolanda artýar we $0<a<1$ bolanda kemelýär. **76.** Logarifmiň esasy birden uly bolsa, onda bolar. **78.** $\dfrac{y-3}{2}$ $(-\infty<y<+\infty)$. **79.** a) $-\sqrt{y}$ $(0\leq y<+\infty)$; b) $\sqrt{y}$ $(0\leq y<+\infty)$. **80.** $\dfrac{1-y}{1+y}$ $(y\neq-1)$. **81.** a) $-\sqrt{1-y^2}$ $(0\leq y\leq1)$; b) $\sqrt{1-y^2}$ $(0\leq y\leq1)$. **82.** $\operatorname{Arsh}y=\ln\left(y+\sqrt{1+y^2}\right)$ $(-\infty<y<+\infty)$. **83.** $\operatorname{Arth}y=(1/2)\ln((1+y)/(1-y))$ $(-1<y<1)$. **84.** $x=y$, $-\infty<y<1$ bolanda;

$x=\sqrt{y}$, $1\le y\le 16$ bolanda; $x=\log_2 y$, $16<y<+\infty$ bolanda. **85.** a) täk; b) jübüt; ç) jübüt; d) täk; e) täk. **87.** a) periodik, $T=2\pi/\lambda$; b) periodik, $T=2\pi$; ç) periodik, $T=6\pi$; d) periodik, $T=\pi$; e) periodik däl; f) periodik, $T=\pi$; g) periodik däl; h) periodik däl. **96.** $t=5/3$ s, $x=-10/3$ m. **98.** $x_0=-b/2a$, $y_0=(4ac-b^2)/4a$. **99.** $y=x-\dfrac{x^2}{36000}$; $9\,km$; $36\ km$. **106.** $x_0=-\dfrac{d}{c}$; $y_0=\dfrac{a}{c}$. **107.** $p=\dfrac{12}{v}$ $(v>0)$. **118.** $k=\dfrac{a}{a_1}$ $m=\dfrac{a_1b-ab_1}{a_1^2}$, $n=\dfrac{c}{a_1}-\dfrac{b_1}{a_1^3}(a_1b-ab_1)$, $x_0=-\dfrac{b_1}{a_1}$. **119.** $y=10/x^2$. **142.** $A=\sqrt{a^2+b^2}$; $\sin x_0=-a/A$, $\cos x_0=b/A$. **217.** $y=2\sin x$, $|x-\pi k|\le\pi/6$ we $y=(-1)^k$, $\dfrac{\pi}{6}<|x-\pi k|<\dfrac{5\pi}{6}$ $(k=0,$ $\pm1,\ \pm2,\ \ldots)$. **218.** a) $y=(x+|x|)/2$; b) we ç) $y=x^2$, $x\ge0$ bolanda; $y=0$, $x<0$ bolanda; d) $y=x$, $x<0$ bolanda; $y=x^4$, $x\ge0$ bolanda. **219.** a) $y=1$; b) $y=1$, $1\le|x|\le\sqrt{3}$; $y=0$, $|x|<1$ ýa-da $|x|>\sqrt{3}$; ç) $y=1$, $|x|\le1$; $y=2$, $|x|>1$; d) $y=-2$, $|x|>2$; $y=2-(2-x^2)^2$, $|x|\le2$. **220.** $x<0$ bolanda alarys: a) 1) $f(x)=1+x$, 2) $f(x)=-(1+x)$; b) 1) $f(x)=-2x-x^2$, 2) $f(x)=2x+x^2$; ç) 1) $f(x)=\sqrt{-x}$, 2) $f(x)=-\sqrt{-x}$; d) 1) $f(x)=-\sin x$, 2) $f(x)=\sin x$; e) 1) $f(x)=e^{-x}$, 2) $f(x)=-e^{-x}$; *f*) 1) $f(x)=\ln(-x)$, 2) $f(x)=-\ln(-x)$. **221.** a) $x=-b/2a$; b) $x=1/2$; ç) $x=(b-a)/2$; d) $x=k\pi$ $(k=0,\ \pm1,\ \pm2,\ \ldots)$. **222.** a) $(x_0,\ ax_0+b)$, x_0-erkin; b) $(-d/c,\ a/c)$; ç) (x_0, y_0), $x_0=-b/3a$ we $y_0=ax_0^3+bx_0^2+cx_0+d$; d) (2, 0); e) (2, 1). **236.** Kökleri: –1,88; 0,35; 1,53. **237.** 2,11; –0,25; –1,86. **238.** 0,25; 1,49. **239.** 0,64. **240.** 1,37; 10. **241.** –0,54. **242.** 0; 4,49. **243.** $x_1=-0{,}57$, $y_1=-1{,}26$; $x_2=-0{,}42$, $y_2=1{,}19$; $x_3=0{,}45$, $y_3=0{,}74$; $x_4=0{,}54$, $y_4=-0{,}68$. **244.** $x_1=-1{,}30$, $y_1=9{,}91$; $x_2=2{,}30$, $y_2=9{,}73$; $x_3=-0{,}62$, $y_3=-9{,}98$; $x_4=1{,}62$, $y_4=-9{,}87$.

§2. Funksiýanyň predeli

246. a) umuman aýdylanda, ýok; b) hawa. **249.** Ýokardan çäkli we aşakdan çäkli däl. **251.** $f(a)$ we (b). **252.** 0; 25. **253.** 0; 1. **254.** 0; 1. **255.** 2; $+\infty$. **256.** –1; 1. **257.** $-\sqrt{2}$; $\sqrt{2}$. **258.** 1/2; 4. **259.** a) 0, 1; b) 0; 2. **260.** 0; 1. **261.** a) 8; b) 0,8; ç) 0,08; d) 0,008. **262.** a) π; b) π; ç) π; d) π. **275.** a) 1; b) 2/3; ç) 1/2. **276.** 6. **277.** 10. **278.** $nm(n-m)/2$. **279.** 5^{-5}. **280.** $(3/2)^{30}$. **281.** $n^{-n(n+1)/2}$. **282.** –1/2. **283.** 1/2. **284.** 1. **285.** 1/4. **286.** 1/3. **287.** $(3/2)^{10}$. **288.** $n(n+1)/2$. **289.** 49/2. **290.** m/n. **291.** $(n(n-1)/2)\cdot a^{n-2}$. **292.** $n(n+1)/2$. **293.** $(m-n)/2$. **294.** $x+a/2$. **295.** $x^2+ax+a^2/3$. **296.** 1. **297.** 1/2. **298.** 3. **299.** $ab/3$. **300.** 1. **301.** $1/\sqrt{2}$. **302.** 4/3. **303.** –2. **304.** $1/\sqrt{2a}$. **305.** –1/16. **306.** 1/144. **307.** 1/4. **308.** 12/5. **309.** $1/n$. **310.** –2. **311.** 1/4. **312.** 2/27. **313.** 3/2. **314.** 112/27. **315.** 7/36. **316.** –1/2. **317.** $\dfrac{\alpha}{m}-\dfrac{\beta}{n}$. **318.** $\dfrac{\alpha}{m}+\dfrac{\beta}{n}$. **320.** n/m. **321.** 1/2. **322.** $1/n!$. **323.** $(1/2)\cdot(a+b)$. **324.** 1/2. **325.** –1/4. **326.** 1. **327.** 2/3. **328.** 2. **329.** 4/3. **330.** –1/4. **331.** $(a_1+a_2+\ldots+a_n)/n$. **332.** 2^n. **333.** $2n$. **334.** $\lim\limits_{a\to0}x_1=\infty$,

$\lim\limits_{a\to 0} x_2 = -c/b$. **335.** $a=1$, $b=-1$. **336.** $a_i=\pm 1$, $b_i=\mp 1/2$ $(i=1,\ 2)$. **337.** 5. **338.** 0. **339.** $(-1)^{m-n}\dfrac{m}{n}$. **340.** 1/2. **341.** 1. **342.** 1/3. **343.** 1/2. **344.** 2. **345.** 4. **346.** $1/p$. **347.** 1/2. **348.** $2/\pi$. **350.** $\cos a$. **351.** $-\sin a$. **352.** $\sec^2 a$ $\left(a \neq (2k+1)\dfrac{\pi}{2},\ k=0, \pm 1, \ldots\right)$. **353.** $-\dfrac{1}{\sin^2 a}$ ($a\neq k\pi$, bu ýerde k – bitin san). **354.** $\dfrac{\sin a}{\cos^2 a}$ $\left(a \neq (2k+1)\dfrac{\pi}{2}\right)$. **355.** $-\dfrac{\cos a}{\sin^2 a}$ ($a\neq k\pi$, bu ýerde k – bitin san). **356.** $-\sin a$. **357.** $-\cos a$. **358.** $\dfrac{2\sin a}{\cos^3 a}\left(a \neq (2k+1)\dfrac{\pi}{2}\right.$, bu ýerde k – bitin san$\Big)$. **359.** $\dfrac{2\cos a}{\sin^3 a}$ ($a\neq k\pi$, k – bitin san). **360.** $3\sin 2a/2$. **361.** –3. **362.** 14. **363.** $1/\sqrt{3}$. **364.** – 24. **365.** $-\dfrac{\cos 2a}{\cos^4 a}$ $\left(a \neq (2k+1)\dfrac{\pi}{2}\right.$, bu ýerde k – bitin san$\Big)$. **366.** 3/4. **367.** 1/4. **368.** 4/3. **369.** –1/12. **370.** $\sqrt{2}$. **371.** 0. **372.** 3. **373.** 0. **374.** a) 1/2; b) $\sqrt{2/3}$; ç) 1. **375.** 0. **376.** 0. **377.** 0. **378.** 0. **379.** 1. **380.** e^3. **381.** 1. **382.** e^{-2}. **383.** e^{2a}. **384.** 0, $a_1<a_2$ bolanda; $+\infty$, $a_1>a_2$ bolanda; $e^{((b_1-b_2)/a_1)}$, $a_1=a_2$ bolanda. **385.** e. **386.** e^{-1}. **387.** 1. **388.** $\sqrt{e}$. **389.** $e^{\operatorname{ctg} a}$ ($a\neq k\pi$, k – bitin san). **390.** $e^{3/2}$. **391.** e^{-1}. **392.** 1. **393.** e^{-2}. **394.** e. **395.** $1/\sqrt{e}$. **396.** e^{x+1}. **397.** $e^{-x^2/2}$. **398.** 1. **399.** 1. **400.** $1/a$. **401.** 0. **402.** 1/5. **403.** –2. **404.** 3/2. **405.** 3/2. **406.** $-\log e/x^2$. **407.** $2a/b$. **408.** $(a/b)^2$. **409.** 0. **410.** n. **411.** $\ln a$. **412.** $a^a\ln(a/e)$. **413.** $a^a\ln ea$. **414.** e^2. **415.** 2/3. **416.** $e^{\beta^2-\alpha^2}$. **417.** α/β. **418.** –2. **419.** e^2. **420.** 1. **421.** $(\alpha/\beta)a^{\alpha-\beta}$. **422.** $a^b\ln a$. **423.** $a^x\ln^2 a$. **424.** $e^{-(a+b)}$. **425.** $\ln x$. **426.** $\ln x$. **427.** $\sqrt[a]{b}$. **428.** $\sqrt{ab}$. **429.** $\sqrt[3]{abc}$. **430.** $(a^a b^b c^c)^{\frac{1}{a+b+c}}$. **431.** $1/\sqrt{ab}$. **432.** $(\ln(a/b))^{-1}$. **433.** $a^{a^a}\ln a$. **434.** a) 0, b) ln3/ln2. **435.** ln8. **436.** –ln2. **439.** a) 1/2; b) 1/2. **440.** 1. **441.** 0. **442.** $\ln a^2$. **443.** 1/8. **444.** 1/2. **445.** –2. **446.** e^2. **447.** $e^{2/\pi}$. **448.** $(\alpha+\beta)/\sqrt{\alpha\beta}$. **449.** a) 1; b) 1/2; ç) 1. **450.** 2/9. **451.** 2sh(1/2). **452.** a) ch$a$; b) sh$a$. **453.** –1. **454.** ln2. **455.** 1. **456.** e^{π^2}. **457.** $-\pi/2$. **458.** $\pi/3$. **459.** $-\pi/2$. **460.** $3\pi/4$. **461.** $1/(1+x^2)$. **462.** 2. **463.** $e^x/(1+x^2)$. **464.** 1/2. **465.** 1. **466.** $e^{2/\pi}$. **467.** 0. **468.** 0. **469.** a) $+\infty$; b) 1/2. **470.** a) –1; b) 1. **471.** $\ln(b^2/a^2)$. **472.** a) $\pi/2$; b) $-\pi/2$. **473.** a) 1; b) 0. **474.** a) 0; b) 1. **477.** 2; 1; 2. **478.** 0; $(-1)^{n-1}$; $(-1)^n$. **479.** 0. **480.** 1. **481.** 0. **482.** 1. **483.** 0. **490.** b) $y=1$, $|x|<1$; $y=0$, $|x|=1$. **491.** b) $y=0$, $0\leq x<1$ bolanda; $y=1/2$, $x=1$ bolanda; $y=1$, $1<x<+\infty$ bolanda. **492.** $y=-1$, $0<|x|<1$ bolanda; $y=0$, $|x|=1$ bolanda; $y=1$, $|x|>1$ bolanda. **493.** $y=|x|$. **494.** $y=1$, $0\leq x\leq 1$ bolanda; $y=x$, $x>1$ bolanda. **495.** $y=1$, $0\leq x\leq 1$ bolanda; $y=x$, $1<x<2$ bolanda; $y=x^2/2$, $x\geq 2$ bolanda. **496.** $y=0$, $0\leq x<2$ bolanda; $y=2\sqrt{2}$, $x=2$ bolanda; $y=x^2$, $x>2$ bolanda. **497.** b) $y=0$, $x\neq(2k+1)\pi/2$ bolanda; $y=1$, $x=(2k+1)\pi/2$ $(k=0, \pm 1, \pm 2, \ldots)$ bolanda. **498.** $y=\ln 2$, $0\leq x\leq 2$ bolanda; $y=\ln x$, $x>2$ bolanda. **499.** $y=0$, $-1<x\leq 1$ bolanda; $y=\pi(x-1)/2$, $x>1$ bolanda. **500.** $y=1$, $x\leq -1$ bolanda; $y=e^{x+1}$, $x>-1$ bolanda. **501.** $y=x$, $x<0$ bolanda; $y=1/2$, $x=0$ bolanda; $y=1$, $x>0$ bolanda. **502.** $1/x$. **503.** $y=\sqrt{x}$, $0\leq x<1$ we $4k-1<x<4k+1$ bolanda; $y=x$, $4k-3<x<4k-2$ we

$4k-2<x<4k-1$ bolanda; $y=(\sqrt{x}+x)/2$, $x=2k-1$ $(k=1, 2, 3, ...)$ bolanda. **504.** $y=0$, x rasional bolanda; $y=x$, x irrasional bolanda. **505.** $\max\{|x|, |y|\}=1$ kwadratyň kontury. **507.** a) $x=1$; $x=-2$, $y=x-1$; b) $y=x+1/2$, $x\to+\infty$ bolanda, $y=-x-1/2$, $x\to-\infty$ bolanda; ç) $y=1/3-x$; d) $y=x$, $x\to+\infty$ bolanda, $y=0$, $x\to-\infty$ bolanda; e) $y=0$, $x\to-\infty$, $y=x$, $x\to+\infty$; f) $y=x+\pi/2$. **508.** 0. **509.** $1/(1-x)$. **510.** $\sin x/x$. **512.** 1/6. **513.** $a/2$. **514.** $\ln a/2$. **515.** $\sqrt{e}$. **516.** $e^{-a^2/6}$. **517.** $(1+\sqrt{1+4a})/2$. **518.** 2/3. **519.** $b/(1-\alpha)$. **520.** $(\sqrt{5}-1)/2$. **521.** $\sqrt{1+x}-1$. **522.** $1-\sqrt{1-x}$. **526.** a) 2; b) $+\infty$; ç) 0; d) 1; e) 2; ä) 1; f) 2sh1. **528.** a) $l=-1$, $L=2$; b) $l=-2$, $L=2$; ç) $l=2$, $L=e$. **529.** a) $l=-1$, $L=1$; b) $l=0$, $L=+\infty$; ç) $l=1/2$, $L=2$; d) $l=0$, $L=+\infty$. **530.** a) birinji tertipli; b) ikinji; ç) birinji; d) üçünji; e) üçünji. **539.** a) $2x$; b) x; ç) $x^2/2$; d) $x^3/2$. **541.** a) $3(x-1)^2$; b) $(1-x)^{1/3}/\sqrt[3]{2}$; ç) $x-1$; d) $e(x-1)$; e) $x-1$. **542.** a) x^2; b) $2x^2$; ç) $x^{2/3}$; d) $x^{1/8}$. **543.** a) $(1/x)^3$; b) $(1/2)(1/x)^{1/2}$; ç) $-(1/4)(1/x)^{3/2}$; d) $(1/x)^2$. **544.** a) $(1/2)\cdot(1/(x-1))$; b) $\sqrt{2}(1/(x-1))^{1/2}$; ç) $(1/\sqrt[3]{3})\cdot$ $\cdot(1/(1-x))^{1/3}$; d) $1/\pi(1-x)$; e) $1/(x-1)$.

§ 3. Üznüksiz funksiýalar

549. a) $9{,}95<x<10{,}05$; b) $9{,}995<x<10{,}005$; ç) $9{,}9995<x<10{,}0005$; d) $\sqrt{100-\varepsilon}<$ $<x<\sqrt{100+\varepsilon}$. **550.** $\Delta<\varepsilon/27$; a) $\Delta<3{,}7$ *mm*; b) $\Delta<0{,}37$ *mm*; ç) $\Delta<0{,}037$ *mm*. **551.** $100[1-$ $-10^{-(n+1)}]^2<x<100[1+10^{-(n+1)}]^2$; a) $81<x<121$; b) $98{,}01<x<102{,}01$; ç) $99{,}8001<x<100{,}2001$; d) $99{,}980001<x<100{,}020001$. **552.** $\delta=\min(\varepsilon/11, 1)$ **553.** $\delta=\varepsilon x_0^2/(1+\varepsilon x_0)\approx 0{,}001x_0^2$ a) $\delta\approx10^{-5}$; b) $\delta\approx10^{-7}$; ç) $\delta\approx10^{-9}$. Ýok. **555.** a) ýok; b) bolýar. **557.** Ýok; x_0 nokatda çäkliligi. **558.** Ýok; eger $f(x)$ funksiýa tükenikli (a, b) interwalda kesgitlenen bolsa, onda ol deňsizlik hemişe ýerine ýetýär, eger-de iň bolmanda a ýa-da b-niň biri ∞ simwola deň bolsa, onda $\lim\limits_{x\to\infty}|f(x)|=+\infty$ bolar. **559.** Ýok; ters funksiýanyň bir bahalylygy we üznüksizligi. **561.** Üznüksiz. **562.** $A=4$ bolanda üznüksiz we $A\neq4$ bolanda $x=2$ nokatda üzülýär. **563.** $x=-1$ nokatda üzülýär. **564.** a) üznüksiz; b) $x=0$ nokatda üzülýär. **565.** $x=0$ nokatda üzülýär. **566.** Üznüksiz. **567.** Üznüksiz. **568.** $x=1$ nokatda üzülýär. **569.** $a=0$ bolanda üznüksiz, $a\neq0$ bolanda üzülýär. **570.** $x=0$ nokatda üzülýär. **571.** $x=k$ (k – bitin san) nokatda üzülýär. **572.** $x=k^2$ $(k=1, 2, ...)$ nokatlarda üzülýär. **573.** $x=-1$ tükeniksiz üzülme nokady. **574.** $x=-1$ aýrylýan üzülme nokady. **575.** $x=-2$ we $x=1$ tükeniksiz üzülme nokatlary. **576.** $x=0$ we $x=1$ aýrylýan üzülme nokatlary; $x=-1$ tükeniksiz üzülme nokady. **577.** $x=0$ aýrylýan üzülme nokady; $x=k\pi$ $(k=\pm1, \pm2, ...)$ tükeniksiz üzülme nokatlary. **578.** $x=\pm2$ aýrylýan üzülme nokatlary. **579.** $x=0$ ikinji görnüşdäki üzülme nokady. **580.** $x=1/k$ $(k=\pm1, \pm2, ...)$ birinji görnüşdäki üzülme nokatlary; $x=0$ ikinji görnüşdäki üzülme nokady. **581.** $x=0$ we $x=2/(2k+1)$ $(k=0, \pm1, ...)$ aýrylýan üzülme nokatlary. **582.** $x=0$ birinji görnüşdäki üzülme nokady. **583.** $x=0$ aýrylýan üzülme nokady. **584.** $x=0$ ikinji görnüşdäki üzülme nokady. **585.** $x=0$ aýrylýan üzülme nokady; $x=1$ tükeniksiz üzülme nokady. **586.** $x=0$ tükeniksiz üzülme nokady; $x=1$ ikinji görnüşdäki üzülme nokady. **587.** $x=k\pi$ $(k=0, \pm1, \pm2, ...)$ birinji görnüşdäki üzülme nokatlary. **588.** $x=k$ $(k=\pm1, \pm2, ...)$ birinji görnüşdäki üzülme nokatlary. **589.** $x=k$ ($k=\pm1, \pm2, ...$) birinji görnüşdäki üzülme nokatlary. **590.** Funksiýa üznüksiz. **591.** $x=\pm\sqrt{n}$ ($n=1, 2, ...$) birinji görnüşdäki üzülme nokatlary. **592.** $x=1/k$ ($k=\pm1,\pm2,...$) birinji görnüşdäki üzülme nokatlary; $x=0$ tükeniksiz üzülme nokady. **593.** $x=1/k$ ($k=\pm1, \pm2, ...$) birinji görnüşdäki üzülme nokatlary; $x=0$ aýrylýan üzülme nokady. **594.** $x=2/(2k+1)\pi$ ($k=0, \pm1, \pm2, ...$) birinji görnüşdäki üzülme nokatlary; $x=0$ ikinji görnüşdäki üzülme nokatlary. **595.** $x=\pm1/k$ we $x=\pm1\big/\sqrt{k}$ ($k=1, 2, ...$) birinji görnüşdäki üzülme nokatlary; $x=0$ ikinji görnüşdäki üzülme nokatlary. **596.** $x=1/k$ ($k=\pm1,\pm2,...$) tükeniksiz üzülme nokatlary; $x=0$ ikinji görnüşdäki üzülme nokatlary. **597.** $x=2/(2k+1)\pi$ ($k=0,\pm1,\pm2,...$) tükeniksiz üzülme nokatlary; $x=0$ ikinji görnüşdäki üzülme nokatlary. **598.** $x=\pm\sqrt{n}$ ($n=1, 2, ...$) birinji görnüşdäki üzülme nokatlary. **599.** $x=0$, $x=1$ we $x=2$ birinji görnüşdäki üzülme nokatlary. **600.** $x=k\pi$ ($k=0, \pm1, \pm2, ...$) tükeniksiz üzülme nokatlary. **601.** $x=\pm\sqrt{k\pi}$ ($k=0, 1, 2, ...$) tükeniksiz üzülme nokatlary. **602.** $x=-1$ we $x=3$ tükeniksiz üzülme nokatlary. **603.** $x=0$ ikinji görnüşdäki üzülme nokatlary. **604.** $x=0$ aýrylýan üzülme nokady. **605.** $x=\pm1$ birinji görnüşdäki üzülme nokatlary. **606.** $y=1$, $0\le x\le 1$ bolanda; $y=1/2$, $x=1$ bolanda; $y=0$, $x>1$ bolanda; $x=1$ birinji görnüşdäki üzülme nokady. **607.** $y=\operatorname{sgn}x$; $x=0$ birinji görnüşdäki üzülme nokady. **608.** $y=1$, $|x|\le 1$ bolanda; $y=x^2$, $|x|>1$ bolanda, funksiýa üznüksiz. **609.** $y=0$, $x\ne k\pi$ bolanda; $y=1$, $x=k\pi$ ($k=0, \pm1, \pm2, ...$) bolanda; $x=k\pi$ birinji görnüşdäki üzülme nokatlary. **610.** $y=x$, $|x-k\pi|<\pi/6$ bolanda; $y=x/2$, $x=k\pi\pm\pi/6$ bolanda; $y=0$, $\pi/6<|x-k\pi|<5\pi/6$ ($k=0, \pm1, ...$) bolanda; $x=k\pi\pm\pi/6$ birinji görnüşdäki üzülme nokatlary. **611.** $y=\pi x/2$, $k\pi<x<k\pi+\pi/2$; $y=-\pi x/2$, $k\pi+\pi/2<x<k\pi+\pi$; $y=0$, $x=k\pi+\pi/2$ ($k=0, \pm1, ...$); $x=k\pi/2$ birinji görnüşdäki üzülme nokatlary. **612.** $y=x$, $x\le0$ bolanda; $y=x^2$, $x>0$ bolanda, funksiýa üznüksiz. **613.** $y=0$, $x\le0$ bolanda we $y=x$, $x>0$ bolanda, funksiýa üznüksiz. **614.** $y=-(1+x)$, $x<0$ bolanda; $y=0$, $x=0$ bolanda we $y=1+x$, $x>0$ bolanda; $x=0$ birinji görnüşdäki üzülme nokady. **615.** Ýok. **616.** $a=1$. **617.** a) funksiýa üznüksiz; b) $x=-1$ birinji görnüşdäki üzülme nokady; ç) $x=-1$ birinji görnüşdäki üzülme nokady; d) $x=k$ ($k=0, \pm1, \pm2, ...$) tükeniksiz üzülme nokatlary; e) $x\ne k$ ($k=0, \pm1, \pm2, ...$) ikinji görnüşdäki üzülme nokatlary. **618.** $d=-x$, $-\infty<x<0$ bolanda; $d=0$, $0\le x\le 1$ bolanda; $d=x-1$, $1\le x\le 3/2$ bolanda; $d=2-x$, $3/2<x<2$ bolanda; $d=0$, $2\le x\le 3$ bolanda; $d=x-3$, $3<x<+\infty$ bolanda, funksiýa üznüksiz. **619.** $S=3y-y^2/2$, $0\le y\le 1$ bolanda; $S=1/2+2y$, $1<y\le 2$ bolanda; $S=5/2+y$, $2<y\le 3$ bolanda; $S=11/2$, $3<y<+\infty$ bolanda, funksiýa üznüksiz; $b=3-y$, $0\le y\le 1$ bolanda; $b=2$, $1<y\le 2$ bolanda; $b=1$, $2<y\le 3$ bolanda; $b=0$, $3<y<+\infty$ bolanda; $x=2$ we $x=3$ birinji görnüşdäki üzülme nokatlary. **621.** $x\ne0$ bolanda üznükli we $x=0$ bolanda üznüksiz. **623.** Argumentiň ähli otrisatel we položitel rasional bahalarynda üznükli. **624.** $f(0)=0{,}5$. **626.** a) 1,5; b) 2; ç) 0; d) e; e) 0; ä) 1; f) 0. **627.** a) hawa; b) ýok. **628.** a) ýok; b) ýok. **629.** Ýok. Mysal: $f(x)=1$, x rasional bolanda, $f(x)=-1$, x irrasional bolanda. **630.** a) $f(g(x))$ üznüksiz, $g(f(x))$ $x=0$ nokatda üznükli; b) $f(g(x))$ üznükli, $x=-1$, $x=0$ we $x=1$ bolanda; $g(f(x))=0$ üznüksiz; ç) $f(g(x))$ we $g(f(x))$ üznüksiz. **631.** $f(\varphi(x))\equiv x$. **645.** $x=(-dy+b)/(cy-a)$; $a+d=0$. **646.** $x=y-k$, $2k\le y<2k+1$ ($k=0, \pm1, \pm2, ...$) bolanda. **650.** $f(f(x))\equiv x$. **653.** $x=-\sqrt{y}$ $(0\le y<+\infty)$; $x=\sqrt{y}$ $(0\le y<+\infty)$. **654.** $x=1-\sqrt{1-y}$ $(-\infty<y\le 1)$; $x=1+\sqrt{1-y}$ $(-\infty<y\le 1)$. **655.** $x=\left(1-\sqrt{1-y^2}\right)\big/y$ $(-1\le y\le 1)$; $x=\left(1+\sqrt{1-y^2}\right)\big/y$ $(0<|y|\le 1)$.

656. $x=(-1)^k\arcsin y+k\pi$ $(k=0, \pm1, \pm2, ...)$ $(-1\le y\le 1)$. **657.** $x=2k\pi\pm\arccos y$ $(k=0, \pm1, \pm2, ...)$ $(-1\le y\le 1)$. **658.** $x=\operatorname{arctg} y+k\pi$ $(k=0, \pm1, \pm2, ...)$ $(-\infty<y<+\infty)$. **662.** $\varepsilon=0$, $xy<1$ bolanda; $\varepsilon=\operatorname{sgn}x$, $xy>1$ bolanda. **665.** a) $y=-\pi/2$, $-1\le x\le 0$ bolanda; $y=2\arcsin x-\pi/2$, $0\le x\le 1$ bolanda; b) $y=-(\pi+4\arcsin x)$, $-1\le x\le -1/\sqrt{2}$ bolanda; $y=0$, $-1/\sqrt{2}<x<1/\sqrt{2}$ bolanda; $y=\pi-4\arcsin x$, $1/\sqrt{2}\le x\le 1$ bolanda. **666.** $y=\pi/2-x$, $(-\pi/2<x<\pi/2)$. **667.** $y=\sqrt{x^2-1}$ $(1\le x<+\infty)$; $y=-\sqrt{x^2-1}$ $(1\le x<+\infty)$. **668.** t-iň $\varphi(t)=x$ bolýan ähli bahalary üçin, (bu ýerde x ululyk $\varphi(t)$ funksiýanyň erkin bahasydyr), $\psi(t)$ funksiýa bolsa şol bir bahany almalydyr. **669.** $\chi(\tau)$ funksiýanyň $\alpha<\tau<\beta$ üçin bahalar köplügi (a, b) interwal bolmaly. **670.** x-iň $\varphi(x)=u$ bolýan ähli bahalary üçin, (bu ýerde u san (A, B) interwalyň erkin bahasydyr), $\psi(x)$ funksiýa bolsa şol bir bahany almalydyr. **671.** $|\delta|\le\varepsilon/20$ *sm*. a) 0,5 *mm*; b) 0,005 *mm*; ç) 0,00005 *mm*. **672.** a) $\delta<1/4$; b) $\delta<2{,}5\cdot 10^{-4}$; ç) $\delta<(5/2)\cdot 10^{-7}$; d) $\delta<\varepsilon^3/4$ $(\varepsilon\le 1)$. **679.** a) hawa; b) ýok. **680.** Deňölçegli üznüksiz. **681.** Deňölçegli üznüksiz däl. **682.** Deňölçegli üznüksiz. **683.** Deňölçegli üznüksiz däl. **684.** Deňölçegli üznüksiz. **685.** Deňölçegli üznüksiz. **686.** Deňölçegli üznüksiz däl. **689.** a) $\delta=\varepsilon/5$; b) $\delta=\varepsilon/8$; ç) $\delta=0{,}01\varepsilon$; d) $\delta=\varepsilon^2$ $(\varepsilon\le 1)$; e) $\delta=\varepsilon/3$; ä) $\delta=\min(\varepsilon/3, \varepsilon^2/(3+\varepsilon))$. **690.** $n\ge 1\,800\,000$. **696.** a) $\omega_f(\delta)\le 3\delta$; b) $\omega_f(\delta)\le\sqrt{\delta}$; $\omega_f(\delta)\le\delta/\sqrt{2a}$; ç) $\omega_f(\delta)\le\delta\sqrt{2}$. **706.** $f(x)=\cos ax$ ýa-da $f(x)=\operatorname{ch}ax$. **707.** $f(x)=\cos ax$; $g(x)=\pm\sin ax$ $(a=\text{const})$.

IV. §1. Funksiýanyň önümi düşünjesi

1. $\Delta x=999$; $\Delta y=3$. **2.** $\Delta x=-0{,}009$; $\Delta y=990\,000$. **3.** a) $\Delta y=a\Delta x$; b) $\Delta y=(2ax+b)\Delta x+a(\Delta x)^2$; ç) $\Delta y=a^x(a^{\Delta x}-1)$. **5.** a) 5; b) 4,1; ç) 4,01; d) $4+\Delta x$; 4. **6.** $3+3h+h^2$; a) 3,31; b) 3,0301; ç) 3,003001; 3. **7.** a) $\vartheta_{ort}=215$ *m/s*; b) $\vartheta_{ort}=210{,}5$ *m/s*; ç) $\vartheta_{ort}=210{,}05$ *m/s*; 210 *m/s*. **8.** a) $2x$; b) $3x^2$; ç) $-1/x^2$ $(x\ne 0)$; d) $1/2\sqrt{x}$ $(x>0)$; e) $1/3\sqrt[3]{x^2}$ $(x\ne 0)$; ä) $1/\cos^2 x$ $(x\ne(2k-1)\pi/2,\ k=0, \pm1, ...)$; f) $-1/\sin^2 x$ $(x\ne k\pi,\ k=0, \pm1, ...)$; g) $1/\sqrt{1-x^2}$ $(|x|<1)$; h) $-1/\sqrt{1-x^2}$ $(|x|<1)$; i) $1/(1+x^2)$. **9.** -8; 0; 0. **10.** 4. **11.** $1+\pi/4$. **12.** $f'(a)$. **14.** $y'=1-2x$; 1, 0, -1, 21. **15.** $y'=x^2+x-2$; a) -2; 1; b) -1; 0; ç) -4; 3. **16.** $10a^3x-5x^4$. **17.** $a/(a+b)$. **18.** $2x-(a+b)$. **19.** $2(x+2)(x+3)^2(3x^2+11x+9)$. **20.** $x\sin 2a+\cos 2a$. **21.** $mn[x^{m-1}+x^{n-1}+(m+n)x^{m+n-1}]$. **22.** $-(1-x)^2(1-x^2)(1-x^3)^2(1+6x+15x^2+14x^3)$. **23.** $-20(17+12x)(5+2x)^9\times(3-4x)^{19}$. **24.** $-\left(\dfrac{1}{x^2}+\dfrac{4}{x^3}+\dfrac{9}{x^4}\right)$ $(x\ne 0)$. **26.** $\dfrac{2(1+x^2)}{(1-x^2)^2}$ $(|x|\ne 1)$. **27.** $\dfrac{2(1-2x)}{(1-x+x^2)^2}$. **28.** $\dfrac{1-x+4x^2}{(1-x)^3(1+x)^4}$ $(|x|\ne 1)$. **29.** $\dfrac{12-6x-6x^2+2x^3+5x^4-3x^5}{(1-x)^3}$ $(x\ne 1)$. **30.** $-\dfrac{(1-x)^{p-1}}{(1+x)^{q+1}}\times\left[(p+q)+(p-q)x\right]$ $(x\ne -1)$. **31.** $\dfrac{x^{p-1}(1-x)^{q-1}}{(1+x)^2}\left[p-(q+1)x-(p+q-1)x^2\right]$ $(x\ne -1)$. **32.** $1+\dfrac{1}{2\sqrt{x}}+\dfrac{1}{3\sqrt[3]{x^2}}$ $(x>0)$. **33.** $-\dfrac{1}{x^2}-\dfrac{1}{2x\sqrt{x}}-\dfrac{1}{3x\sqrt[3]{x}}$ $(x>0)$. **34.** $\dfrac{2}{3\sqrt[3]{x}}+\dfrac{1}{x\sqrt{x}}$ $(x>0)$. **35.** $\dfrac{1+2x^2}{\sqrt{1+x^2}}$. **36.** $\dfrac{6+3x+8x^2+4x^3+2x^4+3x^5}{\sqrt{2+x^2}\sqrt[3]{(3+x^3)^2}}$ $(x\ne\sqrt[3]{-3})$. **37.** $\dfrac{1}{(n+m)}\times$

$\times\dfrac{(n-m)-(n+m)x}{\sqrt[n+m]{(1-x)^n(1+x)^m}}$. **38.** $\dfrac{a^2}{(a^2-x^2)^{3/2}}$ $(|x|<|a|)$. **39.** $\dfrac{2x^2}{1-x^6}\sqrt[3]{\dfrac{1+x^3}{1-x^3}}$ $(|x|\neq-1)$.

40. $-\dfrac{1}{(1+x^2)^{3/2}}$. **41.** $\dfrac{1+2\sqrt{x}+4\sqrt{x}\sqrt{x+\sqrt{x}}}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}$ $(x>0)$. **42.** $\dfrac{1}{27\sqrt[3]{x^2(1+\sqrt[3]{x})^2}}\times$

$\times\dfrac{1}{\sqrt[3]{(1+\sqrt[3]{1+\sqrt[3]{x}})^2}}$ $(x\neq 0,\ x\neq-1,\ x\neq-8)$. **43.** $-2\cos x\,(1+2\sin x)$. **44.** $x^2\sin x$.

45. $-\sin 2x\cdot\cos(\cos 2x)$. **46.** $n\sin^{n-1}x\cdot\cos(n+1)x$. **47.** $\cos x\cdot\cos(\sin x)\cdot\cos[\sin(\sin x)]$.

48. $\dfrac{2\sin x(\cos x\sin x^2-x\sin x\cos x^2)}{\sin^2 x^2}$ $(x^2\neq k\pi;\ k=1,\ 2,\ \ldots)$. **49.** $-\dfrac{1+\cos^2 x}{2\sin^3 x}$ $(x\neq k\pi;$ $k=0,\ \pm1,\ \pm2,\ \ldots)$. **50.** $\dfrac{n\sin x}{\cos^{n+1}x}$ $\left(x\neq\dfrac{2k-1}{2}\pi,\ k\right.$ – bitin san). **51.** $\dfrac{x^2}{(\cos x+x\sin x)^2}$.

52. $\dfrac{2}{\sin^2 x}$; $(x\neq k\pi;\ k=0,\ \pm1,\ \pm2,\ \ldots)$. **53.** $1+\mathrm{tg}^6x$ $\left(x\neq(2k+1)\dfrac{\pi}{2};\ k=0,\ \pm1,\ \ldots\right)$.

54. $\dfrac{8}{3\sin^4 x\sqrt[3]{\mathrm{ctg}x}}$ $(x\neq k\pi;\ k$ – bitin san). **55.** $\dfrac{-16\cos(2x/a)}{a\sin^3(2x/a)}$ $\left(x\neq\dfrac{k\pi a}{2},\ k\right.$ – bitin san).

56. $-3\mathrm{tg}^2x\cdot\sec^2x\cdot\sin(2\mathrm{tg}^3x)\cdot\cos[\cos^2(\mathrm{tg}^3x)]$ $\left(x\neq\dfrac{\pi}{3}+k\pi,\ k\right.$ – bitin san). **57.** $-2xe^{-x^2}$.

58. $-\dfrac{1}{x^2}2^{\mathrm{tg}(1/x)}\sec^2\dfrac{1}{x}\ln 2$. **59.** x^2e^x. **60.** $x^2e^{-x}\sin x$. **61.** $\dfrac{e^x(\sin x-\cos x)}{2\sin^2(x/2)}$ $(x\neq 2k\pi;\ k$ – bitin san). **62.** $-\dfrac{1+\ln^2 3}{3^x}\sin x$. **63.** $\sqrt{a^2+b^2}\,e^{ax}\sin bx$. **64.** $e^x[1+e^{e^x}(1+e^{e^{e^x}})]$. **65.** $y\left(\ln\dfrac{a}{b}-\right.$ $\left.-\dfrac{a-b}{x}\right)$ $(x>0)$. **66.** $a^ax^{a^a-1}+ax^{a-1}a^{x^a}\ln a+a^xa^{a^x}\ln^2a$. **67.** $\dfrac{6}{x}\lg e\lg^2x^2$ $(x\neq0)$. **68.** $\dfrac{1}{x\ln x\ln(\ln x)}$ $(x>e)$. **69.** $\dfrac{6}{x\ln x\ln(\ln^3x)}$ $(x>e)$. **70.** $\dfrac{1}{(1+x)^2(1+x^2)}$ $(x>-1)$. **71.** $\dfrac{x}{x^4-1}$ $(|x|>1)$.

72. $\dfrac{1}{x(1+x^4)^2}$ $(x\neq0)$. **73.** $\dfrac{1}{3x^2-2}$ $(|x|>\sqrt{2/3})$. **74.** $\dfrac{2}{(1-x^2)(1-kx^2)}$ $(|x|<1)$. **75.** $\dfrac{1}{2}\times$ $\times\dfrac{1}{(1+\sqrt{x+1})}$ $(x>-1)$. **76.** $\dfrac{1}{\sqrt{x^2+1}}$. **77.** $\ln(x+\sqrt{x^2+1})$. **78.** $\ln^2(x+\sqrt{x^2+1})$.

79. $\sqrt{x^2+a^2}$. **80.** $\dfrac{1}{a-bx^2}$ $(|x|<\sqrt{a/b})$. **81.** $-\dfrac{8}{x^5\sqrt{1-x^2}}$ $(0<x<1)$. **82.** $\dfrac{1}{\sin x}$ $(0<x-$ $-2k\pi<\pi,\ k$ – bitin san) **83.** $\dfrac{1}{\cos x}$ $\left(|x-2k\pi|<\dfrac{\pi}{2},\ k\right.$ – bitin san). **84.** $-\mathrm{ctg}^3x$ $(0<x-$ $-2k\pi<\pi,\ k$ – bitin san). **85.** $-\dfrac{1}{\cos x}$ $\left(x\neq\dfrac{2k-1}{2}\pi,\ k\right.$ – bitin san). **86.** $\dfrac{\cos^2x}{\sin^3x}$ $(0<x-2k\pi<$ $<\pi,\ k$ – bitin san). **87.** $\dfrac{\sqrt{b^2-a^2}}{a+b\cos x}$. **88.** $-\dfrac{\ln^3x}{x^2}$ $(x>0)$. **89.** $\dfrac{1}{x^5}\ln x$ $(x>0)$. **90.** $\dfrac{2x}{1+\sqrt[3]{1+x^2}}$.

91. $-\dfrac{1}{(1+x\ln(1/x))}\cdot\dfrac{1+x+1/x+\ln(1/x)}{1+x\ln(1/x+\ln(1/x))}$. **92.** $2\sin(\ln x)$ $(x>0)$. **93.** $\sin x\cdot\ln\operatorname{tg}x$ $(0<$ $<x-2k\pi<\dfrac{\pi}{2}$, k – bitin san). **94.** $\dfrac{1}{\sqrt{4-x^2}}$ $(|x|<2)$. **95.** $\dfrac{1}{\sqrt{1+2x-x^2}}$ $(|x-1|<\sqrt{2})$. **96.** $\dfrac{2ax}{x^4+a^2}$ $(a\neq 0)$. **97.** $\dfrac{1}{x^2+2}$ $(x\neq 0)$. **98.** $\dfrac{\sqrt{x}}{2(1+x)}$ $(x\geq 0)$. **99.** $-\dfrac{x}{\sqrt{1-x^2}}\arccos x$ $(|x|<1)$. **100.** $\arcsin\sqrt{\dfrac{x}{1+x}}$ $(x\geq 0)$. **101.** $\dfrac{1}{|x|\sqrt{x^2-1}}$ $(|x|>1)$. **102.** $\operatorname{sgn}(\cos x)$ $\left(x\neq\dfrac{2k-1}{2}\pi\right.$, k – bitin san). **103.** $\dfrac{2\operatorname{sgn}(\sin x)\cdot\cos x}{\sqrt{1+\cos^2 x}}$ $(x\neq k\pi$, k – bitin san). **104.** $\dfrac{\sin x+\cos x}{\sqrt{\sin 2x}}$ $(0<x-$ $-k\pi<\dfrac{\pi}{2}$, k – bitin san). **105.** $\dfrac{\operatorname{sgn}x}{\sqrt{1-x^2}}$ $(0<|x|<1)$. **106.** $\dfrac{1}{1+x^2}$ $(x\neq 1)$. **107.** 1 $\left(x\neq\dfrac{\pi}{4}+k\pi\right.$, k – bitin san). **108.** $\dfrac{1}{a+b\cos x}$. **109.** $-\dfrac{2\operatorname{sgn}x}{1+x^2}$ $(x\neq 0)$. **110.** $\dfrac{4x}{\sqrt{1-x^4}\arccos^3(x^2)}$ $(|x|<1)$. **111.** $\dfrac{1+x^4}{1+x^6}$. **112.** $-2\cos x\cdot\operatorname{arctg}(\sin x)$. **113.** $\dfrac{1}{2x\sqrt{x-1}\arccos(1/\sqrt{x})}$ $(x>1)$. **114.** $\dfrac{1}{(x+a)}\times$ $\times\dfrac{a^2+b^2}{(x^2+b^2)}$ $(x>-a)$. **115.** $\sqrt{a^2-x^2}$. **116.** $\dfrac{1}{x^3+1}$ $(x\neq -1)$. **117.** $\dfrac{1}{x^4+1}$ $(|x|\neq 1)$. **118.** $(\arcsin x)^2$ $(|x|<1)$. **119.** $-\dfrac{\arccos x}{x^2}$ $(0<|x|<1)$. **120.** $\dfrac{x\ln x}{(x^2-1)^{3/2}}$ $(x>1)$. **121.** $\dfrac{x\arcsin x}{(1-x^2)^{3/2}}$ $(|x|<1)$. **122.** $\dfrac{x^3}{x^6+1}$ $\left(|x|\neq\dfrac{1}{\sqrt{2}}\right)$. **123.** $\dfrac{12x^5}{(1+x^{12})^2}$. **124.** $-\dfrac{1}{(1-x)\sqrt[3]{x}}$ $(x<1)$. **125.** $\dfrac{1}{2\sqrt{1-x^2}}$ $(|x|<1)$. **126.** $\dfrac{1}{\sqrt{ax-x^2}}$ $(0<x<a)$. **127.** $\dfrac{x^2}{\sqrt{1-2x-x^2}}$ $(|x+1|<\sqrt{2})$. **128.** $\dfrac{1}{\sqrt[4]{1+x^4}}$. **129.** $\dfrac{\sin 2x}{\sin^4 x+\cos^4 x}$ $\left(x\neq\dfrac{2k-1}{2}\pi\right.$, k – bitin san). **130.** $\dfrac{\sqrt{1-x^2}}{x}-\dfrac{x}{\sqrt{1-x^2}}\ln\sqrt{\dfrac{1-x}{1+x}}$ $(|x|<1)$. **131.** $\dfrac{x^2}{1+x^2}\operatorname{arctg}x$. **132.** $\dfrac{e^x}{\sqrt{1+e^{2x}}}$. **133.** $\dfrac{1}{2(1+x^2)}$. **134.** $\dfrac{\sin a\operatorname{sgn}(\cos x-\cos a)}{1-\cos a\cos x}$ $(\cos x\neq\cos a)$. **135.** $\dfrac{1}{(x^4-1)\sqrt{x^2+2}}$ $(0<|x|<1)$. **136.** $\dfrac{\sqrt{1+x^4}}{1-x^4}$ $(|x|\neq 1)$. **137.** $\dfrac{4}{(1+x^2)^2\sqrt{1-x^2}}$ $(|x|<1)$. **138.** $\dfrac{2x(\cos x^2+\sin x^2)}{\sqrt{\sin(2x^2)}}$ $(0<|x|<\sqrt{(k+1/2)\pi}$, $k=0, 1, ...)$. **139.** $2x[\operatorname{sgn}(\cos x^2)+$ $+\operatorname{sgn}(\sin x^2)]$ $\left(|x|\neq\dfrac{k\pi}{2}\right.$, $k=0, 1, 2, ...)$. **140.** $\dfrac{2m}{\sqrt{1-x^2}}e^{m(\arcsin x)}\cos m(\arcsin x)$ $(|x|<1)$. **141.** $\dfrac{e^x-1}{e^{2x}+1}$. **142.** $\dfrac{x^3}{6\sqrt{1+\sqrt[3]{1+\sqrt[4]{1+x^4}}}\cdot\sqrt[3]{(1+\sqrt[4]{1+x^4})^2}\cdot\sqrt[4]{(1+x^4)^3}}$. **143.** $\dfrac{1}{x^3}\times$

$\times \dfrac{1}{\cos(1/x^2)} \cdot \dfrac{1}{(\sin(1/x^2)+\cos(1/x^2))}$. **144.** $\dfrac{2^{1+\sqrt[3]{x}}\ln 2 \cdot \sin\left(2^{\sqrt[3]{x}}\right)\cdot \ln\left(\sec 2^{\sqrt[3]{x}}\right)}{3\sqrt[3]{x^2}\cos^2\left(2^{\sqrt[3]{x}}\right)}$. **145.** $1+x^x \times$ $\times(1+\ln x)+x^x x^{x^x}(1/x+\ln x+\ln^2 x)$ $(x>0)$. **146.** $x^{a-1}x^{x^a}(1+a\ln x)+a^x x^{a^x}\left(\dfrac{1}{x}+\ln a \times\right.$ $\left.\times \ln x\right)+x^x a^{x^x}\ln a\left(1+\ln x\right)$ $(x>0)$. **147.** $x^{1/x-2}(1-\ln x)$ $(x>0)$. **148.** $(\sin x)^{1+\cos x}(\mathrm{ctg}^2 x -$ $-\ln\sin x)-(\cos x)^{1+\sin x}(\mathrm{tg}^2 x-\ln\cos x)$ $(0<x-2k\pi<\dfrac{\pi}{2}$, k – bitin san). **149.** $\dfrac{(\ln x)^{x-1}}{x^{\ln x+1}}[x -$ $-2\ln^2 x+x\ln x\cdot\ln\left(\ln x\right)]$ $(x>1)$. **150.** $y' = 2y\cdot\left\{\dfrac{\mathrm{arctg}x}{1+x^2}\ln\dfrac{\arcsin(\sin^2 x)}{\arccos(\cos^2 x)}+\mathrm{arctg}^2 x \times\right.$ $\left.\times\left[\dfrac{\sin x\cdot \mathrm{sgn}(\cos x)}{\arcsin(\sin^2 x)\sqrt{1+\sin^2 x}}-\dfrac{\cos x\cdot\mathrm{sgn}(\sin x)}{\arccos(\cos^2 x)\sqrt{1+\cos^2 x}}\right]\right\}$ $(|x|\neq\dfrac{k\pi}{2},\ k=0,\ \pm1,\ \ldots)$.
151. $-\dfrac{1}{x}(\log_x e)^2$ $(x>0,\ x\neq1)$. **152.** $\mathrm{th}^3 x$. **153.** $-\dfrac{2}{\mathrm{sh}^3 x}$ $(x>0)$. **154.** $\dfrac{1}{\mathrm{ch}2x}$. **155.** $\dfrac{\mathrm{sgn}(\mathrm{sh}x)}{\mathrm{ch}x}$ $(x\neq0)$. **156.** $\dfrac{a+b\,\mathrm{ch}x}{b+a\,\mathrm{ch}x}$. **157.** $-\dfrac{\sin 2x}{\sqrt{1+\cos^4 x}}$. **158.** $-\dfrac{2}{\sqrt{1-x^2}}\arccos x\cdot\ln(\arccos x)$ $(|x|<1)$.
159. $-\dfrac{x^{-1}}{\sqrt[4]{(1+x^4)^3}}$. **160.** $-\dfrac{2xe^{-x^2}\arcsin\left(e^{-x^2}\right)}{\left(1-e^{-2x^2}\right)^{3/2}}$ $(x\neq0)$. **161.** $\dfrac{4a^{2x}\ln a}{(1+a^{2x})^2}\mathrm{arctg}a^{-x}$ $(a>0)$.
162. a) $\mathrm{sgn}x$ $(x\neq0)$; b) $2|x|$; ç) $1/x$ $(x\neq0)$. **163.** a) $(x-1)(x+1)^2(5x-1)\mathrm{sgn}(x+1)$; b) $\dfrac{3}{2}\sin 2x \times$ $\times|\sin x|$; ç) $\dfrac{1}{x\sqrt{x^2-1}}$ $(|x|>1)$; d) $\pi[x]\sin 2\pi x$. **164.** $y'=-1$, $-\infty<x<1$; $y'=2x-3$, $1\le x\le 2$; $y'=1$, $2<x<+\infty$. **165.** $y'=2(x-a)(x-b)(2x-a-b)$, $x\in[a,\ b]$; $y'=0$, $x\notin[a,\ b]$. **166.** $y'=1$, $x<0$; $y'=\dfrac{1}{1+x}$, $0\le x<+\infty$. **167.** $y'=\dfrac{1}{1+x^2}$, $-1<x\le1$; $y'=1/2$, $|x|>1$. **168.** $y'=2xe^{-x^2}\times$ $\times(1-x^2)$, $|x|\le1$; $y'=0$, $|x|>1$. **169.** a) $\dfrac{1-x-x^2}{x(1-x^2)}$; b) $\dfrac{54-36x+4x^2+2x^3}{3x(1-x)(9-x^2)}$ $(x\neq0,\ x\neq1,$ $x\neq\pm3)$; ç) $\sum\limits_{i=1}^{n}\dfrac{a_i}{x-a_i}$; d) $\dfrac{n}{\sqrt{1+x^2}}$. **170.** a) $\dfrac{\varphi(x)\varphi'(x)+\psi(x)\psi'(x)}{\sqrt{\varphi^2(x)+\psi^2(x)}}$ $(\varphi^2(x)+\psi^2(x)\neq0)$; b) $\dfrac{\varphi'(x)\psi(x)-\varphi(x)\psi'(x)}{\varphi^2(x)+\psi^2(x)}$ $(\varphi^2(x)+\psi^2(x)\neq0)$; ç) $\sqrt[\varphi(x)]{\psi(x)}\cdot\left\{\dfrac{1}{\varphi(x)}\dfrac{\psi'(x)}{\psi(x)}-\dfrac{\varphi'(x)}{\varphi^2(x)}\ln\psi(x)\right\}$; d) $\dfrac{\psi'(x)}{\psi(x)}\dfrac{1}{\ln\varphi(x)}-\dfrac{\varphi'(x)}{\varphi(x)}\dfrac{\ln\psi(x)}{\ln^2\varphi(x)}$. **171.** a) $2xf'(x^2)$; b) $\sin 2x[f'(\sin^2 x)-f'(\cos^2 x)]$; ç) $e^{f(x)}\times$ $\times[e^x f'(e^x)+f'(x)f(e^x)]$; d) $f'(x)f'[f(x)]f'\{f[f(x)]\}$. **172.** 1000! **174.** $3x^2+15$. **175.** $6x^2$. **178.** a) $n>0$; b) $n>1$; ç) $n>2$. **179.** a) $n\ge m+1$; b) $1<n<m+1$. **180.** $\varphi(a)$. **181.** $f'_-(a)=$ $=-\varphi(a)$, $f'_+(a)=\varphi(a)$. **185.** a) $x=1$ bolanda differensirlenmeýär; b) $x=\dfrac{2k-1}{2}\pi$ (k – bitin san) bolanda differensirlenmeýär, ç) hemme ýerde differensirlenýär; d) $x=k\pi$ (k – bi-

tin san) bolanda differensirlenmeýär; e) $x=-1$ bolanda differensirlenmeýär. **186.** $x\neq 0$ bolanda $f_-'(x)=f_+'(x)=\operatorname{sgn}x$ we $f_-'(0)=-1$, $f_+'(0)=1$. **187.** x bitin däl bolanda $f_-'(x)=f_+'(x)=$ $=\pi[x]\cos\pi x$, $f_-'(k)=\pi(k-1)(-1)^k$, $f_+'(x)=\pi k(-1)^k$, k – bitin bolanda. **188.** $f_-'(x)=f_+'(x)=$ $=\left(\cos\frac{\pi}{x}+\frac{\pi}{x}\sin\frac{\pi}{x}\right)\operatorname{sgn}\left(\cos\frac{\pi}{x}\right)$, $x\neq\frac{2}{2k+1}$, k – bitin san bolanda; $f_-'\left(\frac{2}{2k+1}\right)=-(2k+$ $+1)\frac{\pi}{2}$, $f_+'\left(\frac{2}{2k+1}\right)=(2k+1)\frac{\pi}{2}$. **189.** $f_-'(x)=f_+'(x)=\frac{x\cos x^2}{\sqrt{\sin x^2}}$, $\sqrt{2k\pi}<|x|<\sqrt{\pi}\times$ $\times\sqrt{(2k+1)}$ $(k=0,\ 1,\ 2,\ ...)$ bolanda; $f_-'(0)=-1$, $f_+'(0)=1$; $f_\mp'\left(\sqrt{(2k+1)\pi}\right)=\mp\infty$, $f_\pm'(\sqrt{2k\pi})=\pm\infty$ $(k=1,\ 2,\ ...)$. **190.** $f_-'(x)=f_+'(x)=\frac{1+(1+1/x)e^{1/x}}{(1+e^{1/x})^2}$, $x\neq 0$ bolanda; $f_-'(0)=1$, $f_+'(0)=0$. **191.** $f_-'(x)=f_+'(x)=\frac{xe^{-x^2}}{\sqrt{1-e^{-x^2}}}$, $x\neq 0$ bolanda; $f_-'(0)=-1$, $f_+'(0)=1$. **192.** $f_-'(x)=f_+'(x)=\frac{\varepsilon}{x}$, $\varepsilon=-1$, $0<|x|<1$ bolanda we $\varepsilon=1$, $1<|x|<+\infty$ bolanda; $f_-'(\mp 1)=$ $=-1$, $f_+'(\mp 1)=1$. **193.** $f_-'(x)=f_+'(x)=\frac{2\operatorname{sgn}(1-x^2)}{1+x^2}$, $x\neq\mp 1$ bolanda; $f_-'(\mp 1)=\mp 1$, $f_+'(\mp 1)=\mp 1$. **194.** $f_-'(x)=f_+'(x)=\operatorname{arctg}\frac{1}{x-2}-\frac{x-2}{(x-2)^2+1}$, $x\neq 2$ bolanda; $f_\mp'(2)=$ $=\mp\pi/2$. **196.** a) $f_-'(0)=-1/2$, $f_+'(0)=1/2$; b) $f_-'(1)=f_+'(1)=1/2$; ç) $f_-'(0)=f_+'(0)=0$. **197.** $a=2x_0$; $b=-x_0^2$. **198.** $a=f_-'(x_0)$; $b=f(x_0)-x_0f_-'(x_0)$. **199.** $A=\frac{k_1+k_2}{(b-a)^2}$, $c=\frac{ak_1+bk_2}{k_1+k_2}$. **200.** $a=\frac{3m^2}{2c}$, $b=-\frac{m^2}{2c^3}$. **201.** a) bolup biler; b) bolup bilmez. **202.** a) bolup bilmez, b) bolup bilmez. **203.** a), b), ç) $F(x)$ funksiýanyň $F'(x)$ önümi bolup hem biler, bolman hem biler. **204.** $x=k\pi$ $(k=0,\ \pm1,\ \pm2,\ ...)$ **205.** a) bolup bilmez; b) bolup biler. **206.** a) hökman däl; b) hökman. **207.** Hökman däl. **208.** Ýerine ýetmez. **209.** Ýerine ýetmez. **210.** Umuman, bolmaz. **211.** $P_n=\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$; $Q_n=\frac{1}{(1-x)^3}\cdot(1+x-$ $-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2})$. **212.** $S_n=\frac{\sin\frac{nx}{2}\sin\frac{n+1}{2}x}{\sin(x/2)}$; $T_n=\frac{1}{2}\times$ $\times\frac{n\sin\frac{x}{2}\sin\frac{2n+1}{2}x-\sin^2\frac{nx}{2}}{2\sin^2(x/2)}$. **213.** $S_n=\frac{n\operatorname{sh}\frac{x}{2}\operatorname{sh}(n+1/2)x-\operatorname{sh}^2\frac{nx}{2}}{2\operatorname{sh}^2(x/2)}$. **214.** $S_n=\frac{1}{2^n}\times$ $\times\operatorname{ctg}\frac{x}{2^n}-\operatorname{ctg}x$. **217.** $40\pi\ sm^2/s$. **218.** $25\ m^2/s$; $0{,}4\ m/s$. **219.** $50\ km/sag$. **220.** $S(x)=\frac{x^2}{2}$, $0\leq x\leq 2$ bolanda; $S(x)=x^2-2x+2$, $x>2$ bolanda; $S'(x)=x$, $0\leq x\leq 2$ bolanda; $S'(x)=$ $=2x-2$, $x>2$ bolanda. **221.** $S(x)=\frac{|x|}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{|x|}{a}$; $S'(x)=\sqrt{a^2-x^2}\operatorname{sgn}x$

$(0<|x|\le a)$. **222.** $y_x'=\dfrac{1}{3(y^2+1)}$. **223.** $y_x'=\dfrac{1}{1-\varepsilon\cos y}$. **224.** a) $-\infty<y<+\infty$; $x_y'=\dfrac{x}{x+1}$; b) $-\infty<y<+\infty$; $x_y'=\dfrac{1}{1-x+y}$; ç) $-\infty<y<+\infty$; $x_y'=\dfrac{1}{\sqrt{1+y^2}}$; d) $-1<y<1$; $x_y'= =\dfrac{1}{1-y^2}$. **225.** a) $x_1=-\sqrt{1+\sqrt{1-y}}$ $(-\infty<y\le 1)$; $x_2=-\sqrt{1-\sqrt{1-y}}$ $(0\le y\le 1)$; $x_3=\sqrt{1-\sqrt{1-y}}$ $(0\le y\le 1)$; $x_4=\sqrt{1+\sqrt{1-y}}$ $(-\infty<y\le 1)$; $x_i'=\dfrac{1}{4x(1-x^2)}$ $(i=1,$ $2, 3, 4)$. b) $x_1=-\sqrt{\dfrac{y}{1-y}}$ $(0\le y<1)$; $x_2=\sqrt{\dfrac{y}{1-y}}$ $(0\le y<1)$; $x_i'=\dfrac{x^3}{2y^2}$ $(i=1, 2)$ ç) $x_1=-\ln(1+\sqrt{1-y})$ $(-\infty<y\le 1)$; $x_2=\ln\dfrac{1+\sqrt{1-y}}{y}$ $(0<y\le 1)$; $x_i'=-\dfrac{1}{2(e^{-x}-e^{-2x})}$ $(i=1, 2)$. **226.** $y_x'=-\dfrac{3}{2}(1+t)$; -3; $-\dfrac{3}{2}$ we $-\dfrac{9}{2}$; $(-4; 4)$. **227.** $\sqrt[6]{\dfrac{(1-\sqrt{t})^4}{t(1-\sqrt[3]{t})^3}}$ $(t>0,$ $t\ne 1)$. **228.** $y_x'=-1$ $(0<x<1)$. **229.** $y_x'=-(b/a)\mathrm{ctg}\,t$ $(0<|t|<\pi)$. **230.** $y_x'=(b/a)\mathrm{cth}\,t$ $(|t|>0)$. **231.** $y_x'=-\mathrm{tg}\,t$ $\left(t\ne\dfrac{2k+1}{2}\pi,\ k\right.$ – bitin san). **232.** $y_x'=\mathrm{ctg}(t/2)$ $(t\ne 2k\pi,\ k$ – bitin san). **233.** $y_x'=\mathrm{tg}\,t\cdot\mathrm{tg}(t+\pi/4)$ $(t\ne\pi/4+k\pi,\ t\ne\pi/2+k\pi)$ **234.** $y_x'=\mathrm{sgn}\,t$ $(0<|t|<+\infty)$. **236.** $y'=\dfrac{1-x-y}{x-y}$; $\dfrac{5}{2}$; $-\dfrac{1}{2}$. **237.** $\dfrac{p}{y}$. **238.** $-\dfrac{b^2x}{a^2y}$. **239.** $-\sqrt{y/x}$. **240.** $-\sqrt[3]{y/x}$. **241.** $\dfrac{x+y}{x-y}$. **242.** a) $\mathrm{tg}(\varphi+\mathrm{arctg}\varphi)$; b) $-\mathrm{ctg}(3\varphi/2)$ $(\varphi\ne 0,\ \varphi\ne\pm 2\pi/3)$; ç) $\mathrm{tg}(\varphi+\mathrm{arctg}(1/m))$.

§2. Funksiýanyň önüminiň geometrik manysy

243. a) $y=\sqrt[3]{4}(x+1)$; $y=-(\sqrt[3]{2}/2)(x+1)$; b) $y=3$, $x=2$; ç) $x=3$, $y=0$. **244.** a) $(1/2, 9/4)$; b) $(0, 2)$. **246.** $|x|<\pi/3$ we $2\pi/3\le|x|\le\pi$. **247.** $\max|y_1'-y'|=10\pi\approx 31{,}4$. **248.** $\pi/4$. **249.** $\pi/2$; $\mathrm{arctg}(3/4)\approx\mathrm{arc}37^\circ$. **250.** $\mathrm{arctg}2\sqrt{2}\approx\mathrm{arc}70^\circ 30'$. **251.** $n>57{,}3$. **254.** a) $2\mathrm{arctg}(1/|a|)$; b) $\pi/2$. **256.** $|x/n|$. **259.** $y_0^2/|a|$. **261.** $b^2-4ac=0$. **262.** $(p/3)^3+(q/2)^2=0$. **263.** $a=1/2e$. **267.** a) $3x-2y=0$, $2x+3y=0$; b) $3x-y-1=0$, $x+3y-7=0$. **268.** a) $y=x$, $y=-x$; b) $3x-y-4=0$, $x+3y-3=0$; ç) $y=-x$, $y=x$. **269.** $y-2a=(x-at_0)\mathrm{ctg}(t_0/2)$. Egri çyzyga galtaşýan galtaşma nokady bilen togalanýan tegelegiň galtaşýan nokadyny birleşdirýän kesime perpendikulýardyr. **271.** $3x+5y-50=0$, $5x-3y-10{,}8=0$. **272.** $x+2y-3=0$, $2x-y-1=0$.

§3. Funksiýanyň differensialy

273. $\Delta f(1)=\Delta x+3(\Delta x)^2+(\Delta x)^3$; $df(1)=\Delta x$. a) 5, 1; b) 0,131, 0,1; ç) 0,010301, 0,01. **274.** $\Delta x=20\Delta t+5(\Delta t)^2$; $dx=20\Delta t$; a) 25 m, 20 m; b) 2,05 m, 2 m; ç) 0,020005 m, 0,02 m. **275.** $-dx/x^2$ $(x\ne 0)$. **276.** $\dfrac{dx}{a^2+x^2}$. **277.** $\dfrac{dx}{x^2-a^2}$ $(|x|\ne|a|)$. **278.** $\dfrac{dx}{\sqrt{x^2+a}}$. **279.** $\dfrac{\mathrm{sgn}\,a}{\sqrt{a^2-x^2}}dx$

$(|x|<|a|)$. **280.** a) $(1+x)e^x dx$; b) $x\sin x dx$; ç) $-3dx/x^4$ $(x\neq 0)$; d) $\dfrac{2-\ln x}{2x\sqrt{x}}dx$ $(x>0)$; e) $\dfrac{xdx}{\sqrt{a^2+x^2}}$; ä) $\dfrac{dx}{(1-x^2)^{3/2}}$ $(|x|<1)$; f) $-\dfrac{2xdx}{1-x^2}$ $(|x|<1)$; g) $\dfrac{dx}{x\sqrt{x^2-1}}$ $(|x|>1)$ h) $\dfrac{dx}{\cos^3 x}$ $(x\neq \dfrac{\pi}{2}+k\pi,\ k$ – bitin san). **281.** $\upsilon\omega du+u\omega d\upsilon+u\upsilon d\omega$. **282.** $\dfrac{\upsilon du-2ud\upsilon}{\upsilon^3}$ $(\upsilon\neq 0)$.

283. $-\dfrac{udu+\upsilon d\upsilon}{(u^2+\upsilon^2)^{3/2}}$ $(u^2+\upsilon^2>0)$. **284.** $\dfrac{\upsilon du-ud\upsilon}{u^2+\upsilon^2}$ $(u^2+\upsilon^2>0)$. **285.** $(udu+\upsilon d\upsilon)/(u^2+\upsilon^2)$ $(u^2+\upsilon^2>0)$. **286.** a) $1-4x^3-3x^6$; b) $(1/2x^2)(\cos x-\sin x/x)$; ç) $-\mathrm{ctg}x$ $(x\neq k\pi,\ k$ – bitin san); d) $-\mathrm{tg}^2x$ $(x\neq\pi/2+k\pi,\ k$ – bitin san); e) -1 $(|x|<1)$. **287.** a) 104,7 sm^2 ulalar; b) 43,6 sm^2 kiçeler. **288.** 2,23 sm ulaltmaly. **289.** 1,007 (tablisa boýunça: 1,0066). **290.** 0,4849 (tablisa boýunça: 0,4848). **291.** –0,8747 (tablisa boýunça: –0,8746). **292.** 0,8104=arc46º26′ (tablisa boýunça: arc46º24′) **293.** 1,043 (tablisa boýunça: 1,041). **294.** a) 2,25 (tablisa boýunça: 2,24); b) 5,833 (tablisa boýunça: 5,831); ç) 10,9546 (tablisa boýunça: 10,9545). **296.** a) 2,083 (tablisa boýunça: 2,080); b) 2,9907 (tablisa boýunça: 2,9907); ç) 1,938 (tablisa boýunça: 1,931); d) 1,9954 (tablisa boýunça: 1,9953). **297.** 0,24 m^2; 4,2%. **298.** $\delta_R\leq 0{,}33\%$. **299.** a) $\delta_g=\delta_i$; b) $\delta_g=2\delta_T$. **300.** $0{,}43\delta$.

§4. Ýokary tertipli önümler we differensiallar

302. $\dfrac{x(3+2x^2)}{(1+x^2)^{3/2}}$. **303.** $\dfrac{3x}{(1-x^2)^{5/2}}$ $(|x|<1)$. **304.** $2e^{-x^2}(2x^2-1)$. **305.** $\dfrac{2\sin x}{\cos^3 x}$ $(x\neq\dfrac{\pi}{2}\times$ $\times(2k+1),\ k=0,\ \pm1,\ \ldots)$. **306.** $\dfrac{2x}{1+x^2}+2\mathrm{arctg}x$. **307.** $\dfrac{3x}{(1-x^2)^2}+\dfrac{(1+2x^2)\arcsin x}{(1-x^2)^{5/2}}$ $(|x|<1)$. **308.** $\dfrac{1}{x}$ $(x>0)$. **309.** $\dfrac{f(x)f''(x)-f'^2(x)}{f^2(x)}$ $(f(x)>0)$. **310.** $-(2/x)\sin(\ln x)$ $(x>0)$. **311.** $y(0)=1,\ y'(0)=1,\ y''(0)=0$. **312.** $2(uu''+u'^2)$. **313.** $\dfrac{uu''-u'^2}{u^2}-\dfrac{\upsilon\upsilon''-\upsilon'^2}{\upsilon^2}$ $(u\upsilon>0)$.

314. $\dfrac{(u^2+\upsilon^2)(uu''+\upsilon\upsilon'')+(u'\upsilon-u\upsilon')^2}{(u^2+\upsilon^2)^{3/2}}$ $(u^2+\upsilon^2>0)$. **315.** $y''=u^{\upsilon}\left[\left(\upsilon\dfrac{u'}{u}+\upsilon'\ln u\right)^2+\right.$ $\left.+\upsilon\dfrac{uu''-u'^2}{u^2}+\dfrac{2u'\upsilon'}{u}+\upsilon''\ln u\right]$. **316.** $y''=4x^2f''(x^2)+2f'(x^2)$; $y'''=8x^3f'''(x^2)+12xf''(x^2)$.

317. $y''=\dfrac{1}{x^4}f''\left(\dfrac{1}{x}\right)+\dfrac{2}{x^3}f'\left(\dfrac{1}{x}\right)$; $y'''=-\dfrac{1}{x^6}f'''\left(\dfrac{1}{x}\right)-\dfrac{6}{x^5}f''\left(\dfrac{1}{x}\right)-\dfrac{6}{x^4}f'\left(\dfrac{1}{x}\right)$. **318.** $y''=e^{2x}\times$ $\times f''(e^x)+e^xf'(e^x)$; $y'''=e^{3x}f'''(e^x)+3e^{2x}f''(e^x)+e^xf'(e^x)$. **319.** $y''=\dfrac{1}{x^2}[f''(\ln x)-f'(\ln x)]$; $y'''=\dfrac{1}{x^3}[f'''(\ln x)-3f''(\ln x)+2f'(\ln x)]$. **320.** $y''=\varphi'^2(x)f''(\varphi(x))+\varphi''(x)f'(\varphi(x))$; $y'''=\varphi'^3(x)\times$ $\times f'''(\varphi(x))+3\varphi'(x)\varphi''(x)f''(\varphi(x))+\varphi''(x)f'(\varphi(x))$; **321.** a) e^xdx^2; b) $e^x(dx^2+d^2x)$. **322.** $\dfrac{dx^2}{(1+x^2)^{3/2}}$.

323. $\frac{2\ln x-3}{x^3}dx^2$ $(x>0)$. **324.** $x^x[(1+\ln x)^2+1/x]dx^2$. **325.** $ud^2\upsilon+2dud\upsilon+\upsilon d^2u$.

326. $\frac{(\upsilon d^2u-ud^2\upsilon)-2d\upsilon(\upsilon du-ud\upsilon)}{\upsilon^2}$ $(\upsilon>0)$. **327.** $u^{m-2}\upsilon^{n-2}\{[m(m-1)\upsilon^2du^2+2mnu\upsilon dud\upsilon+n\times$ $\times(n-1)u^2d\upsilon^2]+u\upsilon(m\upsilon d^2u+nud^2\upsilon)\}$. **328.** $a^u\ln a(du^2\ln a+d^2u)$. **329.** $[(\upsilon^2-u^2)du^2-$ $-4u\upsilon dud\upsilon+(u^2-\upsilon^2)d\upsilon^2+(u^2+\upsilon^2)(ud^2u+\upsilon d^2\upsilon)](u^2+\upsilon^2)^{-2}$ $(u^2+\upsilon^2>0)$. **330.** $[-2u\upsilon du^2+$ $+2(u^2-\upsilon^2)dud\upsilon+2u\upsilon d\upsilon^2+(u^2+\upsilon^2)(\upsilon d^2u-ud^2\upsilon)](u^2+\upsilon^2)^{-2}$ $(u^2+\upsilon^2>0)$. **331.** $y''=\frac{3}{4(1-t)}$; $y'''=\frac{3}{8(1-t)^3}$ $(t\neq 1)$. **332.** $y''=-\frac{1}{a\sin^3 t}$; $y'''=-\frac{3\cos t}{a^2\sin^5 t}$ ($t\neq k\pi$, k – bitin san) **333.** $y''=$ $=-\frac{1}{4a\sin^4\left(t/2\right)}$; $y'''=\frac{\cos(t/2)}{4a^2\sin^7(t/2)}$ ($t\neq 2k\pi$, k–bitin san). **334.** $y''=\frac{e^{-t}}{\sqrt{2}\cos^3(t+\pi/4)}$; $y'''=\frac{e^{-2t}(2\sin t+\cos t)}{\sqrt{2}\cos^5(t+\pi/4)}$ $(t\neq\frac{\pi}{4}+k\pi,\ k=0,\ \pm1,\ \ldots)$. **335.** $y''=\frac{1}{f''(t)}$; $y'''=-\frac{f'''(t)}{f''^3(t)}$ $(f''(t)\neq 0)$. **336.** $x'=\frac{1}{y'}$; $x''=-\frac{y''}{y'^3}$; $x'''=-\frac{y'y'''-3y''^2}{y'^5}$; $x^{\mathrm{IV}}=-\frac{1}{y'^7}\left(y'^2y^{\mathrm{IV}}-10y'y''y'''+\right.$ $\left.+15y''^3\right)$ $(y'\neq 0)$. **337.** $-\frac{x}{y}$, $-\frac{25}{y^3}$, $-\frac{75x}{y^5}$; $-\frac{3}{4}$, $-\frac{25}{64}$, $-\frac{225}{1024}$. **338.** $\frac{p}{y}$, $-\frac{p^2}{y^3}$, $\frac{3p^3}{y^5}$.

339. $y'=\frac{2x-y}{x-2y}$, $y''=\frac{6}{(x-2y)^3}$, $y'''=\frac{54x}{(x-2y)^5}$. **340.** $y'=\frac{2x^3y}{1+y^2}$; $y''=\frac{2x^2y}{(1+y^2)^3}\times$ $\times\left[3\left(1+y^2\right)^2+2x^4\left(1-y^2\right)\right]$. **341.** $y'=\frac{x+y}{x-y}$; $y''=\frac{2(x^2+y^2)}{(x-y)^3}$. **342.** $a=\frac{1}{2}f''(x_0)$; $b=f'(x_0)$; $c=f(x_0)$. **343.** $20-10t$, -10; 0, -10. **344.** $\vartheta=-\frac{2\pi a}{T}\sin\frac{2\pi}{T}t$, $j=-\frac{4\pi^2a}{T^2}\cos\frac{2\pi}{T}t$.

345. $x=\vartheta_0 t\cos\alpha$, $y=\vartheta_0 t\sin\alpha-\frac{gt^2}{2}$; $\vartheta=\sqrt{\vartheta_0^2-2\vartheta_0 gt\sin\alpha+g^2t^2}$; $j=g$; $y=x\mathrm{tg}\alpha-$ $-\frac{gx^2}{2\vartheta_0^2\cos^2\alpha}$; $\frac{\vartheta_0^2\sin^2\alpha}{2g}$; $\frac{\vartheta_0^2}{g}\sin 2\alpha$. **346.** $x^2+y^2=25$; $5|\omega|$, $5\omega^2$. **347.** $y^{(6)}=4\cdot 6!$; $y^{(7)}=0$.

348. $y'''=-\frac{am(m+1)(m+2)}{x^{m+3}}$ $(x\neq 0)$. **349.** $y^{(10)}=-\frac{17!!}{2^{10}x^9\sqrt{x}}$ $(x>0)$, $17!!=1\cdot 3\cdot 5\ldots 17$.

350. $y^{(8)}=\frac{8!}{(1-x)^9}$ $(x\neq 1)$. **351.** $y^{(100)}=\frac{197!!(399-x)}{2^{100}(1-x)^{100}\sqrt{1-x}}$ $(x<1)$. **352.** $y^{(20)}=2^{20}e^{2x}\times$ $\times(x^2+20x+95)$. **353.** $y^{(10)}=e^x\sum_{i=0}^{10}\left(-1\right)^i\frac{A_{10}^i}{x^{i+1}}$, bu ýerde $A_{10}^i=10\cdot 9\cdot\ldots(11-i)$ we $A_{10}^0=1$. **354.** $y^{(5)}=-\frac{6}{x^4}$ $(x>0)$. **355.** $y^{(5)}=\frac{274}{x^6}-\frac{120}{x^6}\ln x$ $(x>0)$. **356.** $y^{(50)}=2^{50}\times$

$\times\left(-x^2\sin 2x+50x\cos 2x+\frac{1225}{2}\sin 2x\right)$. **357.** $y''' = \frac{27(1-3x)^2-36}{(1-3x)^{7/3}}\sin 3x-\cos 3x\times$ $\times\frac{27\left(1-3x\right)^2-28}{\left(1-3x\right)^{10/3}}\ \left(x\neq\frac{1}{3}\right)$. **358.** $y^{(10)}=-2^8\sin 2x-2^{18}\sin 4x+2^8\cdot 3^{10}\sin 6x$. **359.** $y^{(100)}=x\times$ $\times\operatorname{sh}x+100\operatorname{ch}x$. **360.** $y^{\mathrm{IV}}=-4e^x\cos x$. **361.** $y^{(6)}=-\frac{60}{x^6}+\left(\frac{144}{x^5}-\frac{160}{x^3}+\frac{96}{x}\right)\sin 2x+\left(\frac{60}{x^6}-\right.$ $\left.-\frac{180}{x^4}+\frac{120}{x^2}+32\ln x\right)\cos 2x$. **362.** $120dx^5$. **363.** $-\frac{15}{8x^3\sqrt{x}}dx^3$ $(x>0)$. **364.** $-1024(x\cos 2x+$ $+5\sin 2x)dx^{10}$. **365.** $e^x\left(\ln x+\frac{4}{x}-\frac{6}{x^2}+\frac{8}{x^3}-\frac{6}{x^4}\right)dx^4$. **366.** $8\sin x\operatorname{sh}x\,dx^6$. **367.** $2ud^{10}u+$ $+20dud^9u+90d^2ud^6u+240d^3ud^7u+420d^4ud^6u+252(d^2u)^2$. **368.** $e^u(du^4+6du^2d^2u+$ $+4dud^3u+3d^2u^2+d^4u)$. **369.** $\frac{2du^2}{u^3}-\frac{3dud^2u}{u^2}+\frac{d^3u}{u}$. **370.** $d^2y=y''dx^2+y'd^2x$; $d^3y=y'''dx^3+$ $+3y''dxd^2x+y'd^3x$; $d^4y=y^{\mathrm{IV}}dx^4+6y'''dx^2d^2x+4y''dxd^3x+3y''d^2x^2+y'd^4x$. **371.** $y''=\frac{\begin{vmatrix} dx & dy \\ d^2x & d^2y\end{vmatrix}}{dx^3}$

$y'''=\frac{dx\begin{vmatrix} dx & dy \\ d^3x & d^3y\end{vmatrix}-3d^2x\begin{vmatrix} dx & dy \\ d^2x & d^2y\end{vmatrix}}{dx^5}$. **378.** $P^{(n)}(x)=a_0n!$ **379.** $\frac{(-1)^{n-1}n!c^{n-1}(ad-bc)}{(cx+d)^{n+1}}$.

380. $n!\left[\frac{(-1)^n}{x^{n+1}}+\frac{1}{(1-x)^{n+1}}\right]$. **381.** $(-1)^n n!\left[\frac{1}{(x-2)^{n+1}}-\frac{1}{(x-1)^{n+1}}\right]$. **382.** $\frac{1\cdot 3...(2n-1)}{(1-2x)^{n+1/2}}$ $\left(x<\frac{1}{2}\right)$. **383.** $\frac{(-1)^{n+1}\cdot 1\cdot 4...(3n-5)(3n+2x)}{3^n(1+x)^{n+1/3}}$ $(n\geq 2;\ x\neq -1)$. **384.** $-2^{n-1}\cos(2x+$ $\left.+\frac{n\pi}{2}\right)$. **385.** $2^{n-1}\cos\left(2x+\frac{n\pi}{2}\right)$. **386.** $\frac{3}{4}\sin\left(x+\frac{n\pi}{2}\right)-\frac{3^n}{4}\sin\left(3x+\frac{n\pi}{2}\right)$. **387.** $\frac{3}{4}\cos(x+$ $\left.+\frac{n\pi}{2}\right)+\frac{3^n}{4}\cos\left(3x+\frac{n\pi}{2}\right)$. **388.** $\frac{(a-b)^n}{2}\cos\left[(a-b)x+\frac{n\pi}{2}\right]-\frac{(a+b)^n}{2}\cos[(a+b)\times$ $\left.\times x+\frac{n\pi}{2}\right]$. **389.** $\frac{(a-b)^n}{2}\cos\left[(a-b)x+\frac{n\pi}{2}\right]+\frac{(a+b)^n}{2}\cos\left[(a+b)x+\frac{n\pi}{2}\right]$. **390.** $\frac{1}{2}\times$ $\times(a-b)^n\sin\left[(a-b)x+\frac{n\pi}{2}\right]+\frac{(a+b)^n}{2}\sin\left[(a+b)x+\frac{n\pi}{2}\right]$. **391.** $\frac{b^n}{2}\cos\left(bx+\frac{n\pi}{2}\right)-$ $-\frac{(2a-b)^n}{4}\cos\left[(2a-b)x+\frac{n\pi}{2}\right]-\frac{(2a+b)^n}{4}\cos\left[(2a+b)x+\frac{n\pi}{2}\right]$. **392.** $4^{n-1}\cos(4x+$ $\left.+\frac{n\pi}{2}\right)$. **393.** $a^nx\cos\left(ax+\frac{n\pi}{2}\right)+na^{n-1}\sin\left(ax+\frac{n\pi}{2}\right)$. **394.** $a^n\left[x^2-\frac{n(n-1)}{a^2}\right]\sin(ax+$ $\left.+\frac{n\pi}{2}\right)-2na^{n-1}x\cos\left(ax+\frac{n\pi}{2}\right)$. **395.** $(-1)^ne^{-x}[x^2-2(n-1)x+(n-1)(n-2)]$. **396.** $e^x\left\{\frac{1}{x}+\right.$

$+\sum_{k=1}^{n}(-1)^k\dfrac{n(n-1)...(n-k+1)}{x^{k+1}}\Big\}$. **397.** $e^x 2^{n/2}\cos\left(x+\dfrac{n\pi}{4}\right)$. **398.** $e^x 2^{n/2}\sin\left(x+\dfrac{n\pi}{4}\right)$.

399. $\dfrac{(n-1)!b^n}{(a^2-b^2x^2)^n}[(a+bx)^n+(-1)^{n-1}(a-bx)^n]$ $(|x|<|a/b|)$. **400.** $e^{ax}[a^nP(x)+C_n^1a^{n-1}\times$ $\times P'(x)+...+P^{(n)}(x)]$. **401.** $\dfrac{1}{2}\{[(x+n)-(-1)^n(x-n)]\mathrm{ch}x+[(x+n)+(-1)^n(x-n)]\times$ $\times\,\mathrm{sh}x\}$. **402.** $d^ny=e^x\left[x^n+n^2x^{n-1}+\dfrac{n^2(n-1)^2}{2!}x^{n-2}+...+n!\right]dx^n$. **403.** $\dfrac{(-1)^nn!}{x^{n+1}}\{\ln x-$ $-\sum_{i=1}^{n}\dfrac{1}{i}\Big\}dx^n$ $(x>0)$. **405.** a) $(a^2+b^2)^{n/2}\left[\cos\left(n\varphi-\dfrac{n\pi}{2}\right)\mathrm{ch}ax\cos\left(bx+\dfrac{n\pi}{2}\right)-\sin(n\varphi-n\times\right.$ $\left.\times\dfrac{\pi}{2}\right)\mathrm{sh}ax\sin\left(bx+\dfrac{n\pi}{2}\right)\Big]$; b) $(a^2+b^2)^{n/2}\left[\cos\left(n\varphi-\dfrac{n\pi}{2}\right)\mathrm{ch}ax\sin\left(bx+\dfrac{n\pi}{2}\right)+\sin(n\varphi-\right.$ $\left.-\dfrac{n\pi}{2}\right)\mathrm{sh}ax\cos\left(bx+\dfrac{n\pi}{2}\right)\Big]$, bu ýerde $\cos\varphi=\dfrac{a}{\sqrt{a^2+b^2}}$, $\sin\varphi=\dfrac{b}{\sqrt{a^2+b^2}}$. **406.** $f^{(n)}(x)=$ $=\sum_{k=0}^{p-1}(-1)^{p+k}2^{n-2p+1}(p-k)^nC_{2p}^k\cos\left[(2p-2k)x+\dfrac{n\pi}{2}\right]$. **407.** a) $\sum_{k=0}^{p}\{(-1)^{p+k}\sin[(2p-$ $-2k+1)x+\dfrac{n\pi}{2}\Big]\dfrac{(2p-2k+1)^n}{2^{2p}}C_{2p+1}^k\Big\}$; b) $\sum_{k=0}^{p-1}\Big\{2^{n-2p+1}(p-k)^n\cos\left[(2p-2k)x+\dfrac{n\pi}{2}\right]\times$ $\times C_{2p}^k\Big\}$; ç) $\sum_{k=0}^{p}\left\{\dfrac{(2p-2k+1)^n}{2^{2p}}C_{2p+1}^k\cos\left[(2p-2k+1)x+\dfrac{n\pi}{2}\right]\right\}$. **409.** $\dfrac{(-1)^{n-1}(n-1)!}{(1+x^2)^{n/2}}\times$ $\times\sin(n\,\mathrm{arctg}x)$ $(x\neq0)$. **410.** a) $\dfrac{n!}{3}[2^{n+1}+(-1)^n]$; b) $\dfrac{n(2n-3)!!}{2^{n-1}}$ $(n>1)$. **411.** a) $n(n-1)a^{n-2}$; b) $f^{(2k)}(0)=0$, $f^{(2k+1)}(0)=(-1)^k(2k)!$ $(k=0,\ 1,\ 2,\ ...)$; ç) $f^{(2k)}(0)=0$, $f^{(2k+1)}(0)=[1\cdot3...(2k-1)]^2$ $(k=0,\ 1,\ 2,\ ...)$. **412.** a) $f^{(2k)}(0)=(-1)^km^2(m^2-2^2)...[m^2-(2k-2)^2]$, $f^{(2k-1)}(0)=0$; b) $f^{(2k)}(0)=0$, $f'(0)=m$, $f^{(2k+1)}(0)=(-1)^km(m^2-1^2)...[m^2-(2k-1)^2]$ $(k=1,\ 2,\ ...)$. **413.** a) $f^{(2k)}(0)=$ $=(-1)^{k-1}\cdot2(2k-1)!\left(1+\dfrac{1}{3}+...+\dfrac{1}{2k-1}\right)$, $f^{(2k-1)}(0)=0$ $(k=1,\ 2,\ ...)$; b) $f^{(2k)}(0)=2^{2k-1}\times$ $\times[(k-1)!]^2$, $f^{(2k-1)}(0)=0$ $(k=1,\ 2,\ ...)$; **414.** $n!\varphi(a)$. **419.** $L_m(x)=(-1)^m[x^m-m^2x^{m-1}+$ $+\dfrac{m^2(m-1)^2}{1\cdot2}x^{m-2}+...+(-1)^mm!\Big]$. **422.** $H_m(x)=(2x)^m-\dfrac{m}{1!}(m-1)(2x)^{m-2}+\dfrac{m}{2!}\times$ $\times(m-1)(m-2)(m-3)(2x)^{m-4}-...$.

V. §1. Funksiýanyň orta bahasy hakyndaky teoremalar

2. $x=0$ bolanda funksiýanyň tükenikli $f'(x)$ önümi ýok. **10.** $A(-1,\ -1)$, $C(1,\ 1)$. **11.** Dogry däl. **12.** a) $\theta=1/2$; b) $\theta=\left(\sqrt{x^2+x\Delta x+(\Delta x)^2/3}-x\right)\Big/\Delta x$ $(x\geq0,\ \Delta x>0)$;

ç) $\theta = x(\sqrt{1+\Delta x/x} - 1)/\Delta x$ $(x(x+\Delta x)>0)$; d) $\theta = \ln((e^{\Delta x}-1)/\Delta x)/\Delta x$. **16.** $c=1/2$ ýa-da $\sqrt{2}$. **18.** Umuman aýdylanda, ýok. **29.** $f(x)=c_0+c_1x+...+c_{n-1}x^{n-1}$, bu ýerde c_i $(i=0, 1, ..., n-1)$ hemişelik sanlar.

§2. Monoton we güberçek funksiýalar. Epin nokatlary

37. Funksiýa $-\infty<x<1/2$ bolanda artýar, $1/2<x<+\infty$ bolanda kemelýär. **38.** Funksiýa $-\infty<x<-1$ bolanda kemelýär, $-1<x<1$ bolanda artýar, $1<x<+\infty$ bolanda kemelýär. **39.** Funksiýa $-\infty<x<-1$ bolanda kemelýär, $-1<x<1$ bolanda artýar, $1<x<+\infty$ bolanda kemelýär. **40.** Funksiýa $0<x<100$ bolanda artýar; $100<x<+\infty$ bolanda kemelýär. **41.** Funksiýa artýar. **42.** Funksiýa $(k\pi/2,\ k\pi/2+\pi/3)$ interwallarda artýar; $(k\pi/2+\pi/3,\ k\pi/2+\pi/2)$ $(k=0, \pm1, \pm2, ...)$ interwallarda kemelýär. **43.** Funksiýa $(1/(2k+1),\ 1/2k)$ we $(-1/(2k+1),\ -1/(2k+2))$ interwallarda artýar; $(1/(2k+2),\ 1/(2k+1))$ we $(-1/2k,\ -1/(2k+1))$ $(k=0, 1, 2, ...)$ interwallarda kemelýär. **44.** Funksiýa $-\infty<x<0$ bolanda kemelýär; $0<x<2/\ln2$ bolanda artýar; $2/\ln2<x<+\infty$ bolanda kemelýär. **45.** Funksiýa $0<x<n$ bolanda artýar; $n<x<+\infty$ bolanda kemelýär. **46.** Funksiýa $-\infty<x<-1$ we $0<x<1$ bolanda kemelýär, $-1<x<0$ we $1<x<+\infty$ bolanda artýar. **47.** $(e^{-7\pi/12+2k\pi},\ e^{13\pi/12+2k\pi})$ interwallarda funksiýa artýar; $(e^{13\pi/12+2k\pi},\ e^{17\pi/12+2k\pi})$ $(k=0; \pm1; \pm2; ...)$ interwallarda funksiýa kemelýär. **52.** Hökman däl. **67.** A nokatda aşaklygyna güberçek; B nokatda ýokarlygyna güberçek; C epin nokady. **68.** Grafigi $-\infty<x<1$ bolanda aşaklygyna güberçek; $1<x<+\infty$ bolanda ýokarlygyna güberçek; $x=1$ epin nokady. **69.** $|x|<a/\sqrt{3}$ bolanda ýokarlygyna güberçek; $|x|>a/\sqrt{3}$ bolanda aşaklygyna güberçek, $|x|=\pm a/\sqrt{3}$ epin nokatlary. **70.** $x<0$ bolanda ýokarlygyna güberçek; $x>0$ bolanda aşaklygyna güberçek; $x=0-$epin nokady. **71.** Aşaklygyna güberçek. **72.** $2k\pi<x<(2k+1)\pi$ bolanda ýokarlygyna güberçek; $(2k+1)\pi<x<(2k+2)\pi$ bolanda aşaklygyna güberçek; $x=k\pi-$epin nokatlary $(k=0, \pm1, \pm2, ...)$. **73.** $|x|<\sqrt{1/2}$ bolanda ýokarlygyna güberçek; $|x|>\sqrt{1/2}$ bolanda aşaklygyna güberçek; $|x|=\pm\sqrt{1/2}$ epin nokatlary. **74.** $|x|<1$ bolanda aşaklygyna güberçek; $|x|>1$ bolanda ýokarlygyna güberçek; $|x|=\pm1$ epin nokatlary. **75.** $e^{2k\pi-3\pi/4}<x<e^{2k\pi+\pi/4}$ bolanda aşaklygyna güberçek; $e^{2k\pi+\pi/4}<x<e^{2k\pi+5\pi/4}$ bolanda ýokarlygyna güberçek; $x=e^{k\pi+\pi/4}$ $(k=0, \pm1, \pm2, ...)$ epin nokatlary. **76.** $0<x<+\infty$ bolanda aşaklygyna güberçek. **78.** $h=1/(\sigma\sqrt{2})$. **79.** Ýokarlygyna güberçek ($a>0$ bolanda).

§3. Lopitalyň kesgitsizlikleri açmak düzgünleri

88. a/b. **89.** 1. **90.** 2. **91.** –2. **92.** 1/3. **93.** –1/3. **94.** 1/3. **95.** 1/6. **96.** 1/2. **97.** 1. **98.** $(a-b)/3ab$. **99.** $\ln a/6$. **100.** –2. **101.** 1. **102.** $(a/b)^2$. **103.** 1/6. **104.** 2/3. **105.** 1. **106.** 0. **107.** 0. **108.** 0. **109.** 0. **110.** 0. **111.** 0. **112.** 1. **113.** 1. **114.** –1. **115.** e^k. **116.** e^{-1}. **117.** $e^{2/\pi}$.

118. e^{-1}. **119.** 1. **120.** 1. **121.** 1. **122.** $e^{2/\sin 2a}$ ($a \neq k\pi/2$, k – bitin san). **123.** $e^{(\ln^2 a - \ln^2 b)/2}$. **124.** 1/2. **125.** 1/2. **126.** 0. **127.** –1/2. **128.** $a^a(\ln a - 1)$. **129.** $-e/2$. **130.** $1/a$. **131.** $e^{-2/\pi}$. **132.** 1. **133.** $e^{1/6}$. **134.** $e^{-1/6}$. **135.** $e^{1/3}$. **136.** $e^{-1/3}$. **137.** $e^{-1/6}$. **138.** $e^{-1/2}$. **139.** $e^{-2/\pi}$. **140.** e^{-1}. **141.** $mn/(n-m)$. **142.** $\sqrt{e}$. **143.** 0. **144.** –1/6. **145.** a. **146.** $\mathrm{tg}\alpha$. **149.** $f'(0) = -1/12$. **150.** $y = (x + 1/2)/e$. **151.** a) Lopitalyň düzgüni ulanylmaýar, predel 0-a deňdir; b) Lopitalyň düzgüni ulanylmaýar, predel 1-e deň; ç) Lopitalyň düzgüni formal taýdan ulanylanda 0-a deň bolan ýalňyş netijäni berýär, predeli ýok; d) Lopitalyň düzgünini ulanmaklyk bikanun we 0-a deň ýalňyş netijä getirýär, predeli ýok. **152.** 4/3.

§4. Teýloryň formulasy

153. $5 - 13(x+1) + 11(x+1)^2 - 2(x+1)^3$. **154.** $1 + 2x + 2x^2 - 2x^4 + o(x^4)$; –48. **155.** $1 + 60x + 1950x^2 + o(x^2)$. **156.** $a + \dfrac{x}{ma^{m-1}} - \dfrac{(m-1)x^2}{2m^2a^{2m-1}} + o(x^2)$. **157.** $\frac{1}{6}x^2 + x^3 + o(x^3)$. **158.** $1 + 2x + x^2 - \frac{2}{3}x^3 - \frac{5}{6}x^4 - \frac{1}{15}x^5 + o(x^5)$. **159.** $1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + o(x^4)$. **160.** $x - \frac{x^7}{18} - \frac{x^{13}}{3240} + o(x^{13})$. **161.** $-\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} + o(x^6)$. **162.** $x - \frac{x^3}{3} + o(x^3)$. **163.** $x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5)$. **164.** $-\frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} + o(x^6)$. **165.** $1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + o((x-1)^2)$. **166.** $(x-1) + (x-1)^2 + \frac{1}{2}(x-1)^3 + o((x-1)^3)$. **167.** $y = a + \frac{x^2}{2a} + o(x^2)$. **168.** $\frac{1}{2x} - \frac{1}{8x^3} + o\left(\frac{1}{x^3}\right)$. **169.** $\ln x + \frac{h}{x} - \frac{h^2}{2x^2} + \ldots + (-1)^{n-1}\frac{h^n}{nx^n} + o(h^n)$.

174. a) $\dfrac{3}{(n+1)!}$-den kiçi; b) 1/3480-den uly däl; ç) $2 \cdot 10^{-6}$-den kiçi; d) 1/16-den kiçi.

175. $|x| < 0{,}222 = \mathrm{arc}12^\circ 30'$. **177.** a) 3,1072; b) 3,0171; ç) 1,9961; d) 1,64872; e) 0,309017; ä) 0,182321; f) $0{,}67474 = \mathrm{arc}38^\circ 39' 35''$; g) $0{,}46676 = \mathrm{arc}26^\circ 44' 37''$; h) 1,12117. **178.** a) 2,718281828; b) 0,01745241; ç) 0,98769; d) 2,2361; e) 1,04139. **179.** –1/12. **180.** 1/3. **181.** –1/4. **182.** 1/3. **183.** 1/6. **184.** $\ln^2 a$. **185.** 1/2. **186.** 0. **187.** 1/3. **188.** 19/90. **189.** 1/2. **190.** 1/2. **191.** $x^7/30$. **192.** x^2. **193.** $x/2$. **194.** $a = 4/3$; $b = -1/3$. **195.** $A = -2/5$; $B = -1/15$. **196.** $A = 1/2$; $B = 1/12$; $C = -1/2$; $D = 1/12$. **197.** a) $2x/R^3$; b) $4x/3$; ç) $An/100$; d) $70/x$. **198.** $\alpha = 2/3$; $\beta = 1/3$. **199.** $\alpha^4/180$, bu ýerde α duganyň merkezi burçunyň ýarysy.

§5. Funksiýanyň ekstremumy. Funksiýanyň iň uly we iň kiçi bahalary

200. $x = 1/2$ bolanda, $y = 9/4$ maksimum. **201.** Ekstremum ýok. **202.** $x = 1$ bolanda, $y = 0$ minimum. **203.** $x = 0$ we m jübüt bolanda, $y = 0$ minimum, m täk we $x = 0$ bolanda ekstremum ýok; $x=m/(m+n)$ bolanda, $y=m^m n^n/(m+n)^{m+n}$ maksimum; $x=1$ we n jübüt bolanda, $y=0$ minimum, n täk we $x=1$ bolanda ekstremum ýok. **204.** $x=0$ bolanda, $y=2$ minimum. **205.** $x=-1$ bolanda, $y=0$ minimum; $x=9$ bolanda, $y=10^{10}e^{-9}\approx 1\,234\,000$ maksimum. **206.** $x=0$ we n täk bolanda, $y=1$ maksimum, $x=0$ we n jübüt bolanda, ekstremum ýok. **207.** $x=0$ bolanda, $y=0$ minimum. **208.** $x=1/3$ bolanda, $y=\sqrt[3]{4}/3\approx 0{,}529$ maksimum; $x=1$ bolanda, $y=0$ minimum; $x=0$ bolanda, ekstremum ýok. **209.** Eger $\varphi(x_0)>0$ we n jübüt san bolsa, $f(x_0)=0$ minimum; eger $\varphi(x_0)<0$ we n jübüt bolsa, $f(x_0)=0$ maksimum; eger n täk san bolsa, onda $f(x_0)$ ekstremum däl. **211.** Ýok. **213.** a) $f(0)=0$ minimum; b) $f(0)=0$ minimum. **214.** $f(0)=0$ minimum. **215.** $x=1$ bolanda, $y=0$ maksimum; $x=3$ bolanda, $y=-4$ minimum. **216.** $x=0$ bolanda, $y=0$ minimum; $x=\pm1$ bolanda, $y=1$ maksimum. **217.** $x=(5-\sqrt{13})/6\approx 0{,}23$ bolanda, $y\approx-0{,}76$ minimum; $x=1$ bolanda, $y=0$ maksimum; $x=(5+\sqrt{13})/6\approx 1{,}43$ bolanda, $y\approx-0{,}05$ minimum; $x=2$ bolanda, ekstremum ýok. **218.** $x=-1$ bolanda, $y=-2$ maksimum; $x=1$ bolanda, $y=2$ minimum. **219.** $x=-1$ bolanda, $y=-1$ minimum; $x=1$ bolanda, $y=1$ maksimum. **220.** $x=7/5$ bolanda, $y=-1/24$ minimum. **221.** $x=0$ we $x=2$ bolanda, $y=0$ gyraky minimum; $x=1$ bolanda, $y=1$ maksimum. **222.** $x=3/4$ bolanda, $y=-3\sqrt[3]{2}/8\approx-0{,}46$ minimum; $x=1$ bolanda, ekstremum ýok. **223.** $x=1$ bolanda, $y=e^{-1}\approx 0{,}368$ maksimum. **224.** $x=+0$ bolanda, $y=0$ gyraky maksimum; $x=e^{-2}\approx 0{,}135$ bolanda, $y=-2/e\approx-0{,}736$ minimum. **225.** $x=1$ bolanda, $y=0$ minimum; $x=e^2\approx 7{,}389$ bolanda, $y=4/e^2\approx 0{,}541$ maksimum. **226.** $x=k\pi$ ($k=0, \pm1, \pm2, ...$) bolanda, $y=(-1)^k+1/2$ maksimum; $x=\pm2\pi/3+2k\pi$ ($k=0, \pm1, \pm2, ...$) bolanda, $y=-3/4$ minimum. **227.** $x=k\pi$ ($k=0,\pm1,\pm2,...$) bolanda, $y=10$ maksimum; $x=\pi(k+1/2)$ ($k=0,\pm1,\pm2,...$) bolanda, $y=5$ minimum. **228.** $x=1$ bolanda, $y=\pi/4-\ln 2/2\approx 0{,}439$ maksimum. **229.** $x=-\pi/4+2k\pi$ ($k=0,\pm1,\pm2,...$) bolanda, $y=-(\sqrt{2}/2)e^{-\pi/4+2k\pi}$ minimum; $x=3\pi/4+2k\pi$ ($k=0,\pm1,\pm2,...$) bolanda, $y=(\sqrt{2}/2)e^{3\pi/4+2k\pi}$ maksimum. **230.** $x=-1$ bolanda, $y=e^{-2}\approx 0{,}135$ maksimum; $x=0$ bolanda, $y=0$ minimum (burç nokady); $x=1$ bolanda, $y=1$ maksimum (burç nokady). **231.** 1/2; 32. **232.** 2; 66. **233.** 0; 132. **234.** 2; 100,01. **235.** 1; 3. **236.** 0; $100/e\approx 36{,}8$. **237.** 0; 1. **238.** 0; $(1+\sqrt{2})/2\approx 1{,}2$. **239.** $-(\sqrt{2}/2)e^{-3\pi/4}\approx-0{,}067$; 1. **240.** $-\infty<x\le-3$ bolanda, $m(x)=-1/6$; $-3<x\le-1$ bolanda, $m(x)=(1+x)/(3+x^2)$; $-1<x<+\infty$ bolanda, $m(x)=0$; $-\infty<x\le 1$ bolanda, $M(x)=1/2$; $1<x<+\infty$ bolanda, $M(x)=(1+x)/(3+x^2)$. **242.** a) $14^{10}/2^{14}\approx 1{,}77\cdot 10^7$; b) 1/200; ç) $\sqrt[3]{3}\approx 1{,}44$. **245.** $(9+6\sqrt{3})/4\approx 4{,}85$. **246.** $q=-1/2$. **247.** 4/27. **248.** $g(x)=(x_1+x_2)x-(1/8)(x_1^2+x_2^2+6x_1x_2)$; $\Delta=(1/8)\times(x_1-x_2)^2$. **249.** 2/3. **250.** Bir köki: $(3, +\infty)$. **251.** $h>27$ bolanda, bir köki: $-\infty<x_1<-1$; $-5<h<27$ bolanda, üç köki: $-\infty<x_1<-1$, $-1<x_2<3$ we $3<x_3<+\infty$; $h<-5$ bolanda, bir köki: $3<x_3<+\infty$. **252.** Iki köki: $-\infty<x_1<-1$ we $1<x_2<+\infty$. **253.** $-\infty<a<-4$ bolanda, bir köki: $-\infty<x_1<-1$; $-4<a<4$ bolanda, üç köki: $-\infty<x_1<-1$, $-1<x_2<1$, $1<x_3<+\infty$; $4<a<+\infty$ bolanda, bir köki: $1<x_1<+\infty$. **254.** $-\infty<k<0$ bolanda, bir köki: $0<x_1<1$; $0<k\le 1/e$ bolanda, iki köki:

$0<x_1<1/k$ we $1/k<x_2<+\infty$; $k>1/e$ bolanda, kökleri ýok. **255.** $a<0$ bolanda, kökleri ýok; $0<a<e^2/4$ bolanda, bir köki: $-\infty<x_1<0$; $e^2/4<a<+\infty$ bolanda, üç köki: $-\infty<x_1<0$, $0<x_2<2$ we $2<x_3<+\infty$. **256.** $|a|<3\sqrt{3}/16$ bolanda iki köki bar; $|a|>3\sqrt{3}/16$ bolanda, köki ýok. **257.** $|k|>\mathrm{sh}\xi\approx1{,}50$ bolanda, $\mathrm{cth}x=x$ deňlemäniň položitel $\xi\approx1{,}2$ köki üçin, iki köki: $0<|x_1|<\xi$ we $\xi<|x_2|<+\infty$; $|k|>\mathrm{sh}\xi$ bolanda, köki ýok. **258.** a) $p^3/27+q^2/4>0$; b) $p^3/27+q^2/4<0$.

§6. Häsiýetlendiriji nokatlary boýunça funksiýalaryň grafiklerini gurmak

259. Koordinatalar başlangyjyna görä simmetrik. Funksiýanyň nollary: $x=0$ we $x=\pm\sqrt{3}\approx\pm1{,}73$. $x=-1$ bolanda, minimum $y=-2$; $x=1$ bolanda, maksimum $y=2$. $x=0$, $y=0$ epin nokady. **260.** Oy okuna görä simmetrik. Nollary $x=\pm\sqrt{1+\sqrt{3}}\approx\pm1{,}65$. $x=0$ bolanda, $y=1$ minimum; $x=\pm1$ bolanda, $y=3/2$ maksimum. Epin nokatlary: $x=\pm1/\sqrt{3}\approx\pm0{,}58$; $y=23/18$. **261.** $A(1, 2)$ nokada görä simmetrik. Nollary: $x=-1$ we $x=2$. $x=2$ bolanda, minimum $y=0$; $x=0$ bolanda, maksimum $y=4$. Epin nokady $x=1$, $y=2$. **262.** Oy okuna görä simmetrik. Funksiýanyň nollary: $x=\pm\sqrt{2}\approx\pm1{,}41$. $x=0$ bolanda, maksimum $y=2$; $x=\pm\sqrt{2+\sqrt{5}}\approx\pm2{,}06$ bolanda, minimum $y=1-\sqrt{5}/2\approx-0{,}12$. Epin nokatlary: $x_{1,2}=\pm0{,}77$. $y_{1,2}=1{,}04$; $x_{3,4}\approx\pm2{,}67$, $y_{3,4}\approx-0{,}010$. Asimptotasy $y=0$. **263.** Üzülme nokatlary: $x=2$ we $x=3$. Nollary: $x=\pm1$. $x=(7-\sqrt{24})/5\approx0{,}42$ bolanda, minimum $y=-(10-\sqrt{96})\approx-0{,}20$; $x=(7+\sqrt{24})/5\approx2{,}38$ bolanda, maksimum $y=-(10+\sqrt{96})\approx-19{,}80$. Epin nokady $x\approx-0{,}58$, $y\approx-0{,}07$. Asimptotalary: $x=2$, $x=3$ we $y=1$. **264.** Üzülme nokatlary: $x_1=-1$ we $x_2=1$. Funksiýanyň noly $x=0$. Ekstremum nokatlary ýok. Epin nokatlary $x\approx-0{,}22$, $y\approx-0{,}19$. Asimptotalary: $x=-1$, $x=1$ we $y=0$. **265.** Funksiýanyň noly $x=0$. Üzülme nokady: $x=-1$. $x=0$ bolanda, minimum $y=0$; $x=-4$ bolanda, maksimum $y=-256/27$. Epin nokatlary ýok. Asimptotalary: $x=-1$ we $y=x-3$. **266.** $x=-1$ bolanda, minimum $y=0$. Epin nokady $x=-4$, $y=81/625$. Asimptotalary: $x=1$ we $y=1$. **267.** Maksimumlary $x=-(3+\sqrt{17})/2\approx-3{,}56$ bolanda, $y=-(34\sqrt{17}+142)/32\approx-8{,}82$ we $x=0$ bolanda, $y=0$; $x=(\sqrt{17}-3)/2\approx0{,}56$ bolanda, minimum $y=(34\sqrt{17}-142)/32\approx-0{,}06$. Epin nokady $x=1/5$, $y=-1/45$. Asimptotalary: $x=-1$ we $y=x-3$. **268.** Koordinatalar başlangyjyna görä simmetrik. Ekstremum nokatlary ýok; epin nokady $x=0$, $y=0$. Asimptotalary: $x=-1$, $x=1$ we $y=0$. **269.** $x=5$ bolanda, minimum $y=27/2$. Epin nokady $x=-1$, $y=0$. Asimptotalary: $x=1$ we $y=x+5$. **270.** $x=2$ bolanda, minimum $y=8/3$. $x\approx-2{,}4$ bolanda, maksimum $y\approx-3{,}2$. Epin nokady $x=0$, $y=8$. Asimptotalary: $x=-1$ we $y=x$. **271.** Oy okuna görä simmetrik. Funksiýanyň nollary: $x=\pm\sqrt{10}/4\approx\pm0{,}79$. Ekstremum nokatlary ýok. Epin nokatlary: $x=\pm\sqrt{1/2}\approx\pm0{,}71$, $y=-8/3$. Asimptotalary: $x=-1$, $x=0$, $x=1$ we $y=0$. **272.** Kesgitlenen ýaýlasy: $0\le x<+\infty$. Nollary: $x=0$ we $x=3$. $x=1$ bolanda, minimum $y=-2$; $x=0$ bolanda, gyraky maksimum

$y=0$. Aşaklygyna güberçek. **273.** Kesgitlenen ýaýlasy: $|x| \leq 2\sqrt{2} \approx 2,83$. Koordinatalar başlangyjyna we koordinatalar oklaryna görä simmetrik. Nollary: $x=0$ we $x=\pm 2\sqrt{2}$. $x=\pm 2$ bolanda, maksimum $|y|=4$; $x=0$ bolanda, minimum $|y|=0$; $x=+2\sqrt{2}$ bolanda, gyraky minimum $|y|=0$. Epin nokatlary ýok. **274.** Funksiýanyň noly $x=2$. $x=-0,5$ bolanda, minimum $y=-\sqrt{5}\approx -2,24$. Epin nokady $x_1=-(3+\sqrt{41})/8\approx -1,18$; $y_1\approx -2,06$ we $x_2=(\sqrt{41}-3)/8\approx 0,42$; $y_2\approx -1,46$. Asimptotalary: $x\to -\infty$ bolanda, $y=-1$ we $x\to +\infty$ bolanda $y=1$. **275.** Kesgitlenen ýaýlasy: $1\leq x\leq 2$ we $3\leq x<+\infty$. Nollary: $x=1, x=2$ we $x=3$. $x=(6-\sqrt{3})/3\approx 1,42$ bolanda, maksimum $|y|=\sqrt[4]{12}/3\approx 0,62$; $x=1$, 2 we 3 bolanda, gyraky minimumlary $|y|=0$. **276.** $x=1$ bolanda, minimum $y=0$; $x=-1/3$ bolanda, maksimum $y=2\sqrt[3]{4}/3\approx 1,06$. Epin nokady $x=-1$, $y=0$. Asimptotasy: $y=x-1/3$. **277.** *Oy* okuna görä simmetrik. $x=0$ bolanda, minimum $y=-1$. Ýokarlygyna güberçek. Asimptotasy: $y=0$. **278.** Koordinatalar başlangyjyna görä simmetrik. Funksiýanyň noly: $x=0$. $x=-2$ bolanda, minimum $y=-\sqrt[3]{16}\approx -2,52$; $x=2$ bolanda, maksimum $y=\sqrt[3]{16}$. Epin nokady: $x=0$, $y=0$. Asimptotasy: $y=0$. **279.** *Oy* okuna görä simmetrik. $x=\pm 1$ bolanda, minimum $y=\sqrt[3]{4}\approx 1,59$; $x=0$ bolanda, maksimum $y=2$. Ýokarlygyna güberçek. **280.** Koordinatalar başlangyjyna görä simmetrik. Üzülme nokady: $x=\pm 1$. Funksiýanyň noly: $x=0$. $x=\sqrt{3}$ bolanda, minimum $y=\sqrt{3}/\sqrt[3]{2}\approx 1,38$; $x=-\sqrt{3}$ bolanda, maksimum $y=-\sqrt{3}/\sqrt[3]{2}$. Epin nokatlary: $x_1=0, y_1=0$ we $x_{2,3}=\pm 3, y_{2,3}=\pm 3/2$. **281.** Funksiýanyň kesgitlenen ýaýlasy: $|x|\geq 1$. *Oy* okuna görä simmetrik. $x=\pm 1$ bolanda, gyraky minimum $y=0$. Ýokarlygyna güberçek. Asimptotalary: $x\to +\infty$ bolanda, $y=x/2$ we $x\to -\infty$ bolanda, $y=-x/2$. **282.** Funksiýanyň kesgitlenen ýaýlasy: $x>0$. $x=1/2$ bolanda, minimum $y=3\sqrt{3}/2\approx 2,60$. Aşaklygyna güberçek. Asimptotalary: $y=x+3/2$ we $x=0$. **283.** Kesgitlenen ýaýlasy: $x\geq 0$ we $x<-3$. Funksiýanyň noly $x=(5+\sqrt{13})/2\approx 4,30$. $x=-4$ bolanda, minimum $y=13$; $x=0$ bolanda, gyraky maksimum $y=1$. Aşaklygyna güberçek. Asimptotalary: $x\to -\infty$ bolanda, $y=5/2-2x$; $x\to +\infty$ bolanda, $y=-1/2$; $x\to -3-0$ bolanda, $x=-3$. **284.** $x=0$ bolanda, minimum $y=0$; $x=-2$ bolanda, maksimum $y=-\sqrt[3]{4}\approx -1,59$. Epin nokatlary: $x_1=-(2-\sqrt{3})\approx -0,27$, $y_1=\sqrt[3]{(\sqrt{27}-5)/2}\approx 0,46$; $x_2=-(2+\sqrt{3})\approx -3,73$, $y_2=-\sqrt[3]{(5+\sqrt{27})/2}\approx -1,72$. Asimptotasy $x=-1$. **285.** *Oy* okuna görä simmetrik. Funksiýa položitel. $x=0$ bolanda, maksimum $y=\sqrt{3}\approx 1,73$; $x=\pm 1$ bolanda, minimum $y=\sqrt{2}\approx 1,41$. Epin nokatlary $x_{1,2}\approx \pm 0,47$; $y_{1,2}\approx \pm 1,14$ we $x_{3,4}\approx \pm 4,58$; $y_{3,4}\approx 4,55$. Asimptotalary $y=\pm x$. **286.** Funksiýanyň periody: $T=2\pi$; esasy ýaýlasy $0\leq x\leq 2\pi$. Funksiýanyň nollary: $x_1=\pi+\arcsin((\sqrt{5}-1)/2)\approx 1,21\pi$, $x_2=2\pi-\arcsin((\sqrt{5}-1)/2)\approx 1,79\pi$. Minimumlary $x=\pi/2$ bolanda, $y=1$ we $x=3\pi/2$ bolanda $y=-1$; $x=\pi/6$ we $x=5\pi/6$ bolanda, maksimum $y=5/4$. Epin nokatlary: $x_1=\arcsin((1+\sqrt{33})/8)\approx 0,32\pi$, $y_1=(19+$ $+3\sqrt{33})/32\approx 1,13$; $x_2=\pi-\arcsin\dfrac{1+\sqrt{33}}{8}\approx 0,68\pi$, $y_2=\dfrac{19+3\sqrt{33}}{32}$; $x_3=\pi+$

$+\arcsin\frac{\sqrt{33}-1}{8}\approx 1,20\pi,\quad y_3=\frac{19-3\sqrt{33}}{32}\approx 0,055;\quad x_4=2\pi-\arcsin\frac{\sqrt{33}-1}{8}\approx$ $\approx 1,80\pi,\ y_4=(19-3\sqrt{33})/32$. **287.** Funksiýanyň periody 2π; esasy ýaýlasy $-\pi\le x\le\pi$. Koordinatalar başlangyjyna görä simmetrik. Nollary: $x_1=0$ we $x_{2,3}=\pm\pi$. $x=-\arccos(1/4)\approx$ $\approx-0,42\pi$ bolanda, minimum $y=-15\sqrt{15}/8\approx-7,3$; $x=\arccos(1/4)\approx 0,42\pi$ bolanda, maksimum $y=15\sqrt{15}/8\approx 7,3$. Epin nokatlary: $x_1=0$, $y_1=0$; $x_{2,3}=\pm\arccos(-7/8)\approx$ $\approx\pm 0,84\pi$; $y_{2,3}=\pm 21\sqrt{15}/32\approx\pm 2,54$; $x_{4,5}=\pm\pi$, $y_{4,5}=0$. **288.** Funksiýanyň periody: $T=2\pi$; esasy ýaýlasy $-\pi\le x\le\pi$. Koordinatalar başlangyjyna görä simmetrik. Nollary: $x_1=0$ we $x_{2,3}=\pm\pi$. Minimumlary: $x=-3\pi/4$ we $x=-\pi/4$ bolanda, $y=-2\sqrt{2}/3\approx-0,94$; $x=\pi/2$ bolanda, $y=2/3$. Maksimumlary: $x=-\pi/2$ bolanda $y=-2/3$; $x=\pi/4$ we $x=3\pi/4$ bolanda, $y=2\sqrt{2}/3$. Epin nokatlary: $x_1=0$, $y_1=0$; $x_{2,3}=\pm\arcsin\sqrt{5/6}\approx\pm 0,37\pi$, $y_{2,3}=\pm 4\sqrt{30}/27\approx\pm 0,81$; $x_{4,5}=\pm(\pi-\arcsin\sqrt{5/6})\approx\pm 0,63\pi$, $y_{4,5}=\pm 4\sqrt{30}/27$; $x_{6,7}=\pm\pi$, $y_{6,7}=0$. **289.** Funksiýanyň periody: $T=2\pi$; esasy ýaýlasy $[-\pi,\pi]$. *Oy* okuna görä simmetrik. Funksiýanyň nollary: $x_{1,2}=\pm\arccos((1-\sqrt{3})/2)\approx\pm 0,62\pi$. Minimumlary: $x=0$ bolanda, $y=1/2$; $x=\pm\pi$ bolanda, $y=-3/2$. Maksimumlary: $x=\pm\pi/3$ bolanda, $y=3/4$. Epin nokatlary: $x_{1,2}=\pm\arccos((1+\sqrt{33})/8)\approx\pm 0,18\pi$, $y_{1,2}\approx 0,63$; $x_{3,4}=\pm\arccos\big((1-$ $-\sqrt{33})/8\big)\approx\pm 0,70\pi$, $y_{3,4}\approx-0,44$. **290.** Funksiýanyň periody: $T=\pi/2$; esasy ýaýlasy $[-\pi/4,\pi/4]$. *Oy* okuna görä simmetrik. Funksiýa položitel. $x=0$ bolanda, maksimum $y=1$; $x=\pm\pi/4$ bolanda, minimum $y=1/2$. Epin nokatlary $x_{1,2}=\pm\pi/8$, $y_{1,2}=3/4$. **291.** Funksiýanyň periody $T=\pi$; esasy ýaýlasy $[-\pi/2,\pi/2]$. *Oy* okuna görä simmetrik. Funksiýanyň nollary: $x_1=0$ we $x_{2,3}=\pm\pi/3$. Minimumlary: $x=0$ bolanda, $y=0$ we $x=\pm\pi/2$ bolanda, $y=-1$. $x=\pm\arccos(1/4)\approx\pm 0,21\pi$ bolanda, maksimum $y=9/16$. Epin nokatlary $x_{1,2}=\pm\frac{1}{2}\arccos\frac{1+\sqrt{129}}{16}\approx\pm 0,11\pi$, $y_{1,2}\approx 0,29$; $x_{3,4}=\pm\frac{1}{2}\arccos\frac{1-\sqrt{129}}{16}\approx\pm 0,36\pi$; $y_{3,4}\approx-0,24$. **292.** Funksiýanyň periody $T=\pi$, esasy ýaýlasy $0\le x\le\pi$. Üzülme nokady: $x=3\pi/4$. Nollary: $x_1=0$, $x_2=\pi$. Ekstremumlary ýok, artýan funksiýa. Epin nokady: $x=\pi/4$, $y=\sqrt{2}/2$. Asimptotasy $x=3\pi/4$. **293.** Funksiýanyň periody $T=2\pi$, esasy ýaýlasy $[-\pi,\pi]$. *Oy* okuna görä simmetrik. Funksiýanyň nollary: $x_{1,2}=\pm\pi/2$. $x=0$ bolanda, minimum $y=1$; $x=\pm\pi$ bolanda, maksimum $y=-1$. Epin nokatlary: $x_{1,2}=\pi/2$; $y_{1,2}=0$. Asimptotalary $x=\pm\pi/4$ we $x=\pm 3\pi/4$. **294.** Funksiýanyň periody $T=2\pi$, esasy ýaýlasy $-\pi\le x\le\pi$. Funksiýa täk. $x=-2\pi/3$ bolanda, minimum $y=-\sqrt{3}/3\approx-0,58$; $x=2\pi/3$ bolanda, maksimum $y=\sqrt{3}/3\approx 0,58$. Epin nokatlary: $x_1=0$, $y_1=0$; $x_{2,3}=\mp\pi$, $y_{2,3}=0$. **295.** Simmetrik merkezleri: $(k\pi, 2k\pi)$. Funksiýanyň nollary: $x_1=0$, $x_{2,3}\approx\pm 0,37\pi,\dots$. Maksimumlary: $x=\pi/4+k\pi$ bolanda, $y=\pi/2-1+2k\pi$; minimumlary: $x=-(\pi/4+k\pi)$ bolanda, $y=-(\pi/2-1+2k\pi)$. Epin nokatlary: $x=k\pi$, $y=2k\pi$. Asimptotalary: $x=(2k+1)\pi/2$ (k – bitin san). **296.** $x=1$ göni çyzyga görä simmetrik. Funksiýa položitel. $x=1$ bolanda, maksimum $y=e$. Epin nokatlary $x_{1,2}=1\pm\sqrt{2}/2$, $y_{1,2}=\sqrt{e}\approx 1,65$. Asimptotasy $y=0$. **297.** *Oy* okuna görä simmetrik. Funksiýa položitel. $x=0$ bolanda, maksimum $y=1$. Epin nokatlary: $x_{1,2}=\pm\sqrt{3/2}\approx\pm 1,22$, $y_{1,2}=(5/2)e^{-3/2}\approx 0,56$. Asimptotasy $y=0$. **298.** Funksiýa položitel. $x=0$ bolanda, minimum

$y=1$. Aşaklygyna güberçek. Asimptotasy $x\to+\infty$ bolanda, $y=x$. **299.** Funksiýa otrisatel däl; noly $x=0$. $x=0$ bolanda, minimum $y=0$; $x=2/3$ bolanda, maksimum $y=\sqrt[3]{4/9}\,e^{-2/3}\approx 0,39$. Epin nokatlary: $x_1=(2-\sqrt{6})/3\approx-0,15$, $y_1\approx0,34$ we $x_2=(2+\sqrt{6})/3\approx1,48$, $y_2\approx0,30$. Asimptotasy $x\to+\infty$ bolanda, $y=0$. **300.** Funksiýa otrisatel däl. $x=k\pi$ ($k=0, \pm1, \pm2, ...$) bolanda, $y=0$ minimum; $x=\pi/4+k\pi$ bolanda, maksimumlary $y=e^{-(2k+1/2)\pi}/2$. Epin nokatlary: $x_k=(-1)^k\pi/6+k\pi$, $y_k=e^{-[2k+(-1)^k/3]\pi}/4$. **301.** $x>-1$ bolanda funksiýa položitel we $x<-1$ bolanda, funksiýa otrisatel. $x=0$ bolanda, minimum $y=1$. $x>-1$ bolanda, aşaklygyna güberçek we $x<-1$ bolanda, ýokarlygyna güberçek. **302.** *Oy* okuna görä simmetrik. Funksiýa otrisatel däl; noly $x=0$. $x=0$ bolanda, minimum $y=0$ (burç nokady). Ýokarlygyna güberçek. **303.** Funksiýanyň kesgitlenen ýaýlasy: $x>0$. Funksiýanyň noly $x=1$. $x=e^2\approx7,39$ bolanda, maksimum $y=2/e\approx0,74$. Epin nokady: $x=e^{8/3}\approx14,33$, $y=8/3e^{-4/3}\approx0,70$. Asimptotalary: $x\to+0$ bolanda, $x=0$ we $x\to+\infty$ bolanda, $y=0$. **304.** Koordinatalar başlangyjyna görä simmetrik. Noly $x=0$. Ekstremum nokatlary ýok; artýan funksiýa. Epin nokady: $x=0$, $y=0$. **305.** Koordinatalar başlangyjyna görä simmetrik. Funksiýanyň noly $x=0$. Artýan funksiýa. $x>0$ bolanda, aşaklygyna güberçek we $x<0$ bolanda, ýokarlygyna güberçek; $O(0; 0)$ – epin nokady. **306.** Funksiýanyň kesgitlenen ýaýlasy: $|x|<1$. Koordinatalar başlangyjyna görä simmetrik. Artýan funksiýa. $x>0$ bolanda, aşaklygyna güberçek we $x<0$ bolanda, ýokarlygyna güberçek; epin nokady: $x=0$, $y=0$. Asimptotalary: $x=\pm1$. **307.** Koordinatalar başlangyjyna görä simmetrik. Funksiýanyň noly $x=0$. Ekstremum nokatlary ýok, artýan funksiýa. Epin nokady: $x=0$, $y=0$. Asimptotalary $x\to-\infty$ bolanda, $y=x-\pi/2$ we $x\to+\infty$ bolanda $y=x+\pi/2$. **308.** $x\approx-5,95$ funksiýanyň noly. $x=1$ bolanda, minimum $y=1/2+\pi/4\approx1,285$; $x=-1$ bolanda, maksimum $y=-1/2+3\pi/4\approx1,856$. $x>0$ bolanda, aşaklygyna güberçek we $x<0$ bolanda, ýokarlygyna güberçek; epin nokady $x=0$, $y=\pi/2$. Asimptotalary: $x\to-\infty$ bolanda, $y=x/2+\pi$ we $x\to+\infty$ bolanda, $y=x/2$. **309.** *Oy* okuna görä simmetrik. Funksiýa otrisatel däl. Noly $x=0$. $x=0$ bolanda, minimum $y=0$. Aşaklygyna güberçek. Asimptotalary: $x\to-\infty$ bolanda, $y=-\pi x/2-1$ we $x\to+\infty$ bolanda, $y=\pi x/2-1$. **310.** Koordinatalar başlangyjyna görä simmetrik. Funksiýanyň noly $x=0$. $x=1$ bolanda, minimum $y=-\pi/2$ (burç nokady); $x=1$ bolanda, maksimum $y=\pi/2$ (burç nokady). Epin nokady: $x=0$, $y=0$. Asimptotasy $y=0$. **311.** *Oy* okuna görä simmetrik. Funksiýa otrisatel däl; Noly $x=0$. $x=0$ bolanda, minimum $y=0$ (burç nokady). Ýokarlygyna güberçek. Asimptotasy $y=\pi$. **312.** Funksiýanyň üzülme nokady $x=0$. Funksiýanyň noly $x=-2$. $x=2$ bolanda, minimum $y=4\sqrt{e}\approx6,59$; $x=-1$ bolanda, maksimum $y=1/e\approx0,37$. Epin nokady: $x=-2/5$, $y=8e^{-5/2}/5\approx0,13$. Asimptotalary $x=0$ we $y=x+3$. **313.** Funksiýanyň kesgitlenen ýaýlasy $|x|\geq1$. *Oy* okuna görä simmetrik. $x=\pm1$ bolanda, gyraky maksimum $y=2^{\sqrt{2}}\approx2,67$. Aşaklygyna güberçek. Asimptotasy $y=1$. **314.** Funksiýanyň kesgitlenen ýaýlasy $x<1$ we $x>2$. Koordinatalar oklary bilen kesişme nokatlary (0, ln2) we (1/3, 0). $x=(1-\sqrt{10})/3\approx-0,72$ bolanda, maksimum $y\approx1,12$. Asimptotalary: $x=1$, $x=2$ we $y=0$. **315.** Funksiýanyň kesgitlenen ýaýlasy $|x|\leq a$. Koordinatalar oklary bilen kesişme nokatlary $(0, -a)$ we $(0,67a, 0)$ (takmynan). Artýan funksiýa. $x=-a$ bolanda, gyraky minimum $y=-\pi a/2$ we $x=a$ bolanda, gyraky maksimum $y=\pi a/2$. Aşaklygyna güberçek. **316.** Funksiýanyň kesgitlenen ýaýlasy $x\leq0$ we $x\geq2/3$. $x=0$ bolanda, gyraky minimum $y=0$; $x=2/3$ bolanda, gyraky maksimum $y=\pi$. $x\leq0$ bolanda, ýokarlygyna güberçek we $x\geq2/3$

bolanda, aşaklygyna güberçek. Asimptotasy $y=\pi/3$. **317.** Funksiýanyň kesgitlenen ýaýlasy $x>0$. Funksiýa položitel. $x=1/e\approx0{,}368$ bolanda, minimum $y=(1/e)^{1/e}\approx0{,}692$; $x=\pm0$ bolanda, gyraky maksimum $y=1$. Aşaklygyna güberçek. **318.** Funksiýanyň kesgitlenen ýaýlasy $x>0$. $x=+0$ bolanda, gyraky minimum $y=0$; $x=e$ bolanda, maksimum $y=e^{1/e}\approx1{,}44$. Asimptotasy $y=1$. **319.** Funksiýanyň kesgitlenen ýaýlasy $x>-1$, $x\neq0$. Funksiýa položitel. Aýrylýan üzülme nokady: $x=0$. Ekstremum nokatlary ýok; kemelýän funksiýa. Aşaklygyna güberçek. Asimptotalary: $x=-1$ we $y=1$. **320.** $x>0$ bolanda funksiýa monoton. $x=+\,0$ bolanda, gyraky minimum $y=0$. Asimptotasy $y=e(x-1/2)$. **321.** Funksiýa položitel. Oy okuna görä simmetrik. Üzülme nokatlary $x=\pm1$. $x=0$ bolanda, minimum $y=e$; $x=\pm\sqrt{3}$ bolanda, maksimum $y=1/4\sqrt{e}\approx0{,}15$. Dört epin nokatlary bar. Asimptotalary: $x\to-1+0$ bolanda, $x=-1$; $x\to1-0$ bolanda, $x=1$; $x\to\infty$ bolanda, $y=0$. **322.** x we y funksiýalar otrisatel däl; $t=-1$ bolanda, $x_{\min}=0$; $t=1$ bolanda, $y_{\min}=0$. $t>-1$ bolanda aşaklygyna güberçek we $t<-1$ bolanda ýokarlygyna güberçek. **323.** Koordinatalar oklary bilen kesişýän nokatlary: $t=0$ bolanda, $(0,0)$; $t=\pm\sqrt{3}$ bolanda, $(\pm2\sqrt{3}-3,\ 0)$ we $t=2$ bolanda, $(0,-2)$. $t=1$ bolanda, $x_{\max}=1$ we $y_{\max}=2$ (dolanma nokady); $t=-1$ bolanda, $y_{\min}=-2$. $t<1$ bolanda aşaklygyna güberçek we $t>1$ bolanda, ýokarlygyna güberçek. **324.** Koordinatalar oklary bilen kesişme nokady: $t=0$ bolanda, $(0,0)$; $t=0$ bolanda, $x_{\max}=0$; $t=2$ bolanda, $x_{\min}=4$. t artanda y kemelýär. $t\approx-0{,}32$ bolanda, epin nokady $(-0{,}08;\ 0{,}3)$ (takmynan). Asimptotalary: $y=0$, $x=-1/2$ we $y=x/2-3/4$. **325.** Oy oky bilen kesişýän nokady: $t=0$ bolanda, $(0,1)$; Ox oky bilen kesişýän nokady $t=\infty$ bolanda $(-1,\ 0)$. Gyraky ekstremumlary: $t=0$ bolanda $x_{\min}=0$ we $y_{\max}=1$; $t=\infty$ bolanda, $x_{\max}=-1$ we $y_{\min}=0$. Epin nokatlary ýok. Asimptotasy $y=1/2$. $|t|>1$ bolanda aşaklygyna güberçek we $|t|<1$ bolanda, ýokarlygyna güberçek. **326.** x we y funksiýalar položitel; $t=0$ (dolanma nokady) bolanda, $x_{\min}=1$ we $y_{\min}=1$; $t<0$ bolanda, aşaklygyna güberçek we $t>0$ bolanda, ýokarlygyna güberçek. Asimptotasy $t\to+\infty$ bolanda, $y=2x$. **327.** Esasy ýaýlasy: $[0,\pi]$. Koordinatalar oklary bilen kesişýän nokatlary: $t=\pi/6$ bolanda, $(a/2,\ 0)$; $t=\pi/4$ bolanda, $(0,\ -a/\sqrt{2})$; $t=\pi/2$ bolanda, $(-a,\ 0)$; $t=3\pi/4$ bolanda $(0,\ a/\sqrt{2})$; $t=5\pi/6$ bolanda, $(a/2,\ 0)$. Ekstremumlary: $t=0$ bolanda, $x_{\max}=a$ we $y_{\max}=a$; $t=\pi/3$ bolanda, $y_{\min}=-a$; $t=\pi/2$ bolanda, $x_{\min}=-a$; $t=2\pi/3$ bolanda, $y_{\max}=a$; $t=\pi$ bolanda, $x_{\max}=a$ we $y_{\min}=-a$. $0<t<\pi/2$ bolanda, aşaklygyna güberçek, $\pi/2<t<\pi$ bolanda, ýokarlygyna güberçek. **328.** x we y otrisatel däl we periodik funksiýalar; esasy ýaýlasy $0\le t\le\pi/2$. Ekstremumlary: $t=\pi/2$ bolanda, $x_{\min}=0$ we $y_{\max}=1$; $t=0$ bolanda, $x_{\max}=1$ we $y_{\min}=0$. Aşaklygyna güberçek. **329.** Kesgitlenen ýaýlasy $t>0$. $x+y=0$ göni çyzyga görä simmetrik. Ekstremumlary: $t=1/e$ bolanda, $x_{\min}=-1/e\approx-0{,}37$, $y_{\max}=-e\approx-2{,}72$; $t=e$ bolanda, $y_{\max}=1/e$, $x=e$. Epin nokatlary: $t=e^{-\sqrt{2}}\approx0{,}24$ bolanda, $x_1=-\sqrt{2}\,e^{-\sqrt{2}}\approx-0{,}34$, $y_1=-\sqrt{2}\,e^{\sqrt{2}}\approx-5{,}82$ we $t=e^{\sqrt{2}}\approx4{,}10$ bolanda, $x_2=\sqrt{2}\,e^{\sqrt{2}}$, $y_2=\sqrt{2}\,e^{-\sqrt{2}}$. $t=1/e$ bolanda, güberçekligiň alamatynyň üýtgemegi. Asimptotalary: $x=0$ we $y=0$. **330.** x we y funksiýalar periodik, periody $T=2\pi$, esasy ýaýlasy $-\pi\le t\le\pi$. Egri çyzyk koordinatalar oklaryna görä simmetrik. Egri çyzygyň 2 sany şahasy bar. Ekstremumlary: $t=0$ bolanda, $x_{\min}=a$, $y=0$; $t=\pm\pi$ bolanda, $x_{\max}=-a$, $y=0$. $-\pi<t<-\pi/2$ we $0<t<\pi/2$ bolanda, aşaklygyna güberçek; $-\pi/2<t<0$ we $\pi/2<t<\pi$ bolanda, ýokarlygyna güberçek. **331.** Oy okuna görä simmetrik; $t=0$ bolanda, $y_{\min}=0$, $x=0$. Ýokarlygyna güberçek. **332.** Parametrik deňlemesi: $x=3at/(1+t^3)$, $y=3at^2/(1+t^3)$ $(-\infty<t<+\infty)$. $y=x$ göni çyzyga görä simmetrik. Koordinatalar oklary bilen kesişýän nokady $O(0,\ 0)$ (ikigat nokat). $y = a\sqrt[3]{2} \approx 1,2a$ bolanda, $x_{\max} = a\sqrt[3]{4} \approx 1,59a$; $x = a\sqrt[3]{2}$ bolanda, $y_{\max} = a\sqrt[3]{4}$. Asimptotasy $x+y+a=0$. **333.** Koordinatalar başlangyjyna, koordinatalar oklaryna we koordinatalar burçlarynyň bissektrisalaryna görä simmetrik. $O(0,\ 0)$ – üzňe nokady. Koordinatalar oklary bilen kesişýän nokatlary: $(\pm1,\ 0)$ we $(0,\ \pm1)$. $y=0$ bolanda, $|x|_{\min}=1$; $|\,y\,| = \sqrt{1/2} \approx 0,71$ bolanda, $|\,x\,|_{\max} = \sqrt{(1+\sqrt{2})/2} \approx 1,10$; $x=0$ bolanda, $|y|_{\min}=1$; $|\,x\,| = \sqrt{1/2}$ bolanda, $|\,y\,|_{\max} = \sqrt{(1+\sqrt{2})/2}$. **334.** Parametrik deňlemesi: $x=(1-t^3)/t^2$, $y=(1-t^3)/t$, bu ýerde $t=y/x$ $(-\infty<t<+\infty)$. Egri çyzygyň iki şahasy bar. $x+y=0$ göni çyzyga görä simmetrik. Ekstremumlary: $t=-\sqrt[3]{2} \approx -1,26$ bolanda, $x_{\min} = 3\sqrt[3]{2}/2 \approx 1,89$, $y = -3\sqrt[3]{4}/2 \approx -2,38$; $t = -\sqrt[3]{1/2} \approx -0,79$ bolanda, $y_{\max} = -3\sqrt[3]{2}/2$, $x = 3\sqrt[3]{4}/2$. Epin nokatlary: $t = -\sqrt[3]{(7+3\sqrt{5})/2} \approx -1,90$ bolanda, $x_1 \approx 2,18$, $y_1 \approx -4,14$; $t = -\sqrt[3]{(7-3\sqrt{5})/2} \approx -0,53$ bolanda, $x_2 \approx 4,14$, $y_2 \approx -2,18$; $t = -\sqrt[3]{2}$ bolanda güberçeklik ugrunyň alamatynyň üýtgemegi. **335.** Egri çyzyk $y=x$ göni çyzykdan we $x=(1+t)^{1/t}$, $y=(1+t)^{1+1/t}$ $(-1<t<+\infty)$ giperbolik şahadan ybarat. $(e,\ e)$ – ikigat nokat. $x\neq y$ bolanda aşaklygyna güberçek. Asimptotalary: $x=1$ we $y=1$. **336.** Kesgitlenen ýaýlasy: $|\,x\,| \geq \ln(1+\sqrt{2}) \approx 0,88$. Koordinatalar oklaryna görä simmetrik. $x = \pm\ln(1+\sqrt{2})$ bolanda, gyraky minimum $|y|=0$. $y>0$ bolanda, ýokarlygyna güberçek we $y<0$ bolanda, aşaklygyna güberçek. Asimptotalary: $y=x$ we $y=-x$. **337.** Funksiýanyň kesgitlenen ýaýlasy: $r\geq0$, $|\varphi|\leq\alpha$, bu ýerde $\alpha=\arccos(-a/b)$. Ýapyk egri çyzyk. Polýar okuna görä simmetrik. $\varphi=0$ bolanda, maksimum $r=a+b$; $\varphi=\pm\alpha$ bolanda, gyraky minimum $r=0$. **338.** Kesgitlenen ýaýlasy: $0\leq\varphi\leq\pi/3$; $2\pi/3\leq\varphi\leq\pi$, $4\pi/3\leq\varphi\leq5\pi/3$. Funksiýa r – periodik, periody $2\pi/3$ deň. Ýapyk egri çyzyk hem-de üç meňzeş ýaprakly. Simmetrik oklary: $\varphi=\pi/6$, $\varphi=5\pi/6$ we $\varphi=3\pi/2$. Koordinatalar başlangyjy $O(0,\ 0)$ – üçgat nokat. $0\leq\varphi\leq\pi/3$ üçin, $\varphi=\pi/6$ bolanda, maksimum $r=a$ we $\varphi=0$ we $\varphi=\pi/3$ bolanda, minimum $r=0$. **339.** Kesgitlenen ýaýlasy: $|\varphi|<\pi/6$ we $\pi/2<|\varphi|<5\pi/6$; periody $2\pi/3$. $\varphi=0$ we $\varphi=\pm2\pi/3$ bolanda, minimum $r=a$. Asimptotalary: $\varphi=\pm\pi/6$, $\varphi=\pm\pi/2$ we $\varphi=\pm5\pi/6$. **340.** Koordinatalar başlangyjy özüniň asimptotik nokady bolan spiral; φ artanda r kemelýär. Asimptotasy $\varphi=1$. **341.** Kesgitlenen ýaýlasy: $r\geq(\sqrt{5}-1)/2 \approx 0,62$. $r=(\sqrt{5}-1)/2$ bolanda, gyraky maksimum $\varphi=\pi$; $r=2$ bolanda, minimum $\varphi = \arccos(1/4) \approx \mathrm{arc}\,75^{\circ}30'$. Asimptotasy $r\to+\infty$ bolanda, $r\cos\varphi=1$. **342.** $(1,\ a-1)$ depeli parabolalar maşgalasy (minimumlar). Koordinatalar oklary bilen kesişýän nokatlary $(0,\ a)$ we $(1\mp\sqrt{1-a},\ 0)$ $(a\leq1$ bolanda). Aşaklygyna güberçek. **343.** $a\neq0$ bolanda, giperbolalar maşgalasy we $a=0$ bolanda, $y=x$ göni çyzyk. $x=|a|$ bolanda, minimumlar $y=2|a|$ we $x=-|a|$ $(a\neq0)$ bolanda, maksimumlar $y=-2|a|$. Asimptotalary $y=x$ we $x=0$. **344.** $0<a<+\infty$ bolanda, ellipsler maşgalasy; $-\infty<a<0$ bolanda, giperbolalar maşgalasy; $a=0$ bolanda, $y=x$ göni çyzyk. Hemme egri çyzyklar $(-1,\ -1)$ we $(1,\ 1)$ nokatlar arkaly geçýärler. $y\geq x$ üçin 1) $x = 1/\sqrt{1+a}$ we $a>0$ bolanda, maksimum $y=\sqrt{1+a}$; $x=-1/\sqrt{1+a}$ we $-1<a<0$ bolanda, maksimum $y=-\sqrt{1+a}$; $x=\mp1$ $(a\neq0)$ bolanda, gyraky minimumlar $y=\mp1$. 2) ýokarlygyna güberçek. $y\leq x$ üçin 1) $x=-1/\sqrt{1+a}$ we $a>0$ bolanda, minimum

$y=-\sqrt{1+a}$; $x=1/\sqrt{1+a}$ we $-1<a<0$ bolanda, minimum $y=\sqrt{1+a}$; $x=\mp 1$ bolanda, gyraky maksimumlar $y=\mp 1$; 2) Aşaklygyna güberçek. $a<0$ bolanda, asimptotalary: $y=(1+\sqrt{-a})x$ we $y=(1-\sqrt{-a})x$. **345.** $a\neq 0$ bolanda, görkezijili egri çyzyklar; $a=0$ bolanda, $y=1+x/2$ göni çyzyk. Egri çyzyklaryň umumy nokady (0, 1). $x=(\ln 2a)/a$ we $a>0$ bolanda, minimumlary $y=(1+\ln 2a)/2a$; $a\leq 0$ bolanda y artýar. Asimptotasy $y=x/2$. **346.** (0, 0) nokatdan geçýän we şol nokatda $y=x$ göni çyzyk bilen umumy galtaşýany bolan egri çyzyklar. $x=a$ we $a>0$ bolanda, maksimum $y=ae^{-1}\approx 0{,}37a$; $x=a$ we $a<0$ bolanda, minimum $y=ae^{-1}$. Epin nokady $x=2a$, $y=2ae^{-2}\approx 0{,}27a$. Asimptotasy $y=0$.

§7. Funksiýalaryň maksimumlaryny we minimumlaryny tapmaklyga degişli meseleler

349. $\dfrac{a^{m+n}m^m n^n}{(m+n)^{m+n}}$. **350.** $(m+n)\left(\dfrac{a^{mn}}{m^m n^n}\right)^{\frac{1}{m+n}}$. **351.** Logarifmik sistemalarynyň esasy $e^{1/e}\approx 1{,}445$-den uly bolmaly däl. **352.** $\sqrt{S}$ taraply kwadrat. **353.** Üçburçlugyň ýiti burçlary 30º we 60º. **354.** Bankanyň $H=2\sqrt[3]{V/2\pi}$ beýikligi onuň esasynyň diametrine deňdir; doly üsti $P=\sqrt[3]{54\pi V^2}$. **355.** $\cos\varphi=\dfrac{\cos\alpha+\sqrt{\cos^2\alpha+8}}{4}$, bu ýerde 2α – segmentiň dugasy we 2φ – gönüburçlugyň tarapy bilen dartylýan duga. **356.** Gönüburçlugyň taraplary $a\sqrt{2}$ we $b\sqrt{2}$. **357.** Eger $h>b$ bolsa, onda esasy x we beýikligi y bolan içinden çyzylan gönüburçlugyň P perimetriniň $y=h$ bolanda gyraky maksimumy bardyr; eger $h<b$ bolsa, onda P-iň $y=0$ bolanda gyraky maksimumy bardyr; eger-de $h=b$ bolsa, onda perimetr P hemişelikdir. **358.** $b=d/\sqrt{3}$, $h=d\sqrt{2/3}$. **359.** Parallelepipediň ölçegleri $2R/\sqrt{3}$, $2R/\sqrt{3}$ we $R/\sqrt{3}$. **360.** $4\pi R^3/3\sqrt{3}$. **361.** Şaryň üstüniň $\pi R^2(1+\sqrt{5})\approx 81\%$ bölegi. **362.** Konusyň göwrümi şaryň göwrüminiň iki essesine deňdir. **363.** $2\pi l^3/9\sqrt{3}$. **364.** Eger $\mathrm{tg}\alpha<1/2$ bolsa, onda silindriň doly üsti maksimum bahasyna $r=R/2(1-\mathrm{tg}\alpha)$ bolanda ýetýär, bu ýerde r – silindriň esasynyň radiusy. Eger $\mathrm{tg}\alpha\geq 1/2$ bolsa, onda $r=R$ bolanda gyraky maksimumy alýar. **365.** $p(\sqrt[3]{2}-1)\sqrt{(2+\sqrt[3]{2})/2}$. **366.** 1; 3. **367.** Eger $b\leq a/\sqrt{2}$ bolsa, onda hordanyň $MB=a^2/c$ ($M=M(x,\ y)$) uzynlygy maksimum bahany $x=\pm a^2\sqrt{a^2-2b^2}/c^2$; $y=b^3/c^2$ bolanda alýar, bu ýerde $c=\sqrt{a^2-b^2}$; eger $b>a/\sqrt{2}$ bolsa, onda hordanyň $MB=2b$ uzynlygy $x=0$, $y=b$ bolanda, gyraky maksimumy alýar. **368.** $x=a/\sqrt{2}$, $y=b/\sqrt{2}$; ab. **369.** Üst iň kiçi bahany $r=h=\sqrt[3]{3V/5\pi}$ bolanda alýar, bu ýerde r – silindriň esasynyň radiusy we h – onuň beýikligi. **370.** $\varphi=60º$. **371.** Töweregiň daşyndan çyzylan trapesiýa. Gapdal taraplary $AB=CD=a\sec^2(\alpha/2)$. **372.** $\alpha=2\pi\sqrt{2/3}\approx \mathrm{arc}294°$, bu ýerde α – galan sektoryň merkezi burçy. **373.** Eger $\arccos(q/p)\geq \mathrm{arctg}(a/b)$ bolsa, onda $\varphi=\arccos(q/p)$, eger $\arccos(q/p)<\mathrm{arctg}(a/b)$ bolsa, onda $\varphi=\mathrm{arctg}(a/b)$. **374.** $\dfrac{|a\vartheta\mp bu|\sin\theta}{\sqrt{u^2+\vartheta^2-2u\vartheta\cos\theta}}$. **375.** $AK=a(1+\sqrt[3]{S_2/S_1})^{-1}$. **376.** $a\geq r+R\sqrt{R/r}$ bolanda, uly şaryň merkezinden ýalpyldaýan nokada çenli uzaklyk $x=a/(1+(r/R)^{3/2})$, eger $r+R<a<r+R\sqrt{R/r}$ bolsa, onda $x=a-r$, bu ýerde – a şarlaryň arasyndaky uzaklyk. **377.** $a/\sqrt{2}$. **378.** $(a^{2/3}+b^{2/3})^{3/2}$. **379.** $\vartheta=\sqrt[3]{a/2k}$, bu ýerde k – proporsionallyk koeffisiýenti. **380.** arctg$k$. **381.** $l\le 4a$ bolanda, sterženiň gyşarma burçy $\cos\alpha=\left(l+\sqrt{l^2+128a^2}\right)/16a$ formula boýunça hasaplanýar; $l>4a$ bolanda, deňagramlylyk ýagdaýy ýok.

§8. Egri çyzyklaryň galtaşmasy. Egriligiň tegelegi. Ewolýuta

382. $k=-3$; $b=3$; $y=3(1-x)$. **383.** $a=e^{x_0}/2$; $b=e^{x_0}(1-x_0)$; $c=e^{x_0}\left(1-x_0+x_0^2/2\right)$. **384.** a) birinji; b) ikinji; ç) ikinji. **386.** a) $\sqrt{2}$, (2, 2); b) 500 000, (150, 500 000) (takmynan). **387.** $p(1+2x/p)^{3/2}$. **388.** $\dfrac{(a^2-\varepsilon^2x^2)^{3/2}}{ab}$, bu ýerde $\varepsilon=\dfrac{\sqrt{a^2-b^2}}{a}$ – ellipsiň ekssentrisiteti. **389.** $\dfrac{(\varepsilon^2x^2-a^2)^{3/2}}{ab}$, bu ýerde $\varepsilon=\dfrac{\sqrt{a^2+b^2}}{a}$ – giperbolanyň ekssentrisiteti. **390.** $3|axy|^{\frac{1}{3}}$. **391.** $\dfrac{a^2}{b}(1-\varepsilon^2\cos^2 t)^{\frac{3}{2}}$, bu ýerde ε – ellipsiň ekssentrisiteti. **392.** $2\sqrt{2ay}$. **393.** at. **395.** $\dfrac{(r^2+r'^2)^{3/2}}{|r^2+2r'^2-rr''|}$. **396.** $\dfrac{(a^2+r^2)^{3/2}}{2a^2+r^2}$. **397.** $r\sqrt{1+m^2}$. **398.** $2\sqrt{2ar}/3$. **399.** $a^2/3r$. **400.** $(1/\sqrt{2},-\ln 2/2)$. **401.** $x_0\approx 680\,m$. **402.** Ýarymkubik parabola $27p\eta^2=8(\xi-p)^3$. **403.** Astroida $(a\xi)^{2/3}+(b\eta)^{2/3}=c^{4/3}$, bu ýerde $c^2=a^2-b^2$. **404.** Astroida $(\xi+\eta)^{2/3}+(\xi-\eta)^{2/3}=2a^{2/3}$. **405.** Zynjyr çyzygy $\eta=a\,\mathrm{ch}(\xi/a)$. **406.** Logarifmik spiraly $\rho=mae^{m(\varphi-\pi/2)}$. **407.** $\xi=\pi a+a(\tau-\sin\tau)$; $\eta=-2a+a(1-\cos\tau)$, bu ýerde $\tau=t-\pi$.

§ 9. Deňlemeleriň takmyny çözüwi

408. $x_1=-2{,}602$; $x_2=0{,}340$; $x_3=2{,}262$. **409.** $x_1=-0{,}724$; $x_2=1{,}221$. **410.** $x=2{,}087=$ $=$ arc119º35′ **411.** ± 0,824. **412.** $x_1=0{,}472$; $x_2=9{,}999$. **413.** $x_1=2{,}5062$. **414.** $x_1=4{,}730$; $x_2=7{,}853$. **415.** $x=-0{,}56715$. **416.** $x=\pm 1{,}199678$. **417.** $x_1=4{,}493$; $x_2=7{,}725$; $x_3=10{,}904$. **418.** $x_1=2{,}081$; $x_2=5{,}940$.

VI. §1. Kesgitsiz integral we integrirlemek usullary

Şu bölümdäki jogaplarda gysgalyk üçin hemişelik C ýazylmady.

1. $27x-9x^3+\frac{9}{5}x^5-\frac{1}{7}x^7$. **2.** $\frac{625}{3}x^3-125x^4+30x^5-\frac{10}{3}x^6+\frac{1}{7}x^7$. **3.** $x-3x^2+$ $+\frac{11}{3}x^3-\frac{3}{2}x^4$. **4.** $x-\frac{1}{x}-2\ln|x|$. **5.** $a\ln|x|-\frac{a^2}{x}-\frac{a^3}{2x^2}$. **6.** $\frac{2}{3}x\sqrt{x}+2\sqrt{x}$. **7.** $\frac{4}{5}x^4\sqrt{x}-$ $-\frac{24}{17}x\sqrt[12]{x^5}+\frac{4}{3}\sqrt[4]{x^3}$. **8.** $-\frac{3}{\sqrt[3]{x}}\left(1+\frac{3}{2}x-\frac{3}{5}x^2+\frac{1}{8}x^3\right)$. **9.** $\frac{4(x^2+7)}{7\sqrt[4]{x}}$. **10.** $2x-\frac{12}{5}\times$

$\times\sqrt[6]{72x^5}+\frac{3}{2}\sqrt[3]{9x^2}$. **11.** $\ln|x|-\frac{1}{4x^4}$. **12.** $x-\operatorname{arctg}x$. **13.** $-x+\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$. **14.** $x+2\times$
$\times\ln\left|\frac{x-1}{x+1}\right|$. **15.** $\arcsin x+\ln(x+\sqrt{1+x^2})$. **16.** $\ln\left|\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2+1}}\right|$. **17.** $\frac{4^x}{\ln 4}+2\frac{6^x}{\ln 6}+$
$+\frac{9^x}{\ln 9}$. **18.** $-\frac{2}{\ln 5}\left(\frac{1}{5}\right)^x+\frac{1}{5\ln 2}\left(\frac{1}{2}\right)^x$. **19.** $\frac{1}{2}e^{2x}-e^x+x$. **20.** $x-\cos x+\sin x$. **21.** $2\sqrt{2}\left[\frac{t}{\pi}\right]+$
$+\sqrt{2}\operatorname{sgn}t\left\{\cos\frac{t}{\pi}-\cos t\right\}$, bu ýerde $t=x-\frac{\pi}{4}$ ([ ] – bitin bölegi aňladýar). **22.** $-x-\operatorname{ctg}x$.
23. $-x+\operatorname{tg}x$. **24.** $a\operatorname{ch}x+b\operatorname{sh}x$. **25.** $x-\operatorname{th}x$. **26.** $x-\operatorname{cth}x$. **28.** $\ln|x+a|$. **29.** $\frac{1}{22}(2x-3)^{11}$.
30. $-\frac{1}{4}(1-3x)^{\frac{4}{3}}$. **31.** $-\frac{2}{5}\sqrt{2-5x}$. **32.** $-\frac{2}{15(5x-2)^{3/2}}$. **33.** $-\frac{5}{2}\sqrt[5]{(1-x)^2}$. **34.** $\frac{1}{\sqrt{6}}\times$
$\times\operatorname{arctg}\left(x\sqrt{\frac{3}{2}}\right)$. **35.** $\frac{1}{2\sqrt{6}}\ln\left|\frac{\sqrt{2}+x\sqrt{3}}{\sqrt{2}-x\sqrt{3}}\right|$. **36.** $\frac{1}{\sqrt{3}}\arcsin\left(x\sqrt{\frac{3}{2}}\right)$. **37.** $\frac{1}{\sqrt{3}}\ln|x\sqrt{3}+$
$+\sqrt{3x^2-2}\,|$. **38.** $-\left(e^{-x}+\frac{1}{2}e^{-2x}\right)$. **39.** $-x\sin 5\alpha-\frac{1}{5}\cos 5x$. **40.** $-\frac{1}{2}\operatorname{ctg}\left(2x+\frac{\pi}{4}\right)$.
41. $\operatorname{tg}\frac{x}{2}$. **42.** $-\operatorname{ctg}\frac{x}{2}$. **43.** $-\operatorname{tg}\left(\frac{\pi}{4}-\frac{x}{2}\right)$. **44.** $\frac{1}{2}[\operatorname{ch}(2x+1)+\operatorname{sh}(2x-1)]$. **45.** $2\operatorname{th}\frac{x}{2}$.
46. $-2\operatorname{cth}\frac{x}{2}$. **47.** $-\sqrt{1-x^2}$. **48.** $\frac{1}{4}(1+x^3)^{\frac{4}{3}}$. **49.** $-\frac{1}{4}\ln|3-2x^2|$. **50.** $-\frac{1}{2(1+x^2)}$.
51. $\frac{1}{4}\operatorname{arctg}\frac{x^2}{2}$. **52.** $\frac{1}{8\sqrt{2}}\ln\left|\frac{x^4-\sqrt{2}}{x^4+\sqrt{2}}\right|$. **53.** $2\operatorname{arctg}\sqrt{x}$. **54.** $\cos\frac{1}{x}$. **55.** $-\ln\left|\frac{1+\sqrt{x^2+1}}{x}\right|$.
56. $-\arcsin\frac{1}{|x|}$. **57.** $\frac{x}{\sqrt{x^2+1}}$. **58.** $-\frac{1}{\sqrt{x^2-1}}$. **59.** $\frac{1}{8}\sqrt[3]{8x^3+27}$. **60.** $2\operatorname{sgn}x\ln(\sqrt{|x|}+$
$+\sqrt{|1+x|})$ $(x(1+x)>0)$. **61.** $2\arcsin\sqrt{x}$. **62.** $-\frac{1}{2}e^{-x^2}$. **63.** $\ln(2+e^x)$. **64.** $\operatorname{arctg}e^x$.
65. $-\ln(e^{-x}+\sqrt{1+e^{-2x}})$. **66.** $\frac{1}{3}\ln^3 x$. **67.** $\ln|\ln(\ln x)|$. **68.** $\frac{1}{6}\sin^6 x$. **69.** $\frac{2}{\sqrt{\cos x}}$.
70. $-\ln|\cos x|$. **71.** $\ln|\sin x|$. **72.** $\frac{3}{2}\sqrt[3]{1-\sin 2x}$. **73.** $\frac{\sqrt{a^2\sin^2 x+b^2\cos^2 x}}{a^2-b^2}$ $(a^2\neq b^2)$. **74.** $-\frac{1}{\sqrt{2}}\times$
$\times\ln|\sqrt{2}\cos x+\sqrt{\cos 2x}\,|$. **75.** $\frac{1}{\sqrt{2}}\arcsin(\sqrt{2}\sin x)$. **76.** $\frac{1}{\sqrt{2}}\ln(\sqrt{2}\operatorname{ch}x+\sqrt{\operatorname{ch}2x})$.
77. $-\frac{4}{3}\sqrt[4]{\operatorname{ctg}^3 x}$. **78.** $\frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{\operatorname{tg}x}{\sqrt{2}}\right)$. **79.** $\ln|\operatorname{tg}\frac{x}{2}|$. **80.** $\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$. **81.** $\ln|\operatorname{th}\frac{x}{2}|$.
82. $2\operatorname{arctg}e^x$. **83.** $\frac{1}{2\sqrt{2}}\ln\left(\frac{\operatorname{ch}2x}{\sqrt{2}}+\sqrt{\operatorname{sh}^4 x+\operatorname{ch}^4 x}\right)$. **84.** $3\sqrt[3]{\operatorname{th}x}$. **85.** $\frac{1}{2}(\operatorname{arctg}x)^2$.
86. $-\frac{1}{\arcsin x}$. **87.** $\frac{2}{3}\ln^{\frac{3}{2}}(x+\sqrt{1+x^2})$. **88.** $\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x^2-1}{x\sqrt{2}}$. **89.** $\frac{1}{2\sqrt{2}}\ln\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}$.
90. $-\frac{1}{15(x^5+1)^3}$. **91.** $\frac{2}{n+2}\ln(x^{\frac{n+2}{2}}+\sqrt{1+x^{n+2}})$ eger, $n\neq-2$; $\frac{1}{\sqrt{2}}\ln|x|$ eger, $n=-2$.

92. $\frac{1}{4}\ln^2\frac{1+x}{1-x}$. **93.** $\frac{1}{\sqrt{2}}\arcsin\left(\sqrt{\frac{2}{3}}\sin x\right)$. **94.** $\frac{1}{2}\operatorname{arctg}(\operatorname{tg}^2 x)$. **95.** $\frac{1}{2(\ln 3-\ln 2)}\times$ $\times\ln\left|\frac{3^x-2^x}{3^x+2^x}\right|$. **96.** $2\sqrt{1+\sqrt{1+x^2}}$. **97.** $\frac{4}{3}x^3-\frac{12}{5}x^5+\frac{9}{7}x^7$. **98.** $-\frac{(1-x)^{11}}{11}+\frac{(1-x)^{12}}{12}$.

99. $-x-2\ln|1-x|$. **100.** $\frac{1}{2}(1-x)^2+\ln|1+x|$. **101.** $9x-\frac{3}{2}x^2+\frac{1}{3}x^3-27\ln|3+x|$.

102. $x+\ln(1+x^2)$. **103.** $\frac{3}{\sqrt{2}}\ln\left|\frac{\sqrt{2}+x}{\sqrt{2}-x}\right|+2\ln|2-x^2|-x$. **104.** $\frac{1}{99(1-x)^{99}}-\frac{1}{49}\times$ $\times\frac{1}{(1-x)^{98}}+\frac{1}{97(1-x)^{97}}$. **105.** $\frac{x^4}{5}-\frac{x^4}{4}+\frac{x^3}{3}-\frac{x^2}{2}+x-\ln|x+1|$. **106.** $\frac{1}{3}\left[(x+1)^{\frac{3}{2}}-\right.$ $\left.-(x-1)^{\frac{3}{2}}\right]$. **107.** $-\frac{8+30x}{375}(2-5x)^{\frac{3}{2}}$. **108.** $-\frac{1+2x}{10}(1-3x)^{\frac{2}{3}}$. **109.** $\frac{3}{14}(1+x^2)^{\frac{7}{3}}-\frac{3}{8}\times$ $\times(1+x^2)^{\frac{4}{3}}$. **110.** $\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right|$. **111.** $\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|$. **112.** $\operatorname{arctg}x-\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}$. **113.** $\frac{1}{10}\times$ $\times\frac{1}{\sqrt{2}}\ln\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right|-\frac{1}{5\sqrt{3}}\operatorname{arctg}\frac{x}{\sqrt{3}}$. **114.** $\ln\frac{|x+3|^3}{(x+2)^2}$. **115.** $\frac{1}{2}\ln\frac{x^2+1}{x^2+2}$. **116.** $-\frac{1}{(a-b)^2}\times$ $\times\frac{2x+a+b}{(x+a)(x+b)}+\frac{2}{(a-b)^3}\ln\left|\frac{x+a}{x+b}\right|$. **117.** $\frac{1}{a^2-b^2}\left(\frac{1}{b}\operatorname{arctg}\frac{x}{b}-\frac{1}{a}\operatorname{arctg}\frac{x}{a}\right)$ $(|a|\neq|b|)$.

118. $\frac{x}{2}-\frac{1}{4}\sin 2x$. **119.** $\frac{x}{2}+\frac{1}{4}\sin 2x$. **120.** $\frac{x}{2}\cos\alpha-\frac{1}{4}\sin(2x+\alpha)$. **121.** $\frac{1}{4}\sin 2x-\frac{1}{16}\times$ $\times\sin 8x$. **122.** $3\sin\frac{x}{6}+\frac{3}{5}\sin\frac{5x}{6}$. **123.** $-\frac{1}{10}\cos\left(5x+\frac{\pi}{12}\right)+\frac{1}{2}\cos\left(x+\frac{5\pi}{12}\right)$. **124.** $-\cos x+$ $+\frac{1}{3}\cos^3 x$. **125.** $\sin x-\frac{1}{3}\sin^3 x$. **126.** $\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x$. **127.** $\frac{3}{8}x+\frac{1}{4}\sin 2x+$ $+\frac{1}{32}\sin 4x$. **128.** $-x-\operatorname{ctg}x$. **129.** $\frac{1}{2}\operatorname{tg}^2x+\ln|\cos x|$. **130.** $-\frac{3}{16}\cos 2x-\frac{3}{64}\cos 4x+$ $+\frac{1}{48}\cos 6x+\frac{3}{128}\cos 8x+\frac{1}{192}\cos 12x$. **131.** $\operatorname{tg}x-\operatorname{ctg}x$. **132.** $-\frac{1}{\sin x}+\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$.

133. $\frac{1}{2\cos^2 x}+\ln|\operatorname{tg}x|$. **134.** $\ln|\sin x|-\frac{1}{2}\sin^2 x$. **135.** $\operatorname{tg}x+\frac{1}{3}\operatorname{tg}^3x$. **136.** $x-\ln(1+e^x)$.
137. $x+2\operatorname{arctg}e^x$. **138.** $-\frac{x}{2}+\frac{1}{4}\operatorname{sh}2x$. **139.** $\frac{x}{2}+\frac{1}{4}\operatorname{sh}2x$. **140.** $\frac{2}{3}\operatorname{sh}^3x$. **141.** $\frac{1}{4}\operatorname{sh}2x+$ $+\frac{1}{8}\operatorname{sh}4x$. **142.** $-(\operatorname{th}x+\operatorname{cth}x)$. **143.** $-\frac{3}{140}(9+12x+14x^2)(1-x)^{4/3}$. **144.** $-\frac{1+55x^2}{6600}\times$ $\times(1-5x^2)^{11}$. **145.** $-\frac{2}{15}(32+8x+3x^2)\sqrt{2-x}$. **146.** $-\frac{1}{15}(8+4x^2+3x^4)\sqrt{1-x^2}$.

147. $-\frac{6+25x^3}{1000}(2-5x^3)^{5/3}$. **148.** $\left(\frac{2}{3}-\frac{4}{7}\sin^2 x+\frac{2}{11}\sin^4 x\right)\sqrt{\sin^3 x}$. **149.** $-\frac{1}{2}\cos^2 x+$

$+\frac{1}{2}\ln\left(1+\cos^2 x\right)$. **150.** $\frac{1}{3}\mathrm{tg}^3 x+\frac{1}{5}\mathrm{tg}^5 x$. **151.** $\frac{2}{3}(-2+\ln x)\sqrt{1+\ln x}$. **152.** $-x-2e^{-x/2}+$ $+2\ln(1+e^{x/2})$. **153.** $x-2\ln(\sqrt{1+e^x})$. **154.** $(\mathrm{arctg}\sqrt{x})^2$. **155.** $\frac{x}{\sqrt{1-x^2}}$. **156.** $\frac{x}{2}\sqrt{x^2-2}+$ $+\ln\left|x+\sqrt{x^2-2}\right|$. **157.** $\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\arcsin x$. **158.** $\frac{x}{a^2\sqrt{a^2+x^2}}$. **159.** $-\sqrt{a^2-x^2}+$ $+a\arcsin\frac{x}{a}$. **160.** $-\frac{3a+x}{2}\sqrt{x(2a-x)}+3a^2\arcsin\sqrt{\frac{x}{2a}}$. **161.** $2\arcsin\sqrt{\frac{x-a}{b-a}}$. **162.** $\frac{2x-(a+b)}{4}\sqrt{(x-a)(b-x)}+\frac{(b-a)^2}{4}\arcsin\sqrt{\frac{x-a}{b-a}}$. **163.** $\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\times$ $\times\ln\left(x+\sqrt{a^2+x^2}\right)$. **164.** $\frac{x}{2}\sqrt{a^2+x^2}-\frac{a^2}{2}\ln(x+\sqrt{a^2+x^2})$. **165.** $\sqrt{x^2-a^2}-2a\times$ $\times\ln\left(\sqrt{x-a}+\sqrt{x+a}\right)$, eger $x>a$ bolsa; $-\sqrt{x^2-a^2}+2a\ln(\sqrt{-x+a}+\sqrt{-x-a})$, eger-de, $x<-a$ bolsa. **166.** $2\ln(\sqrt{x+a}+\sqrt{x+b})$, eger, $x+a>0$ we $x+b>0$; $-2\ln(\sqrt{-x-a}+\sqrt{-x-b})$, eger-de $x+a<0$ we $x+b<0$. **167.** $\frac{2x+a+b}{4}\sqrt{x+a}\times$ $\times\sqrt{x+b}-\frac{(b-a)^2}{4}\ln(\sqrt{x+a}+\sqrt{x+b})$, eger, $x+a>0$ we $x+b>0$. **168.** $x(\ln x-1)$.

169. $\frac{x^{n+1}}{n+1}\left(\ln x-\frac{1}{n+1}\right)$, $(n\neq -1)$. **170.** $-\frac{1}{x}(\ln^2 x+2\ln x+2)$. **171.** $\frac{2}{3}x^{3/2}(\ln^2 x-$ $-\frac{4}{3}\ln x+\frac{8}{9}\Big)$. **172.** $-(x+1)e^{-x}$. **173.** $-\frac{e^{-2x}}{2}\left(x^2+x+\frac{1}{2}\right)$. **174.** $-\frac{x^2+1}{2}e^{-x^2}$. **175.** $x\sin x+$ $+\cos x$. **176.** $-\frac{2x^2-1}{4}\cos 2x+\frac{x}{2}\sin 2x$. **177.** $x\,\mathrm{ch}x-\mathrm{sh}x$. **178.** $(x^3/3+2x/9)\mathrm{sh}3x-$ $-(x^2/3+2/27)\mathrm{ch}3x$. **179.** $x\,\mathrm{arctg}x-\frac{1}{2}\ln(1+x^2)$. **180.** $x\arcsin x+\sqrt{1-x^2}$. **181.** $-\frac{x}{2}+$ $+\frac{1+x^2}{2}\mathrm{arctg}x$. **182.** $-\frac{2+x^2}{9}\sqrt{1-x^2}+\frac{x^3}{3}\arccos x$. **183.** $-\frac{\arcsin x}{x}-\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|$.

184. $x\ln(x+\sqrt{1+x^2})-\sqrt{1+x^2}$. **185.** $x-\frac{1-x^2}{2}\ln\frac{1+x}{1-x}$. **186.** $-\sqrt{x}+(1+x)\times$ $\times\mathrm{arctg}\sqrt{x}$. **187.** $\ln\left|\mathrm{tg}\frac{x}{2}\right|-\cos x\ln\mathrm{tg}x$. **188.** $\frac{1}{3}(x^3-1)e^{x^3}$. **189.** $x(\arcsin x)^2+2\times$ $\times\sqrt{1-x^2}\arcsin x-2x$. **190.** $\frac{1+x^2}{2}(\mathrm{arctg}x)^2-x\,\mathrm{arctg}x+\frac{1}{2}\ln(1+x^2)$. **191.** $-\frac{1}{3}x^2-$ $-\frac{1}{3}\ln\left|1-x^2\right|+\frac{x^3}{3}\ln\left|\frac{1-x}{1+x}\right|$. **192.** $\sqrt{1+x^2}\ln(x+\sqrt{1+x^2})-x$. **193.** $-\frac{x}{2(1+x^2)}+$ $+\frac{1}{2}\mathrm{arctg}x$. **194.** $\frac{x}{2a^2(a^2+x^2)}+\frac{1}{2a^3}\mathrm{arctg}\frac{x}{a}$ $(a\neq 0)$. **195.** $\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{|a|}$, $(a\neq 0)$. **196.** $\frac{x}{2}\sqrt{x^2+a}+\frac{a}{2}\ln\left|x+\sqrt{x^2+a}\right|$. **197.** $\frac{x(2x^2+a^2)}{8}\sqrt{a^2+x^2}-\frac{a^4}{8}\ln(x+$

$+\sqrt{a^2+x^2}\big)$. **198.** $\frac{x^2}{4}-\frac{x}{4}\sin 2x-\frac{\cos 2x}{8}$. **199.** $2(\sqrt{x}-1)e^{\sqrt{x}}$. **200.** $2(6-x)\sqrt{x}\times$ $\times\cos\sqrt{x}-6(2-x)\sin\sqrt{x}$. **201.** $-\frac{(1-x)e^{\operatorname{arctg}x}}{2\sqrt{1+x^2}}$. **202.** $\frac{(1+x)e^{\operatorname{arctg}x}}{2\sqrt{1+x^2}}$. **203.** $\frac{x}{2}[\sin(\ln x)-$ $-\cos(\ln x)]$. **204.** $\frac{x}{2}[\sin(\ln x)+\cos(\ln x)]$. **205.** $\frac{a\cos bx+b\sin bx}{a^2+b^2}e^{ax}$. **206.** $\frac{1}{a^2+b^2}\times$ $\times(a\sin bx-b\cos bx)e^{ax}$. **207.** $\frac{e^{2x}}{8}(2-\sin 2x-\cos 2x)$. **208.** $\frac{x}{2}+\frac{1}{4}\sin 2x-e^x(\cos x+$ $+\sin x)+\frac{1}{2}e^{2x}$. **209.** $-x+\frac{1}{2}\ln(1+e^{2x})-e^{-x}\operatorname{arcctg}(e^x)$. **210.** $-[x+\operatorname{ctg}x\ln(e\sin x)]$.

211. $x\operatorname{tg}x+\ln|\cos x|$. **212.** $\frac{e^x}{x+1}$. **213.** $\frac{1}{\sqrt{ab}}\operatorname{arctg}(x\sqrt{b/a})$, eger $ab>0$; $\frac{\operatorname{sgn}a}{2\sqrt{-ab}}\times$ $\times\ln\left|\frac{\sqrt{|a|}+x\sqrt{|b|}}{\sqrt{|a|}-x\sqrt{|b|}}\right|$, eger-de $ab<0$. **214.** $\frac{2}{\sqrt{7}}\operatorname{arctg}\frac{2x-1}{\sqrt{7}}$. **215.** $\frac{1}{4}\ln\left|\frac{x-1}{3x+1}\right|$. **216.** $\frac{1}{4}\times$ $\times\frac{1}{\sqrt{2}}\ln\left|\frac{x^2-(\sqrt{2}+1)}{x^2+(\sqrt{2}-1)}\right|$. **217.** $\frac{1}{2}\ln(x^2+x+1)+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}$. **218.** $\frac{1}{2}\ln(x^2-$ $-2x\cos\alpha+1)+\operatorname{ctg}\alpha\operatorname{arctg}\frac{x-\cos\alpha}{\sin\alpha}$ ($\alpha\neq k\pi$, k – bitin san). **219.** $\frac{1}{4}\ln(x^4-x^2+2)+$ $+\frac{1}{2\sqrt{7}}\cdot\operatorname{arctg}\frac{2x^2-1}{\sqrt{7}}$. **220.** $\frac{1}{9}\ln\{|x^3+1|\cdot(x^3-2)^2\}$. **221.** $\frac{1}{2}\ln\left|\frac{3\sin x-5\cos x}{\sin x-\cos x}\right|$.

222. $\operatorname{arctg}\frac{(\operatorname{tg}(x/2)+1)}{2}$. **223.** $\frac{1}{\sqrt{b}}\ln(x\sqrt{b}+\sqrt{a+bx^2})$, eger $b>0$; $\frac{1}{\sqrt{-b}}\arcsin(x\times$ $\times\sqrt{-b/a})$ eger $a>0$ we $b<0$. **224.** $\arcsin\frac{x+1}{\sqrt{2}}$. **225.** $\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|$. **226.** $\frac{1}{\sqrt{2}}\times$ $\times\ln\left(x-\frac{1}{4}+\sqrt{x^2-\frac{1}{2}x+1}\right)$. **228.** $-\sqrt{5+x-x^2}+\frac{1}{2}\arcsin\frac{2x-1}{\sqrt{21}}$. **229.** $\frac{1}{2}\ln(x+$ $+\frac{1}{2}+\sqrt{x^2+x+1})+\sqrt{x^2+x+1}$. **230.** $\frac{1}{2\sqrt{2}}\arcsin\frac{4x^2+3}{\sqrt{17}}$. **231.** $\arcsin\frac{2\sin x-1}{3}$.

232. $\frac{1}{2}\ln|x^2-1+\sqrt{x^4-2x^2-1}|+\frac{1}{2}\sqrt{x^4-2x^2-1}$. **233.** $-\frac{1}{2}\sqrt{1+x^2-x^4}+\frac{3}{4}\times$ $\times\arcsin\frac{2x^2-1}{\sqrt{5}}$. **234.** $-\ln\left|\frac{x+2+2\sqrt{x^2+x+1}}{x}\right|$. **235.** $\frac{\sqrt{x^2+x-1}}{x}+\frac{1}{2}\arcsin\frac{x-2}{|x|\sqrt{5}}$ $\left(\left|x+\frac{1}{2}\right|>\frac{\sqrt{5}}{2}\right)$. **236.** $-\frac{1}{\sqrt{2}}\ln\left|\frac{1-x+\sqrt{2(1+x^2)}}{1+x}\right|$. **237.** $\arcsin\left(\frac{1}{\sqrt{2}}\frac{x-2}{|x-1|}\right)$ $(|x|>$ $>\sqrt{2})$. **238.** $\frac{1}{5}\frac{\sqrt{x^2+2x-5}}{x+2}+\frac{1}{5\sqrt{5}}\arcsin\frac{x+7}{|x+2|\sqrt{6}}$ $(|x+1|>\sqrt{6})$. **239.** $\frac{2x-1}{4}\times$ $\times\sqrt{2+x-x^2}+\frac{9}{8}\arcsin\frac{2x-1}{3}$. **240.** $\frac{7}{8}\ln\left(\frac{1}{2}+x+\sqrt{2+x+x^2}\right)+\sqrt{2+x+x^2}\,\frac{2x+1}{4}$.

241. $-\frac{1}{2}\ln|x^2+1+\sqrt{x^4+2x^2-1}|+\frac{x^2+1}{4}\sqrt{x^4+2x^2-1}$. **242.** $-\sqrt{1+x-x^2}+\frac{1}{2}\times$ $\times\arcsin\frac{1-2x}{\sqrt{5}}-\ln\left|\frac{2+x+2\sqrt{1+x-x^2}}{2}\right|\left(\left|x-\frac{1}{2}\right|<\frac{\sqrt{5}}{2}\right)$. **243.** $\ln\left|\frac{x^2-1+\sqrt{x^4+1}}{x}\right|$.

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244. $\ln|x-2|+\ln|x+5|$. **245.** $\frac{1}{2}\ln\left|\frac{(x+2)^4}{(x+1)(x+3)^3}\right|$. **246.** $\frac{x^9}{9}-\frac{x^8}{8}+\frac{3x^7}{7}-\frac{5x^6}{6}+$ $+\frac{11x^5}{5}-\frac{21x^4}{4}+\frac{43x^3}{3}-\frac{85x^2}{2}+171x+\frac{1}{3}\ln\left|\frac{x-1}{(x+2)^{1024}}\right|$. **247.** $x+\frac{1}{6}\ln|x|-\frac{9}{2}\ln|x-$ $-2|+\frac{28}{3}\ln|x-3|$. **248.** $x+\frac{1}{3}\operatorname{arctg}x-\frac{8}{3}\operatorname{arctg}\frac{x}{2}$. **249.** $-\frac{1}{3(x-1)}+\frac{2}{9}\ln\left|\frac{x-1}{x+2}\right|$. **250.** $\frac{1}{x+1}+\frac{1}{2}\ln|x^2-1|$. **251.** $-\frac{5x-6}{x^2-3x+2}+4\ln\left|\frac{x-1}{x-2}\right|$. **252.** $\frac{9x^2+50x+68}{4(x+2)(x+3)^2}+$ $+\frac{1}{8}\ln\left|\frac{(x+1)(x+2)^{16}}{(x+3)^{17}}\right|$. **253.** $\frac{3}{16}\ln\left|\frac{x+1}{x-1}\right|-\frac{3x^2+3x-2}{8(x-1)(x+1)^2}$. **254.** $\operatorname{arctg}x+\frac{5}{6}\times$ $\times\ln\frac{x^2+1}{x^2+4}$. **255.** $\frac{1}{2}\operatorname{arcctg}x+\frac{1}{4}\ln\frac{(x+1)^2}{x^2+1}$. **256.** $-\frac{1}{x-2}-\operatorname{arctg}(x-2)$. **257.** $-\frac{1}{5}\times$ $\times\frac{1}{(x-1)}+\frac{1}{50}\ln\frac{(x-1)^2}{x^2+2x+2}-\frac{8}{25}\operatorname{arctg}(x+1)$. **258.** $\ln\left|\frac{x}{1+x}\right|-\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{1+2x}{\sqrt{3}}$. **259.** $\frac{1}{6}\cdot\ln\frac{(x+1)^2}{x^2-x+1}+\frac{1}{\sqrt{3}}\cdot\operatorname{arctg}\frac{2x-1}{\sqrt{3}}$. **260.** $\frac{1}{6}\cdot\ln\frac{(x-1)^2}{x^2+x+1}+\frac{1}{\sqrt{3}}\cdot\operatorname{arctg}\frac{2x+1}{\sqrt{3}}$. **261.** $\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\operatorname{arctg}x$. **262.** $\frac{1}{4\sqrt{2}}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{1}{2\sqrt{2}}\operatorname{arctg}\frac{x\sqrt{2}}{1-x^2}$. **263.** $\frac{1}{4}\times$ $\times\ln\frac{x^2+x+1}{x^2-x+1}+\frac{1}{2\sqrt{3}}\operatorname{arctg}\frac{x^2-1}{x\sqrt{3}}$. **264.** $\frac{1}{2}\operatorname{arctg}x+\frac{1}{6}\operatorname{arctg}x^3+\frac{1}{4\sqrt{3}}\ln\frac{1+x\sqrt{3}+x^2}{1-x\sqrt{3}+x^2}$. **265.** $-\frac{1}{6(1+x)}+\frac{1}{6}\ln\frac{(1+x)^2}{1-x+x^2}+\frac{1}{2}\operatorname{arctg}x-\frac{1}{3\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}$. **266.** $\frac{1}{6}\ln\frac{(x-1)^2}{x^2+x+1}-$ $-\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}$. **267.** $\frac{2}{5}\ln\frac{x^2+2x+2}{x^2+x+1/2}+\frac{8}{5}\operatorname{arctg}(x+1)-\frac{2}{5}\operatorname{arctg}(2x+1)$. **268.** $a+$ $+2b+3c=0$. **269.** $-\frac{x^2+x+2}{8(x-1)(x+1)^2}+\frac{1}{16}\ln\left|\frac{x+1}{x-1}\right|$. **270.** $\frac{x}{3(x^3+1)}+\frac{1}{9}\ln\frac{(x+1)^2}{x^2-x+1}+$ $+\frac{2}{3\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}$. **271.** $\frac{x(3x^2+5)}{8(x^2+1)^2}+\frac{3}{8}\operatorname{arctg}x$. **272.** $\frac{1}{x^2+2x+2}+\operatorname{arctg}(x+1)$. **273.** $\frac{x}{4(x^4+1)}+\frac{3}{16\sqrt{2}}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}-\frac{3}{8\sqrt{2}}\operatorname{arctg}\frac{x\sqrt{2}}{x^2-1}$. **274.** $\frac{5x+2}{3(x^2+x+1)}+$

$+\frac{1}{9}\ln\frac{(x-1)^2}{x^2+x+1}+\frac{8}{3\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}$. **275.** $\frac{7x^5-11x}{32(x^4-1)^2}+\frac{21}{128}\ln\left|\frac{x-1}{x+1}\right|-\frac{21}{64}\operatorname{arctg}x$. **276.** $\frac{x^3+2x}{6(x^4+x^2+1)}$. **277.** $-\frac{8x^4+8x^2+4x-1}{28(x^3+x+1)^2}$. **278.** $\frac{-x}{x^5+x+1}$ (tutuş integral). **279.** $\frac{2x+1}{3(x^2+x+1)}+\frac{4}{3\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}$. **280.** $a\gamma+c\alpha=2b\beta$. **281.** $-\frac{1}{96(x-1)^{96}}-$ $-\frac{3}{97(x-1)^{97}}-\frac{3}{98(x-1)^{98}}-\frac{1}{99(x-1)^{99}}$. **282.** $\frac{1}{8}\ln\left|\frac{x^2-1}{x^2+1}\right|-\frac{1}{4}\operatorname{arctg}x^2$. **283.** $\frac{1}{4}\times$ $\times\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x^4}{\sqrt{3}}$. **284.** $\frac{1}{12}\ln\frac{(x^2+1)^2}{x^4-x^2+1}+\frac{1}{3}\operatorname{arctg}x^3+\frac{1}{2\sqrt{3}}\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}}$. **285.** $\frac{5}{8}\times$ $\times\ln\frac{x^4}{x^4+2}-\ln\frac{x^4}{x^4+1}$. **286.** $-\frac{1}{100}\left(\frac{1}{2\sqrt{10}}\ln\left|\frac{x^5-\sqrt{10}}{x^5+\sqrt{10}}\right|+\frac{x^5}{x^{10}-10}\right)$. **287.** $\frac{x^4}{4}+\frac{1}{4}\times$ $\times\ln\frac{x^4+1}{(x^4+2)^4}$. **288.** $-\frac{x^5+2}{10(x^{10}+2x^5+2)}-\frac{1}{10}\operatorname{arctg}(x^5+1)$. **289.** $\frac{1}{n}(x^n-\ln|x^n+1|)$ $(n\neq 0)$. **290.** $\frac{1}{2n}\left(\operatorname{arctg}x^n-\frac{x^n}{x^{2n}+1}\right)$ $(n\neq 0)$. **291.** $\frac{1}{20}\ln\frac{x^{10}}{x^{10}+2}$. **292.** $\frac{1}{10}\cdot\frac{1}{(x^{10}+1)}+$ $+\frac{1}{10}\ln\frac{x^{10}}{x^{10}+1}$. **293.** $\frac{1}{7}\ln\frac{|x^7|}{(1+x^7)^2}$. **294.** $\frac{1}{5}\ln\left|\frac{x(x^4-5)}{x^5-5x+1}\right|$. **295.** $\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x^2-1}{x\sqrt{3}}$. **296.** $\frac{1}{\sqrt{5}}\ln\frac{2x^2+(1-\sqrt{5})x+2}{2x^2+(1+\sqrt{5})x+2}$. **297.** $\frac{1}{4\sqrt{2}}\ln\frac{x^4-x^2\sqrt{2}+1}{x^4+x^2\sqrt{2}+1}$. **298.** $\operatorname{arctg}x+\frac{1}{3}\operatorname{arctg}x^3$. **299.** $I_n=\frac{2ax+b}{(n-1)\Delta(ax^2+bx+c)^{n-1}}+\frac{2n-3}{n-1}\frac{2a}{\Delta}I_{n-1}$, bu ýerde $\Delta=4ac-b^2$; $I_3=(2x+$ $+1)\frac{1}{6(x^2+x+1)^2}+\frac{2x+1}{3(x^2+x+1)}+\frac{4}{3\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}$. **300.** $I=\frac{1}{(b-a)^{m+n-1}}\int\left(\frac{1}{t^m}\times\right.$ $\left.\times(1-t)^{m+n-2}\right)dt$; $\frac{1}{625}\left(-\frac{1}{t}+3t-\frac{t^2}{2}-3\ln|t|\right)$, bu ýerde $t=\frac{x-2}{x+3}$. **301.** $-\sum_{k=0}^{n-1}\left(\frac{1}{k!}\times\right.$ $\left.\times\frac{P_n^{(k)}(a)}{(n-k)(x-a)^{n-k}}\right)+\frac{P_n^{(n)}(a)}{n!}\ln|x-a|$. **302.** $R(x)=P(x^2)+\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left[\frac{A_{ij}}{(a_i-x)^{\alpha_j}}+\right.$ $\left.+\frac{A_{ij}}{(a_i+x)^{\alpha_j}}\right]$, bu ýerde P – köpagza, $\pm a_i$ $(i=1, ..., k)$ maýdalawjynyň kökleri we A_{ij} – hemişelik koeffisiýentler. **303.** $-\frac{1}{2n}\sum_{k=1}^{n}\cos\frac{\pi(2k-1)}{2n}\ln\left(1-2x\cos\frac{2k-1}{2n}\pi+x^2\right)+\frac{1}{n}\times$ $\times\sum_{k=1}^{n}\left\{\sin\frac{\pi(2k-1)}{2n}\operatorname{arctg}\frac{x-\cos((2k-1)\pi/2n)}{\sin((2k-1)\pi/2n)}\right\}$.

§3. Irrasional funksiýalaryň integrirlenişi

304. $2\cdot\sqrt{x}-2\cdot\ln(1+\sqrt{x})$. **305.** $\frac{3}{4}\cdot\ln\frac{x\sqrt[3]{x}}{\left(1+\sqrt[6]{x}\right)^2\left(1-\sqrt[6]{x}+2\sqrt[3]{x}\right)^3}-\frac{3}{2\sqrt{7}}\times$ $\times\operatorname{arctg}\frac{4\sqrt[6]{x}-1}{\sqrt{7}}$. **306.** $\frac{3}{4}t^4-\frac{3}{2}t^2-\frac{3}{4}\ln|t-1|+\frac{15}{8}\ln\left(t^2+t+2\right)-\frac{27}{8\sqrt{7}}\operatorname{arctg}\frac{2t+1}{\sqrt{7}}$, $t=\sqrt[3]{2+x}$. **307.** $6t-3t^2-6\operatorname{arctg}t+\frac{3}{2}t^4+\frac{6}{5}t^5-\frac{6}{7}t^7+3\ln\left(1+t^2\right)-2t^3$, bu ýerde $t=\sqrt[6]{x+1}$. **308.** $\frac{2}{(1+\sqrt[4]{x})^2}-\frac{4}{1+\sqrt[4]{x}}$. **309.** $\frac{x^2}{2}-\frac{x\sqrt{x^2-1}}{2}+\frac{1}{2}\ln|x+\sqrt{x^2-1}|$. **310.** $-\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}$. **311.** $-\frac{at^3}{1+t^4}+\frac{a}{4\sqrt{2}}\ln\frac{1+t\sqrt{2}+t^2}{1-t\sqrt{2}+t^2}+\frac{a}{2\sqrt{2}}\operatorname{arctg}\frac{1-t^2}{t\sqrt{2}}$, bu ýerde $t=\sqrt[4]{\frac{a-x}{x}}$. **312.** $-\frac{n}{a-b}\sqrt[n]{\frac{x-b}{x-a}}$. **313.** $\frac{x}{2}+\sqrt{x}-\frac{1}{2}\sqrt{x(1+x)}-\frac{1}{2}\ln(\sqrt{x}+$ $+\sqrt{1+x})$. **315.** $-\frac{3-2x}{4}\sqrt{1+x+x^2}-\frac{1}{8}\ln\left(\frac{1}{2}+x+\sqrt{1+x+x^2}\right)$. **316.** $-\ln\left|\frac{1}{x+1}\times\right.$ $\left.\times\left(2-x+2\sqrt{x^2+x+1}\right)\right|$. **317.** $\frac{2-x}{3(1-x)^2}\sqrt{1-x^2}$. **318.** $R+\ln(x+1+R)-\sqrt{2}\times$ $\times\ln\left|\frac{x+2+\sqrt{2}R}{x}\right|$, bu ýerde $R=\sqrt{x^2+2x+2}$. **319.** $\ln\left|\frac{3+x+2\sqrt{1-x-x^2}}{1+x}\right|+$ $+\arcsin\frac{1+2x}{\sqrt{5}}$. **320.** $\frac{1-2x}{4}\sqrt{1+x-x^2}-\frac{11}{8}\arcsin\frac{1-2x}{\sqrt{5}}$. **321.** $-\frac{19+5x+2x^2}{6}\times$ $\times\sqrt{1+2x-x^2}-4\arcsin\frac{1-x}{\sqrt{2}}$. **322.** $\left(\frac{63}{256}x-\frac{21}{128}x^3+\frac{21}{160}x^5-\frac{9}{80}x^7+\frac{x^9}{10}\right)\sqrt{1+x^2}-$ $-\frac{63}{256}\ln\left(x+\sqrt{1+x^2}\right)$. **323.** $\left(-\frac{a^4x}{16}-\frac{a^2x^3}{24}+\frac{x^5}{6}\right)\sqrt{a^2-x^2}+\frac{a^6}{16}\arcsin\frac{x}{|a|}$. **324.** $\left(\frac{x^2}{3}-\right.$ $\left.-\frac{14x}{3}+37\right)\sqrt{x^2+4x+3}-66\ln|x+2+\sqrt{x^2+4x+3}|$. **325.** $-\frac{1}{2x^2}\sqrt{x^2+1}+\frac{1}{2}\times$ $\times\ln\frac{1+\sqrt{x^2+1}}{|x|}$. **326.** $\frac{2x^2+1}{3x^3}\sqrt{x^2-1}$. **327.** $\frac{3x-5}{20(x-1)^2}\sqrt{x^2+3x+1}-\frac{11}{40\sqrt{5}}\times$ $\times\ln\left|\frac{(x+1)\sqrt{5}+2\sqrt{x^2+3x+1}}{x-1}\right|$. **328.** $\frac{3x+5}{8(x+1)^2}\sqrt{x^2+2x}-\frac{3}{8}\arcsin\frac{1}{|x+1|}$, bu ýerde $x<-2$ ýa-da $x>0$. **329.** $4a(ca_1+bb_1)=8a^2c_1+3b^2a_1$ $(a\neq 0)$. **330.** $\frac{\sqrt{1+2x-x^2}}{2(1-x)}-\frac{1}{\sqrt{2}}\times$ $\times\ln\left|\frac{\sqrt{2}+\sqrt{1+2x-x^2}}{1-x}\right|$. **331.** $\frac{1}{2}\cdot\arcsin\frac{x-3}{|x-1|\sqrt{5}}-\frac{1}{2}\cdot\ln\left|\frac{3x+1-2\sqrt{x^2-x-1}}{x+1}\right|$. **332.** $-\frac{\sqrt{x^2+x+1}}{x+1}+\ln\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+\frac{1}{2}\ln\left|\frac{1-x+2\sqrt{x^2+x+1}}{x+1}\right|$. **333.** $-\frac{1}{2}\times$

$\times(1+x)\sqrt{1+2x-x^2} - 2\arcsin\frac{1-x}{\sqrt{2}} - \frac{1}{\sqrt{2}}\arcsin\frac{x\sqrt{2}}{|1+x|}$. **334.** $-\frac{\sqrt{x^2-4x+3}}{x-1} - 2\times$
$\times\arcsin\frac{1}{|x-2|}$ ($x<1$ ýa-da $x>3$). **335.** $\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x\sqrt{2}}{\sqrt{1-x^2}}$. **336.** $\ln\left|\frac{x\sqrt{2}+\sqrt{x^2-1}}{x\sqrt{2}-\sqrt{x^2-1}}\right|\times$
$\times\frac{1}{2\sqrt{2}}$. **337.** $\frac{x}{2\sqrt{1+x^2}} + \frac{1}{4\sqrt{2}}\ln\left|\frac{\sqrt{1+x^2}+x\sqrt{2}}{\sqrt{1+x^2}-x\sqrt{2}}\right|$. **338.** $\ln\left(x+\sqrt{x^2+2}\right) - \operatorname{arctg}\Big(\frac{1}{x}\times$
$\times\frac{\sqrt{x^2+2}}{x}\Big)$. **339.** $\frac{1}{\sqrt{6}}\ln\left|\frac{(2x+1)\sqrt{2}+\sqrt{3(x^2+x-1)}}{(2x+1)\sqrt{2}-\sqrt{3(x^2+x-1)}}\right|$. **340.** $\arcsin\frac{x-1}{\sqrt{3}} - \frac{\sqrt{2}}{3}\times$
$\times\operatorname{arctg}\frac{\sqrt{2+2x-x^2}}{(1-x)\sqrt{2}} - \frac{1}{\sqrt{6}}\ln\frac{\sqrt{6}+\sqrt{2+2x-x^2}}{\sqrt{6}-\sqrt{2+2x-x^2}}$. **341.** $\frac{2(x-1)}{3\sqrt{x^2+x+1}}$. **342.** $\frac{1}{\sqrt{6}}\times$
$\times\ln\left|\frac{(x+1)\sqrt{2}-\sqrt{3(x^2+x+1)}}{\sqrt{x^2-x+1}}\right| - \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\sqrt{x^2+x+1}}{(x-1)\sqrt{2}}$, $x+1>0$ bolanda. **343.** $-\frac{1}{3}\times$
$\times\operatorname{arctg}\frac{\sqrt{2x^2-2x+5}}{x+1} + \frac{1}{6\sqrt{2}}\ln\frac{\sqrt{2(2x^2-2x+5)}-(x+1)}{\sqrt{2(2x^2-2x+5)}+(x+1)}$. **344.** $\frac{3}{2(2z+1)} + \frac{1}{2}\times$
$\times\ln\frac{z^4}{|2z+1|^3}$, bu ýerde $z = x+\sqrt{x^2+x+1}$. **345.** $\ln\left|\frac{z-1}{z}\right| - 2\operatorname{arctg}z$, bu ýerde $z =$
$= \frac{1+\sqrt{1-2x-x^2}}{x}$. **346.** $\frac{1}{8}\left\{\frac{1}{3}[(z-1)^3+(z-1)^{-3}]+[(z-1)^2-(z-1)^{-2}]+[(z-1)+\right.$
$\left.+(z-1)^{-1}]\right\} + \frac{1}{2}\ln|z-1|$, bu ýerde $z = x+\sqrt{x^2-2x+2}$. **347.** $-\frac{5}{18(z+1)} - \frac{1}{6}\times$
$\times\frac{1}{(z+1)^2} + \frac{3}{4}\ln|z-1| - \frac{16}{27}\ln|z-2| - \frac{17}{108}\ln|z+1|$, bu ýerde $z = \frac{\sqrt{x^2+3x+2}}{x+1}$.
348. $\frac{2(3-4z)}{5(1-z-z^2)} + \frac{2}{5\sqrt{5}}\ln\left|\frac{\sqrt{5}+1+2z}{\sqrt{5}-1-2z}\right|$, bu ýerde $z = -x+\sqrt{x(1+x)}$. **349.** $\frac{x}{4}\times$
$\times\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right) + \frac{1}{4}\ln\left|\frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2-1}}\right|$. **350.** $\frac{1}{3}\sqrt{z} - \frac{1}{3\sqrt[4]{12}}\Big\{\ln\Big[(z\sqrt{3}+\sqrt[4]{12z^2}+$
$+1)\cdot\frac{1}{z\sqrt{3}-\sqrt[4]{12z^2}+1}\Big] - 2\operatorname{arctg}\frac{\sqrt[4]{12z^2}}{z\sqrt{3}-1}\Big\}$, bu ýerde $z = \frac{1+x}{1-x}$. **351.** $\sqrt{1+x} -$
$-\sqrt{1-x} - \frac{1}{\sqrt{2}}\arcsin x$. **352.** $\sqrt{1+x+x^2} + \frac{1}{2}\ln\frac{1+2x+2\sqrt{1+x+x^2}}{\left(2+x+2\sqrt{1+x+x^2}\right)^2}$. **353.** $\frac{2}{3}\times$
$\times\left[(x+1)^{\frac{3}{2}}+x^{\frac{3}{2}}\right] - \frac{2}{5}[(x+1)^{5/2}-x^{5/2}]$. **354.** $-\frac{1}{\sqrt{2}}\arcsin\frac{x\sqrt{2}}{x^2+1}$. **355.** $-\frac{1}{\sqrt{2}}\ln\Big|(x\sqrt{2}+$
$+\sqrt{x^4+1})\cdot\frac{1}{x^2-1}\Big|$. **356.** $\frac{1}{2}\arcsin\frac{x^2-1}{x^2\sqrt{2}}$ ($|x|>\sqrt{\sqrt{2}-1}$). **357.** $\frac{1}{2}\ln[(x^2(2x^2+1+$

$+2\sqrt{x^4+x^2+1})\cdot\dfrac{1}{x^2+2+2\sqrt{x^4+x^2+1}}\Big)\Big]$. **359.** $-\dfrac{1+2x}{8}\sqrt{x+x^2}+\dfrac{1}{3}\sqrt{(x+x^2)^3}+$ $+\dfrac{1}{8}\ln\left(\sqrt{x}+\sqrt{1+x}\right)$, $x>0$ bolanda. **360.** $\dfrac{6}{5}x^{5/6}-4x^{1/2}+18x^{1/6}+\dfrac{3x^{1/6}}{1+x^{1/3}}-21\times$ $\times\operatorname{arctg}x^{1/6}$. **361.** $\dfrac{3}{5}z^5-2z^3+3z$, bu ýerde $z=\sqrt{1+\sqrt[3]{x^2}}$. **362.** $-z+\dfrac{2}{3}z^3-\dfrac{z^5}{5}$, bu ýerde $z=\sqrt{1-x^2}$. **363.** $\dfrac{1}{6}\ln\dfrac{z^2+z+1}{(z-1)^2}-\dfrac{1}{\sqrt{3}}\operatorname{arctg}\dfrac{2z+1}{\sqrt{3}}$, bu ýerde $z=\dfrac{\sqrt[3]{1+x^3}}{x}$. **364.** $\dfrac{1}{4}\ln\left|\dfrac{z+1}{z-1}\right|-\dfrac{1}{2}\operatorname{arctg}z$, bu ýerde $z=\dfrac{\sqrt[4]{1+x^4}}{x}$. **365.** $\dfrac{1}{6}\ln\dfrac{z-1}{z+1}+\dfrac{1}{12}\ln[(z^2+$ $+z+1)\cdot\dfrac{1}{z^2-z+1}\Big]+\dfrac{1}{2\sqrt{3}}\operatorname{arctg}\dfrac{z^2-1}{z\sqrt{3}}$, bu ýerde $z=\sqrt[6]{1+x^6}$. **366.** $\dfrac{5}{4}z^4-\dfrac{5}{9}z^9$, bu ýerde $z=\sqrt[5]{1+\dfrac{1}{x}}$. **367.** $\dfrac{3z}{2(z^3+1)}-\dfrac{1}{4}\ln\dfrac{(z+1)^2}{z^2-z+1}-\dfrac{\sqrt{3}}{2}\operatorname{arctg}\dfrac{2z-1}{\sqrt{3}}$, bu ýerde $z=\dfrac{\sqrt[3]{3x-x^3}}{x}$. **368.** $m=\dfrac{2}{k}$, bu ýerde $k=\pm1, \pm2, \ldots$.

§4. Trigonometrik funksiýalaryň integrirlenişi

369. $\sin x-\dfrac{2}{3}\sin^3x+\dfrac{1}{5}\sin^5x$. **370.** $\dfrac{5}{16}x-\dfrac{1}{4}\sin 2x+\dfrac{3}{64}\sin 4x+\dfrac{1}{48}\sin^3 2x$. **371.** $\dfrac{5}{16}x+$ $+\dfrac{1}{4}\sin 2x+\dfrac{3}{64}\sin 4x-\dfrac{1}{48}\cdot\sin^3 2x$. **372.** $\dfrac{x}{16}-\dfrac{\sin 4x}{64}+\dfrac{\sin^3 2x}{48}$. **373.** $\dfrac{\sin^5 x}{5}-\dfrac{2\sin^7 x}{7}+$ $+\dfrac{\sin^9 x}{9}$. **374.** $-\dfrac{\cos 2x}{64}+\dfrac{\cos^3 2x}{96}-\dfrac{\cos^5 2x}{320}$. **375.** $\dfrac{1}{3\cos^3 x}-\dfrac{1}{\cos x}$. **376.** $-\dfrac{3}{2}\cos x-\dfrac{\cos^3 x}{2\sin^2 x}-$ $-\dfrac{3}{2}\ln\left|\operatorname{tg}\dfrac{x}{2}\right|$. **377.** $-\dfrac{\cos x}{2\sin^2 x}+\dfrac{1}{2}\ln\left|\operatorname{tg}\dfrac{x}{2}\right|$. **378.** $\dfrac{\sin x}{2\cos^2 x}+\dfrac{1}{2}\ln\left|\operatorname{tg}\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)\right|$. **379.** $-8\operatorname{ctg}2x-\dfrac{8}{3}\operatorname{ctg}^3 2x$. **380.** $\dfrac{\operatorname{tg}^4 x}{4}+\dfrac{3\operatorname{tg}^2 x}{2}-\dfrac{\operatorname{ctg}^2 x}{2}+3\ln|\operatorname{tg}x|$. **381.** $\dfrac{1}{\cos x}+\dfrac{1}{3\cos^3 x}+\ln\left|\operatorname{tg}\dfrac{x}{2}\right|$. **382.** $\dfrac{\operatorname{tg}^4 x}{4}-$ $-\dfrac{\operatorname{tg}^2 x}{2}-\ln|\cos x|$. **383.** $-x-\dfrac{\operatorname{ctg}^5 x}{5}+\dfrac{\operatorname{ctg}^3 x}{3}-\operatorname{ctg}x$. **384.** $\dfrac{\operatorname{tg}^5 x}{5}$. **385.** $-2\sqrt{\operatorname{ctg}x}+\dfrac{2}{3}\times$ $\times\sqrt{\operatorname{tg}^3 x}$. **386.** $\dfrac{1}{4}\ln\left|\dfrac{(1+t)^3(1+t^3)}{(1-t)^3(1-t^3)}\right|-\dfrac{\sqrt{3}}{2}\operatorname{arctg}\dfrac{1-t^2}{t\sqrt{3}}$, bu ýerde $t=\sqrt[3]{\sin x}$. **387.** $\dfrac{1}{2}\times$ $\times\dfrac{1}{\sqrt{2}}\ln\dfrac{z^2+z\sqrt{2}+1}{z^2-z\sqrt{2}+1}-\dfrac{1}{\sqrt{2}}\operatorname{arctg}\dfrac{z\sqrt{2}}{z^2-1}$, bu ýerde $z=\sqrt{\operatorname{tg}x}$. **388.** $\dfrac{1}{4}\ln\dfrac{(z^2+1)^2}{z^4-z^2+1}+$ $+\dfrac{\sqrt{3}}{2}\operatorname{arctg}\dfrac{2z^2-1}{\sqrt{3}}$, bu ýerde $z=\sqrt[3]{\operatorname{tg}x}$. **389.** $I_n=-\dfrac{\cos x\sin^{n-1}x}{n}+\dfrac{n-1}{n}I_{n-2}$; $K_n=$ $=\dfrac{\sin x\cos^{n-1}x}{n}+\dfrac{n-1}{n}K_{n-2}$; $I_6=-\dfrac{1}{6}\cos x\sin^5 x-\dfrac{5}{24}\cos x\sin^3 x-\dfrac{5}{16}\cos x\sin x+$

$+\frac{5}{16}x$; $K_8=\frac{1}{8}\sin x\cos^7 x+\frac{7}{48}\sin x\cos^5 x+\frac{35}{192}\sin x\cos^3 x+\frac{35}{128}\sin x\cos x+\frac{35}{128}x$.

390. $I_n=-\frac{\cos x}{(n-1)\sin^{n-1}x}+\frac{n-2}{n-1}I_{n-2}$; $K_n=\frac{\sin x}{(n-1)\cos^{n-1}x}+\frac{n-2}{n-1}K_{n-2}$; $I_5=-\frac{1}{4}\times$ $\times\frac{\cos x}{\sin^4 x}-\frac{3\cos x}{8\sin^2 x}+\frac{3}{8}\ln\left|\operatorname{tg}\frac{x}{2}\right|$; $K_7=\frac{\sin x}{6\cos^6 x}+\frac{5\sin x}{24\cos^4 x}+\frac{5\sin x}{16\cos^2 x}+\frac{5}{16}\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$.

391. $-\frac{1}{8}\cos 4x-\frac{1}{12}\cos 6x$. **392.** $\frac{x}{4}+\frac{\sin 2x}{8}+\frac{\sin 4x}{16}+\frac{\sin 6x}{24}$. **393.** $\frac{3}{2}\cos\frac{x}{6}-\frac{3}{10}\times$ $\times\cos\frac{5x}{6}-\frac{3}{14}\cos\frac{7x}{6}+\frac{3}{22}\cos\frac{11x}{6}$. **394.** $-\frac{1}{2}\cos(a-b)\cos x-\frac{1}{4}\cos(x+a+b)+\frac{1}{12}\times$ $\times\cos(3x+a+b)$. **395.** $\frac{x}{4}+\frac{\sin 2ax}{8a}+\frac{\sin 2bx}{8b}+\frac{\sin 2(a-b)x}{16(a-b)}+\frac{\sin 2(a+b)x}{16(a+b)}$. **396.** $-\frac{3}{16}\times$ $\times\cos 2x+\frac{3}{64}\cos 4x+\frac{1}{48}\cos 6x-\frac{3}{128}\cos 8x+\frac{1}{192}\cos 12x$. **397.** Eger $\sin(a-b)\neq 0$ bolsa, $\frac{1}{\sin(a-b)}\ln\left|\frac{\sin(x+b)}{\sin(x+a)}\right|$. **398.** $\frac{1}{\cos(a-b)}\ln\left|\frac{\sin(x+a)}{\cos(x+b)}\right|$, eger $\cos(a-b)\neq 0$ bolsa.

399. $\frac{2}{\sin(a-b)}\ln\left|\frac{\cos(x+b)}{\cos(x+a)}\right|$, eger $\sin(a-b)\neq 0$ bolsa. **400.** $\frac{1}{\cos a}\ln\left|\left[(\sin(x-a/2)\cdot\frac{1}{\cos((x+a)/2)}\right]\right|$, $(\cos a\neq 0)$. **401.** $\frac{1}{\sin a}\cdot\ln\left|\frac{\cos((x-a)/2)}{\cos((x+a)/2)}\right|$, $(\sin a\neq 0)$. **402.** $-x+$ $+\operatorname{ctg}a\ln\left|\frac{\cos x}{\cos(x+a)}\right|$ $(\sin a\neq 0)$. **403.** $\frac{1}{\sqrt{5}}\operatorname{arctg}\frac{3\operatorname{tg}(x/2)+1}{\sqrt{5}}$. **404.** $\frac{1}{6}\ln\left[\frac{1}{(1+\cos x)^3}(1-\right.$ $\left.-\cos x)(2+\cos x)^2\right]$. **405.** $-\frac{1}{5}(2\sin x+\cos x)+\frac{4}{5\sqrt{5}}\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\operatorname{arctg}2}{2}\right)\right|$. **406.** a) eger $0<\varepsilon<1$ bolsa, $\frac{2}{\sqrt{1-\varepsilon^2}}\operatorname{arctg}\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\operatorname{tg}\frac{x}{2}\right)$; b) eger $\varepsilon>1$ bolsa, $\frac{1}{\sqrt{\varepsilon^2-1}}\ln|(\varepsilon+$ $+\cos x+\sqrt{\varepsilon^2-1}\sin x)\cdot\frac{1}{1+\varepsilon\cos x}\Big|$ **407.** $x-\frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\operatorname{tg}x)$. **408.** $\frac{1}{ab}\operatorname{arctg}\left(\frac{a\operatorname{tg}x}{b}\right)$.

409. $\frac{(2b^2)^{-1}z}{(a^2z^2+b^2)}+\frac{1}{2ab^3}\operatorname{arctg}\frac{az}{b}$ $(ab\neq 0)$, bu ýerde $z=\operatorname{tg}x$. **410.** $\frac{1}{2}(\sin x-\cos x)-$ $-\frac{1}{2\sqrt{2}}\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{8}\right)\right|$. **411.** $-\frac{\cos x}{a(a\sin x+b\cos x)}$. **412.** $-\frac{1}{6}\ln\frac{(\sin x+\cos x)^2}{1-\sin x\cos x}-\frac{1}{\sqrt{3}}\times$ $\times\operatorname{arctg}\left(\frac{2\cos x-\sin x}{\sqrt{3}\sin x}\right)$. **413.** $\frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{\operatorname{tg}2x}{\sqrt{2}}\right)$. **414.** $\frac{1}{4}\left\{\sqrt{2+\sqrt{2}}\operatorname{arctg}\frac{u}{\sqrt{4+2\sqrt{2}}}-\right.$ $\left.-\sqrt{2-\sqrt{2}}\operatorname{arctg}\frac{u}{\sqrt{4-2\sqrt{2}}}\right\}$, bu ýerde $u=\operatorname{tg}2x$. **415.** $\frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2}-\sin 2x}{\sqrt{2}+\sin 2x}$. **416.** $\frac{1}{2}\times$ $\times\operatorname{arctg}(\sin^2 x)$. **417.** $\operatorname{arctg}\left(\frac{1}{2}\operatorname{tg}2x\right)$. **418.** $-\frac{z}{4(z^2+2)}+\frac{3}{4\sqrt{2}}\operatorname{arctg}\frac{z}{\sqrt{2}}$, bu ýerde $z=\operatorname{tg}x$.

419. $\frac{1}{\sqrt{a^2+b^2}}\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\varphi}{2}\right)\right|$, bu ýerde $\cos\varphi=\frac{a}{\sqrt{a^2+b^2}}$ we $\sin\varphi=\frac{b}{\sqrt{a^2+b^2}}$.

421. $-\frac{x}{5}-\frac{3}{5}\ln|\sin x+2\cos x|$. **422.** $0{,}1x+0{,}3\ln|\sin x-3\cos x|$. **423.** $\frac{3x}{34}+\frac{5}{34}\ln|5\sin x+$ $+3\cos x|$. **424.** $-\frac{ab_1-a_1b}{a^2+b^2}\cdot\frac{1}{a\sin x+b\cos x}+\frac{aa_1+bb_1}{(a^2+b^2)^{3/2}}\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\varphi}{2}\right)\right|$, bu ýerde $\cos\varphi=\frac{a}{\sqrt{a^2+b^2}}$ we $\sin\varphi=\frac{b}{\sqrt{a^2+b^2}}$. **426.** $-\frac{3x}{5}+\frac{4}{5}\ln|\sin x-2\cos x+3|-\frac{6}{5}\times$ $\times\operatorname{arctg}\frac{5\operatorname{tg}(x/2)+1}{2}$. **427.** $\frac{x}{2}-\frac{1}{2}\operatorname{tg}\left(\frac{x}{2}-\frac{\pi}{8}\right)-\frac{1}{2}\ln(\sqrt{2}+\sin x+\cos x)$. **428.** $\frac{2}{5}x-\frac{1}{5}\times$ $\times\ln|3\sin x+4\cos x-2|+\frac{4}{5\sqrt{21}}\ln\left|\frac{\sqrt{7}+\sqrt{3}(2\operatorname{tg}(x/2)-1)}{\sqrt{7}-\sqrt{3}(2\operatorname{tg}(x/2)-1)}\right|$. **430.** $-\sin x+3\cos x+$ $+2\sqrt{2}\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{8}\right)\right|$. **431.** $\frac{1}{5}(\sin x+3\cos x)+\frac{8}{5\sqrt{5}}\ln\left|\frac{\sqrt{5}-1+2\operatorname{tg}(x/2)}{\sqrt{5}+1-2\operatorname{tg}(x/2)}\right|$. **433.** $-\frac{2}{\sqrt{3}}\times$ $\times\operatorname{arctg}\left(\frac{\cos x}{\sqrt{3}}\right)-\frac{1}{4}\ln\frac{2+\sin x}{2-\sin x}$. **434.** $\frac{3}{5}\operatorname{arctg}(\sin x-2\cos x)+\frac{1}{10\sqrt{6}}\ln|(\sqrt{6}+2\sin x+$ $+\cos x)\cdot\frac{1}{\sqrt{6}-2\sin x-\cos x}|$ **435.** $\frac{3}{4\sqrt{2}}\ln\left|\frac{\sqrt{2}(\sin x+\cos x)+1}{\sqrt{2}(\sin x+\cos x)-1}\right|-\frac{1}{4\sqrt{6}}\ln|[\sqrt{3}+$ $+\sqrt{2}(\sin x-\cos x)]\cdot\frac{1}{\sqrt{3}-\sqrt{2}(\sin x-\cos x)}|$. **437.** $\frac{2\sin x-\cos x}{10(\sin x+2\cos x)^2}+\frac{1}{10\sqrt{5}}\times$ $\times\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\operatorname{arctg}2}{2}\right)\right|$. **438.** $A=-\frac{b}{(n-1)(a^2-b^2)}$, $B=\frac{(2n-3)a}{(n-1)(a^2-b^2)}$, $C=$ $=-\frac{n-2}{(n-1)(a^2-b^2)}$. **439.** $\frac{1}{\sqrt{2}}\ln\frac{\sqrt{2}+\sqrt{1+\sin^2x}}{|\cos x|}$. **440.** $2\sqrt{\operatorname{tg}x}+\frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\times$ $\times\frac{\sqrt{\operatorname{tg}x}}{\operatorname{tg}x-1})-\frac{1}{2\sqrt{2}}\ln\frac{\operatorname{tg}x+\sqrt{2\operatorname{tg}x}+1}{\operatorname{tg}x-\sqrt{2\operatorname{tg}x}+1}$ $(\operatorname{tg}x>0)$. **441.** $\frac{1}{2}\arcsin\left(\frac{\sin x-\cos x}{\sqrt{3}}\right)-\frac{1}{2}\times$ $\times\ln\left(\sin x+\cos x+\sqrt{2+\sin 2x}\right)$. **442.** $-\frac{\varepsilon\sin x}{(1-\varepsilon^2)(1+\varepsilon\cos x)}+\frac{2}{(1-\varepsilon^2)^{3/2}}\operatorname{arctg}\left(\operatorname{tg}\frac{x}{2}\times\right.$ $\left.\times\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\right)$. **443.** $-\frac{2}{n\cos a}\left(\cos\frac{x+a}{2}\right)^n\left(\sin\frac{x-a}{2}\right)^{-n}$ $(\cos a\neq0)$. **444.** $I_n=2I_{n-1}\cos a-$ $-I_{n-2}+\frac{2\sin a}{n-1}t^{n-1}$, bu ýerde $n>2$ we $t=\sin\frac{x-a}{2}\left(\sin\frac{x+a}{2}\right)^{-1}$.

§5. Dürli transsendent funksiýalaryň integrirlenişi

447. $e^{3x}\left(\frac{x^3}{3}-\frac{x^2}{3}+\frac{2x}{9}-\frac{2}{27}\right)$. **448.** $-e^{-x}(x^2+2)$. **449.** $-\left(\frac{x^5}{5}-\frac{4x^3}{25}+\frac{24x}{625}\right)\cos 5x+$ $+\left(\frac{x^4}{5}-\frac{12x^2}{125}+\frac{24}{3125}\right)\sin 5x$. **450.** $(21-10x^2+x^4)\sin x-(20x-4x^3)\cos x$. **451.** $-\frac{e-x^2}{2}(x^6+$ $+3x^4+6x^2+6)$. **452.** $2e^t(t^5-5t^4+20t^3-60t^2+120t-120)$, bu ýerde $t=\sqrt{x}$. **453.** $e^{ax}\left[\frac{1}{2a}+\right.$ $\left.+\frac{a\cos 2bx+2b\sin 2bx}{2(a^2+4b^2)}\right]$. **454.** $\frac{e^{ax}}{4}\left[\frac{3(a\sin bx-b\cos bx)}{a^2+b^2}-\frac{a\sin 3bx-3b\cos 3bx}{a^2+9b^2}\right]$.

455. $\frac{e^x}{2}[x(\sin x-\cos x)+\cos x]$. **456.** $\frac{e^x}{2}[x^2(\sin x+\cos x)-2x\sin x+(\sin x-\cos x)]$.

457. $e^x\left[\frac{x-1}{2}-\frac{x}{10}(2\sin 2x+\cos 2x)+\frac{1}{50}(4\sin 2x-3\cos 2x)\right]$. **458.** $\frac{1}{4}x^4+\frac{3}{4}x^2+3x^2\times$ $\times\cos x-x\left(6\sin x+\frac{3}{4}\sin 2x\right)-\left(5\cos x+\frac{3}{8}\cos 2x\right)-\frac{1}{3}\cos^3 x$. **459.** $\frac{x}{2}+\frac{1}{2}\sqrt{x}\sin(2\sqrt{x})+$ $+\frac{1}{4}\cos(2\sqrt{x})$. **461.** $x+\frac{1}{1+e^x}-\ln(1+e^x)$. **462.** $e^x-\ln(1+e^x)$. **463.** $-\frac{x}{2}+\frac{1}{3}\ln|e^x-1|+$ $+\frac{1}{6}\ln(e^x+2)$. **464.** $x-3\ln\{(1+e^{x/6})\sqrt{1+e^{x/3}}\}-3\operatorname{arctg}e^{x/6}$. **465.** $x+\frac{8}{1+e^{x/4}}$.

466. $-2\arcsin(-e^{-x/2})$. **467.** $\ln(e^x+\sqrt{e^{2x}-1})+\arcsin(e^{-x})$. **468.** $\sqrt{e^{2x}+4e^x-1}-$ $-\arcsin\left((2e^x-1)/e^x\sqrt{5}\right)+2\ln(e^x+2+\sqrt{e^{2x}+4e^x-1})$. **469.** $-\frac{1}{2}e^{-x}(\sqrt{1+e^x}-$ $-\sqrt{1-e^x})+\frac{1}{4}\ln\frac{(\sqrt{1+e^x}-1)(1-\sqrt{1-e^x})}{(\sqrt{1+e^x}+1)(1+\sqrt{1-e^x})}$. **471.** $a_1+\frac{a_2}{1!}+\frac{a_3}{2!}+\ldots+\frac{a_n}{(n-1)!}=0$.

472. $e^x\left(1-\frac{4}{x}\right)$. **473.** $-e^{-x}-\mathrm{li}(e^{-x})$. **474.** $e^4\mathrm{li}(e^{2x-4})-e^2\mathrm{li}(e^{2x-2})$. **475.** $\frac{e^x}{x+1}$. **476.** $\frac{e^{2x}}{2}(x^2+$ $+3x+\frac{21}{2}-\frac{32}{x-2})+64e^4\,\mathrm{li}(e^{2x-4})$. **477.** $x[\ln^n x-n\ln^{n-1}x+n(n-1)\ln^{n-2}x+\ldots+(-1)^{n-1}\times$ $\times n(n-1)\ldots2\ln x+(-1)^n n!]$. **478.** $\frac{x^4}{4}\left(\ln^3 x-\frac{3}{4}\ln^2 x+\frac{3}{8}\ln x-\frac{3}{32}\right)$. **479.** $-\frac{1}{2x^2}\left(\ln^3 x+\frac{3}{2}\times\right.$ $\left.\times\ln^2 x+\frac{3}{2}\ln x+\frac{3}{4}\right)$. **480.** $\ln(x+a)\ln(x+b)$. **481.** $x\ln^2\left(x+\sqrt{1+x^2}\right)-2\sqrt{1+x^2}\times$ $\times\ln\left(x+\sqrt{1+x^2}\right)+2x$. **482.** $-\frac{x}{2}+x\ln(\sqrt{1-x}+\sqrt{1+x})+\frac{1}{2}\arcsin x$. **483.** $\frac{x\ln x}{\sqrt{1+x^2}}-$ $-\ln\left(x+\sqrt{1+x^2}\right)$. **484.** $-\frac{x}{2}+\frac{1}{2}\ln(x^2+2x+2)+\frac{x^2}{2}\operatorname{arctg}(x+1)$. **485.** $-\frac{x}{3}+\frac{1}{3}\times$ $\times\ln(1+x)+\frac{2x\sqrt{x}}{3}\operatorname{arctg}\sqrt{x}$. **486.** $-\frac{3+x}{4}\sqrt{2x-x^2}+\frac{2x^2-3}{4}\arcsin(1-x)$. **487.** $\frac{1}{2}\times$

$\times\sqrt{x-x^2}+\left(x-\frac{1}{2}\right)\arcsin\sqrt{x}$. **488.** $-\frac{\operatorname{sgn}x}{2}\sqrt{x^2-1}+\frac{x^2}{2}\arccos\frac{1}{x}$. **489.** $2|1-\sqrt{x}|+$ $+(1+x)\arcsin\frac{2\sqrt{x}}{1+x}$. **490.** $\frac{x\arccos x}{\sqrt{1-x^2}}-\ln\sqrt{1-x^2}$. **491.** $\frac{\arccos x}{\sqrt{1-x^2}}+\frac{1}{2}\ln\frac{1+x}{1-x}$.

492. $x-\operatorname{arctg}x+\left(\frac{1+x^2}{2}\operatorname{arctg}x-\frac{x}{2}\right)[\ln(1+x^2)-1]$. **493.** $x-\frac{1-x^2}{2}\ln\frac{1+x}{1-x}$.

494. $-\ln\sqrt{1+x^2}+\frac{x}{\sqrt{1+x^2}}\ln(x+\sqrt{1+x^2})$. **495.** $-\frac{x}{8}+\frac{\operatorname{sh}4x}{32}$. **496.** $\frac{3x}{8}+\frac{\operatorname{sh}2x}{4}+$ $+\frac{\operatorname{sh}4x}{32}$. **497.** $\frac{\operatorname{ch}^3x}{3}-\operatorname{ch}x$. **498.** $\frac{\operatorname{ch}6x}{24}-\frac{\operatorname{ch}4x}{16}-\frac{\operatorname{ch}2x}{8}$. **499.** $\ln\operatorname{ch}x$. **500.** $x-\operatorname{cth}x$.

501. $0,5[\ln(e^{2x}+\sqrt{e^{4x}-1})+\arcsin(e^{-2x})]$. **502.** $\frac{2}{\sqrt{3}}\operatorname{arctg}3^{-1/2}\left(2\operatorname{th}\frac{x}{2}+1\right)$. **503.** $\frac{1}{\sqrt{5}}\times$ $\times\operatorname{arctg}\frac{\operatorname{th}x-2}{\sqrt{5}}$. **504.** $\frac{20}{3\sqrt{11}}\operatorname{arctg}\left(\frac{3\operatorname{th}(x/2)}{\sqrt{11}}\right)$. **505.** $-\frac{4}{7}x-\frac{3}{7}\ln|3\operatorname{sh}x-4\operatorname{ch}x|$.

506. $\frac{a\operatorname{ch}ax\sin bx-b\operatorname{sh}ax\cos bx}{a^2+b^2}$. **507.** $\frac{a\operatorname{ch}ax\cos bx+b\operatorname{sh}ax\sin bx}{a^2+b^2}$.

§6. Dürli görnüşdäki funksiýalary integrirlemegiň mysallary

508. $-\frac{1}{5x^5}+\frac{1}{3x^3}-\frac{1}{x}\operatorname{arctg}x$. **509.** $\frac{1}{8}\frac{x+x^3}{(1-x^2)^2}-\frac{1}{16}\ln\left|\frac{1+x}{1-x}\right|$. **510.** $\frac{1}{4\sqrt{3}}\times$ $\times\ln\frac{1+x\sqrt{3}+x^2}{1-x\sqrt{3}+x^2}-\frac{1}{2\sqrt{3}}\cdot\operatorname{arctg}\frac{1-x^2}{x\sqrt{3}}$. **511.** $2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln(\sqrt[6]{x}+1)$ $(x\geq 0)$. **512.** $\frac{5}{8}\arcsin\sqrt{x}-\frac{1}{24}(15+10x+8x^2)\sqrt{x(1-x)}$ $(0<x<1)$. **513.** $-\frac{2}{x}\sqrt{1-x^2}-$ $-\ln\frac{1+\sqrt{1-x^2}}{|x|}$ $(|x|<1)$. **514.** $-\frac{4}{3}\sqrt{1-x\sqrt{x}}$ $(x>0)$. **515.** $\frac{1}{15}(8-4x^2+3x^4)\sqrt{1+x^2}$. **516.** $\frac{1}{2}\ln\frac{(1+z)^2}{1-z+z^2}-\sqrt{3}\operatorname{arctg}\frac{2z-1}{\sqrt{3}}$, bu ýerde $z=\sqrt[3]{\frac{1-x}{x}}$, **517.** $-\frac{1}{3}\times$ $\times\ln\left|\frac{2+x^3+2\sqrt{1+x^3+x^6}}{x^3}\right|$. **518.** $\frac{1}{2}\arccos\frac{x^2+1}{x^2\sqrt{2}}$. **519.** $-\frac{2+x^2}{x}-\frac{2}{x}\sqrt{1-x^2}-2\times$ $\times\arcsin x$ $(|x|<1)$. **520.** $-\frac{1}{2}(1+x)^2+\frac{5+2x}{4}\sqrt{x+x^2}+\frac{3}{8}\ln\left|x+\frac{1}{2}+\sqrt{x+x^2}\right|$ $(x>0;$ $x<-1)$. **521.** $-\frac{\ln(1+x+x^2)}{1+x}-\frac{1}{2}\ln\frac{(1+x)^2}{1+x+x^2}+\sqrt{3}\operatorname{arctg}\frac{1+2x}{\sqrt{3}}$. **522.** $\frac{-2x+21}{4}\times$ $\times\sqrt{-x^2+3x-2}+\left(x^2+3x-\frac{55}{8}\right)\arccos(2x-3)$ $(1<x<2)$. **523.** $-x^2+\frac{x^2}{2}\ln(4+x^4)+$ $+2\operatorname{arctg}\frac{x^2}{2}$. **524.** $-\frac{\sqrt{1-x^2}}{x}\arcsin x+\ln|x|+\frac{1}{2}(\arcsin x)^2$ $(0<|x|<1)$. **525.** $(1+\sqrt{1+x^2})\times$

$\times \ln\left(1+\sqrt{1+x^2}\right)-\sqrt{1+x^2}$. **526.** $-\frac{x^2+7}{9}\sqrt{x^2+1}+\frac{(x^2+1)^{3/2}}{3}\ln\sqrt{x^2-1}-\frac{1}{3}\times$ $\times\ln\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}$ ($|x|>1$). **527.** $(\sqrt{1-x^2})\left(\frac{3-x}{1-x}-\ln\frac{x}{\sqrt{1-x}}\right)-\frac{1}{2}\arcsin x-\ln\frac{1+\sqrt{1-x^2}}{x}$ ($0<x<1$). **528.** $\frac{\cos x}{3(2+\sin x)}+\frac{4}{3\sqrt{3}}\operatorname{arctg}\frac{2\operatorname{tg}(x/2)+1}{\sqrt{3}}$. **529.** $\frac{1}{\sqrt{2}}\ln\frac{7+4\sqrt{2}+\cos 4x}{7-4\sqrt{2}-\cos 4x}$. **530.** $-\frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2}+\sqrt{1+\cos x}}{\sqrt{2}-\sqrt{1+\cos x}}+\frac{1}{\sqrt{1+\cos x}}$. **531.** $a\left[x\operatorname{arctg}x-\frac{1}{2}\ln\left(x^2+1\right)\right]-\frac{a-b}{2}\times$ $\times(\operatorname{arctg}x)^2$. **532.** $a\left(x\ln\left|\frac{x-1}{x+1}\right|-\ln\left|x^2-1\right|\right)+\frac{a+b}{4}\ln^2\left|\frac{x-1}{x+1}\right|$. **533.** $-\frac{\ln x}{2(1+x^2)}+\frac{1}{4}\times$ $\times\ln\frac{x^2}{1+x^2}$ ($x>0$). **534.** $\sqrt{1+x^2}\operatorname{arctg}x-\ln\left(x+\sqrt{1+x^2}\right)$. **535.** $-\ln\left(\cos^2 x+\sqrt{1+\cos^4 x}\right)$. **536.** $-\frac{6x+x^3}{9}-\frac{2+x^2}{9}\sqrt{1-x^2}\arccos x$ ($|x|<1$). **537.** $\frac{2}{3}\ln(1+x^2)-x^2/6-(x-x^3/3)\times$ $\times\operatorname{arctg}x+\frac{1}{2}(\operatorname{arctg}x)^2$. **538.** $-\frac{x}{4(1+x^2)}-\frac{1-x^2}{4(1+x^2)}\operatorname{arctg}x$. **539.** $\frac{\ln\left(x+\sqrt{1+x^2}\right)}{2(1-x^2)}+$ $+\frac{1}{4\sqrt{2}}\ln\frac{\sqrt{1+x^2}-x\sqrt{2}}{\sqrt{1+x^2}+x\sqrt{2}}$ ($|x|<1$). **540.** $-\frac{x^2}{4}+\frac{x}{2}\sqrt{1-x^2}\arcsin x+\frac{1}{4}(\arcsin x)^2$ ($|x|<1$). **541.** $\frac{x}{4}+\frac{x^3}{12}+\frac{1}{4}(1+x^2)^2\operatorname{arctg}x$. **542.** x^x ($x>0$). **543.** $-\ln\left(1+\sqrt{1-e^{2x}}\right)+x-e^{-x}\arcsin\left(e^x\right)$ ($x<0$). **544.** $x-\ln(1+e^x)-2e^{-x/2}\operatorname{arctg}e^{x/2}-(\operatorname{arctg}e^{x/2})^2$. **545.** $-\frac{\operatorname{cth}1}{4}\left[x-\ln\left(1+e^x\operatorname{ch}1\right)\right]-$ $-\frac{e^{-x}}{4\operatorname{sh}1}$. **546.** $-2\ln\left(\operatorname{th}x+\sqrt{1+\operatorname{th}^2 x}\right)+\frac{1}{\sqrt{2}}\ln\frac{\sqrt{1+\operatorname{th}^2 x}+\sqrt{2}\operatorname{th}x}{\sqrt{1+\operatorname{th}^2 x}-\sqrt{2}\operatorname{th}x}$. **547.** $e^x\operatorname{tg}\frac{x}{2}$. **548.** $\frac{x|x|}{2}$. **549.** $\frac{x^2|x|}{3}$. **550.** $\frac{2x^2}{3}(x+|x|)$. **551.** $\frac{(1+x)|1+x|}{2}+\frac{(1-x)|1-x|}{2}$. **552.** e^x-1, eger $x<0$ bolsa; $1-e^{-x}$, eger $x\geq 0$ bolsa. **553.** x, eger $|x|\leq 1$ bolsa; $\frac{x^3}{3}+\frac{2}{3}\operatorname{sgn}x$, eger $|x|>1$ bolsa. **554.** $\frac{x}{4}+\frac{1}{4}\left((x)-\frac{1}{2}\right)\{1-2|(x)-1/2|\}$, bu ýerde $(x)=x-[x]$. **555.** $[x]/\pi\{[x]-(-1)^{[x]}\cos\pi x\}$. **556.** $x-x^3/3$, eger $|x|\leq 1$ bolsa; $x-\frac{x}{2}|x|+\frac{1}{6}\operatorname{sgn}x$, eger $|x|>1$ bolsa. **557.** x, eger $-\infty<x\leq 0$ bolsa; $x^2/2+x$, eger $0\leq x\leq 1$ bolsa; $x^2+1/2$, eger $x>1$ bolsa. **558.** $xf'(x)-f(x)$. **559.** $f(2x)/2$. **560.** $f(x)=2\sqrt{x}$. **561.** $x-x^2/2$. **562.** $f(x)=x$, $-\infty<x\leq 0$ bolanda; $f(x)=e^x-1$, $0<x<+\infty$ bolanda.

VII. §1. Kesgitli integral we integrirlemek usullary

1. $12\frac{1}{2}$. **2.** a) $\underline{S_n}=16\frac{1}{4}-\frac{175}{2n}+\frac{125}{4n^2}$, $\overline{S_n}=16\frac{1}{4}+\frac{175}{2n}+\frac{125}{4n^2}$; b) $\underline{S_n}=\frac{1}{n}\sum_{i=0}^{n-1}\sqrt{i/n}$, $\overline{S_n}=\frac{1}{n}\sum_{i=1}^{n}\sqrt{i/n}$; ç) $\underline{S_n}=\frac{10230}{n(2^{10/n}-1)}$, $\overline{S_n}=\frac{10230\cdot 2^{10/n}}{n(2^{10/n}-1)}$. **3.** $\underline{S_n}=31\cdot\frac{\sqrt[n]{2}-1}{\sqrt[n]{32}-1}$; $\frac{31}{5}$.

4. $\vartheta_0 T+\frac{1}{2}gT^2$. **5.** 3. **6.** $(a-1)/\ln a$. **7.** 1. **8.** $\sin x$. **9.** $1/a-1/b$. **10.** $(b^{m+1}-a^{m+1})/(m+1)$. **11.** $\ln(b/a)$. **12.** a) $|\alpha|<1$ bolanda 0-a deň; b) $|\alpha|>1$ bolanda $\pi\ln\alpha^2$-e deň. **17.** $((b-a)/2)\times$ $\times[f(a)-f(b)]$. **25.** Umuman aýdylanda, ýok. **27.** Hökman däl. **30.** 45/4. **31.** 2. **32.** $\pi/6$. **33.** $\pi/3$. **34.** 1. **35.** 1. **36.** $\frac{\pi}{2\sin\alpha}$. **37.** $\frac{2\pi}{\sqrt{1-\varepsilon^2}}$. **38.** $\frac{1}{\sqrt{ab}}\ln\frac{1+\sqrt{ab}}{1-\sqrt{ab}}$. **39.** $\frac{\pi}{2|ab|}$. **40.** a) integral astyndaky $1/x$ funksiýa we onuň $\ln|x|$ asyl funksiýasy $[-1, 1]$ integrirleme kesimde üznükli; b) $(1/\sqrt{2})\mathrm{arctg}(\mathrm{tg}x/\sqrt{2})$ asyl funksiýa $0\le x\le 2\pi$ bolanda üznükli; ç) $\mathrm{arctg}(1/x)$ funksiýa $x=0$ bolanda üznükli. **41.** 2/3. **42.** $200\sqrt{2}$. **43.** 1/2. **44.** ln2. **45.** $\pi/4$. **46.** $2/\pi$. **47.** $1/(p+1)$. **48.** $2(2\sqrt{2}-1)/3$. **49.** $1/e$. **50.** $[1/(b-a)]\times$ $\times\int_a^b f(x)dx$. **51.** $5\pi/6$. **52.** $\pi/\sqrt{3}$. **53.** $x+1/2$. **54.** 1/ln2. **55.** a) 0; b) $-\sin a^2$; ç) $\sin b^2$. **56.** a) $2x\sqrt{1+x^4}$; b) $\frac{3x^2}{\sqrt{1+x^{12}}}-\frac{2x}{\sqrt{1+x^8}}$; ç) $(\sin x-\cos x)\cos(\pi\sin^2 x)$. **57.** a) 1; b) $\pi^2/4$; ç) 0. **58.** A. **60.** 1. **62.** a) 5/6; b) $t/2$. **63.** a) $\frac{1}{3}-\frac{\alpha}{2}$, $\alpha<0$ bolanda; $\frac{1}{3}-\frac{\alpha}{2}+\frac{\alpha^3}{3}$, $0\le\alpha\le 1$ bolanda; $\frac{\alpha}{2}-\frac{1}{3}$, $\alpha>1$ bolanda; b) $\frac{\pi}{2}$, $|\alpha|\le 1$ bolanda; $\frac{\pi}{2\alpha^2}$, $|\alpha|>1$ bolanda; ç) 2, $|\alpha|\le 1$ bolanda; $\frac{2}{|\alpha|}$, $|\alpha|>1$ bolanda. **64.** $\frac{1}{2}\ln\frac{e}{2}$. **65.** π. **66.** 4π. **67.** $2\left(1-\frac{1}{e}\right)$. **68.** 1. **69.** $\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$. **70.** $\frac{1}{6}$. **71.** $\frac{\pi a^4}{16}$. **72.** $\frac{1}{\sqrt{2}}\ln\frac{9+4\sqrt{2}}{7}$. **73.** $2-\frac{\pi}{2}$. **74.** $\frac{\pi^2}{4}$. **75.** $\frac{\pi}{\sqrt{2}}$. **76.** a) berlen funksiýanyň $x=\pm t^{3/2}$ ters funksiýasy iki bahaly; b) $x=1/t$ funksiýa $t=0$ nokatda üzülýär; ç) $x=\mathrm{arctg}t$ funksiýanyň bahalary $(0;\pi)$ interwalda üýtgeýän üznüksiz birbahaly şahasy ýok. **77.** Ýok. **78.** Bolup biler. **81.** $f(x+b)-f(x+a)$. **85.** $\frac{3}{2}e^{5/2}$. **86.** $\int_0^1[f(\arcsin t)-f(\pi-\arcsin t)]dt+\int_{-1}^0[f(2\pi+\arcsin t)-f(\pi-\arcsin t)]dt$. **87.** $4n$. **88.** $\frac{\pi^2}{4}$. **89.** $\mathrm{arctg}\frac{32}{27}-2\pi$. **93.** $315\frac{1}{26}$. **94.** $\frac{1}{2}\ln 3-\frac{\pi}{2\sqrt{3}}$. **95.** $\frac{5}{27}e^3-\frac{2}{27}$. **96.** $-66\frac{6}{7}$. **97.** $-\frac{\pi}{3}$. **98.** $\frac{29}{270}$. **99.** $\frac{4}{3}\pi-\sqrt{3}$. **100.** $2\pi\left(\frac{1}{\sqrt{3}}-\frac{1}{2\sqrt{2}}\right)$. **101.** $2\pi\sqrt{2}$. **102.** $\frac{1}{6}$. **103.** $\frac{\pi^3}{6}-\frac{\pi}{4}$. **104.** $\frac{3}{5}(e^{\pi}-1)$. **105.** $\frac{3}{8}\ln 2-\frac{225}{1024}$. **106.** $I_n=\frac{(2k-1)!!}{(2k)!!}\cdot\frac{\pi}{2}$, $n=2k$ bolanda; $I_n=\frac{(2k)!!}{(2k+1)!!}$, $n=2k+1$ bolanda. **107.** 106-a seret. **108.** $(-1)^n\left[\frac{\pi}{4}-\left(1-\frac{1}{3}+\frac{1}{5}-...+\frac{(-1)^{n-1}}{2n-1}\right)\right]$. **109.** $2^{2n}\frac{(n!)^2}{(2n+1)!}$. **110.** 106-a seret. **111.** $I_n = \dfrac{(-1)^n n!}{(m+1)^{n+1}}$. **112.** $I_n = (-1)^n \left\{-\ln\sqrt{2} + \frac{1}{2}\left[1 - \frac{1}{2} + \ldots + (-1)^{n-1}\frac{1}{n}\right]\right\}$.

115. $\dfrac{\pi(2m)!(2n)!}{2^{2m+2n+1}m!n!(m+n)!}$ **116.** 0, n jübüt bolanda; π, n täk bolanda. **117.** $(-1)^n\pi$.

118. $\frac{\pi}{2^n}$. **119.** $\frac{\pi}{2^n}\sin\frac{n\pi}{2}$. **120.** 0. **121.** 0. **122.** $\dfrac{1-e^{-2a\pi}}{2^{2n}a}\left[C_{2n}^n + 2\sum_{k=0}^{n-1} C_{2n}^k \dfrac{a^2}{a^2+(2n-2k)^2}\right]$

123. $\frac{\pi}{4n}(-1)^{n-1}$. **124.** $\dfrac{(m-1)!(n-1)!}{(m+n-1)!}$. **127.** $f(x)$ funksiýanyň üzülme nokatlarynda $F'(x)$ önüm bolup hem biler, bolman hem biler. a) $f(1/n)=1$ ($n=\pm1, \pm2, \ldots$) we $f(x)=0$, $x\neq 1/n$; b) $f(x)=\mathrm{sgn}x$. **128.** $|x|+C$. **129.** $\arccos(\cos x)+C$. **130.** $x[x] - \dfrac{[x](|x|+1)}{2} + C$. **131.** $\dfrac{x^2[x]}{2} - - \dfrac{[x]([x]+1)(2[x]+1)}{12} + C$. **132.** $C + \frac{1}{\pi}\arccos(\cos\pi x)$. **133.** $(|l+x|-|l-x|)/2+C$. **134.** -1. **135.** $14-\ln 7!$. **136.** $30/\pi$. **137.** $-\pi^2/4$. **138.** $\ln n!$. **139.** $-\mathrm{th}(\pi/2)$. **140.** 8/3.

§2. Orta baha hakyndaky teoremalar

141. a) –; b) +; ç) +; d) –. **142.** a) ikinji; b) ikinji; ç) birinji. **143.** a) $\frac{1}{3}$; b) $6\frac{2}{3}$; ç) 10; d) $\frac{1}{2}\cos\varphi$. **144.** $\dfrac{p}{\sqrt{1-\varepsilon^2}} = b$ – ellipsiň kiçi ýarym oky. **145.** $\vartheta_{\text{ort}} = \frac{1}{2}(\vartheta_0 + \vartheta_1)$. bu ýerde ϑ_1 – jisimiň ahyrky tizligi. **146.** $\frac{1}{2}i_0^2$. **147.** A. **148.** a) $\theta = \sqrt[n]{\dfrac{1}{1+n}}$; b) $\theta = \frac{1}{e}$; ç) $\theta = \frac{1}{x}\ln\dfrac{e^x-1}{x}$, $\lim\limits_{x\to 0}\theta = \frac{1}{2}$, $\lim\limits_{x\to+\infty}\theta = 1$. **149.** $\frac{8\pi}{3} \pm \frac{4\pi}{3}\theta$ ($|\theta|<1$). **150.** $\dfrac{1}{10\sqrt{2}}$-dan $\frac{1}{10}$-e çenli aralykda ýerleşýär. **151.** $0{,}01-0{,}005\theta$ ($0<\theta<1$). **153.** a) 1; b) $f(0)\ln\frac{b}{a}$. **155.** $\frac{\theta}{50\pi}$ ($0<\theta<1$). **156.** $\frac{2}{a}\theta$ ($|\theta|<1$). **157.** $\frac{\theta}{a}$ ($|\theta|<1$).

§3. Hususy däl integrallar

161. $\frac{1}{a}$. **162.** -1. **163.** π. **164.** π. **165.** $\frac{2}{3}\ln 2$. **166.** $\dfrac{4\pi}{3\sqrt{3}}$. **167.** $\dfrac{2\pi}{3\sqrt{3}}$. **168.** $\frac{\pi}{\sqrt{2}}$. **169.** $\frac{\pi}{2}$. **170.** $\frac{1}{5}\ln\left(1+\frac{2}{\sqrt{3}}\right)$. **171.** 0. **172.** $\frac{\pi}{2} - 1$. **173.** $\dfrac{a}{a^2+b^2}$. **174.** $\dfrac{b}{a^2+b^2}$. **175.** $I_n = n!$ **176.** $I_n = \dfrac{(2n-3)!!}{(2n-2)!!}\cdot\dfrac{\pi a^{n-1}\mathrm{sgn}a}{(ac-b^2)^{n+1/2}}$. **177.** $I_n = n!\sum_{k=1}^{n}(-1)^{k+1}C_n^k\ln(k+1)$, bu ýerde C_n^k – n elementiň k boýunça utgaşmasy. **178.** $I_n = \dfrac{(n-1)!!}{n!!}\cdot\frac{\pi}{2}$, eger n jübüt bolsa we $I_n = \frac{(n-1)!!}{n!!}$, eger n täk bolsa. **179.** $I_n = \frac{(n-1)!!}{n!!}\pi$, eger n jübüt bolsa, $I_n = \frac{(n-1)!!}{n!!}$, eger n täk bolsa. **180.** a) $-\frac{\pi}{2}\ln 2$; b) $-\frac{\pi}{2}\ln 2$. **181.** $\frac{2\sqrt[4]{8}\cdot e^{-\pi/8}}{1-e^{-\pi}}$.

183. a) 1; b) $\frac{\pi}{2}$; ç) 0. **184.** a) 1; b) $\frac{1}{3}$; ç) 1; d) $\frac{1}{\alpha}f(0)$. **185.** Ýygnanýar. **186.** Ýygnanýar. **187.** Dargaýar. **188.** $p>0$ bolanda ýygnanýar. **189.** $p>-1$, $q>-1$ bolanda ýygnanýar. **190.** $m>-1$; $n-m>1$ bolanda ýygnanýar. **191.** $1<n<2$ bolanda ýygnanýar. **192.** $1<n<2$ bolanda ýygnanýar. **193.** $m>-2$; $n-m>1$ bolanda ýygnanýar. **194.** $n>0$ $(a\neq 0)$ bolanda ýygnanýar. **195.** Dargaýar. **196.** $p<1$, $q<1$ bolanda ýygnanýar. **197.** $n>-1$ bolanda ýygnanýar. **198.** Ýygnanýar. **199.** $\min(p, q)<1$, $\max(p, q)>1$ bolanda ýygnanýar. **200.** Ýygnanýar. **201.** Ýygnanýar. **202.** $p>1$, $q<1$ bolanda ýygnanýar. **203.** $p>1$, erkin q, $r<1$ we $p=1$, $q>1$, $r<1$ bolanda ýygnanýar. **204.** $p_i<1$ $(i=1, 2, ..., n)$, $\sum_{i=1}^{n} p_i > 1$ bolanda ýygnanýar. **205.** $\alpha>-1$, $\beta>-1$, $\alpha+\beta<-1$ bolanda ýygnanýar. **206.** $P_n(x)$-iň $[0, +\infty)$ aralykda köki ýok we $n>m+1$ bolanda ýygnanýar. **207.** Absolýut däl ýygnanýar. **208.** Absolýut däl ýygnanýar. **209.** $-1<(p+1)/q<0$ bolanda absolýut ýygnanýar; $0\leq(p+1)/q<1$ bolanda bolsa şertli ýygnanýar. **210.** Ýygnanýar. **211.** Ýygnanýar. **212.** $p>-2$, $q>p+1$ bolanda absolýut ýygnanýar, $p>-2$, $p<q\leq p+1$ bolanda bolsa şertli ýygnanýar. **213.** $0<n<2$ bolanda şertli ýygnanýar. **214.** $n>m+1$ bolanda absolýut ýygnanýar, $m<n\leq m+1$ bolanda bolsa şertli ýygnanýar. **217.** Ýok. **224.** $\ln(1/2)$. **225.** 0. **226.** π. **227.** 0.

§4. Kesgitli integrallaryň geometriýada ulanylyşlary

229. $a^2/3$. **230.** 9/2. **231.** 9/2. **232.** $9{,}9-8{,}1\lg e\approx 6{,}38$. **233.** $2-1/\ln 2\approx 0{,}56$. **234.** $1/3+2/\pi\approx 0{,}97$. **235.** $\pi/2$. **236.** πa^2. **237.** πab. **238.** $\frac{4}{3}a^3$. **239.** $\frac{88}{15}\sqrt{2}p^2$.

240. $\frac{\pi}{\sqrt{AC-B^2}}$. **241.** $3\pi a^2$. **242.** $\frac{\pi a^2}{2}$. **243.** $\frac{2\pi}{n+2}$. **244.** $\frac{1}{2}\operatorname{cth}\frac{\pi}{2}\approx 0{,}546$. **245.** $(3\pi+2):(9\pi-2)$. **246.** $x=\operatorname{ch}S$, $y=\operatorname{sh}S$. **247.** $3\pi a^2$. **248.** 8/15. **249.** $\frac{a^2}{3}(4\pi^2+3\pi)$. **250.** $6\pi a^2$.

251. $\frac{3\pi c^4}{8ab}$. **252.** $\pi a^2(16/\sqrt{3}-9)$. **253.** a^2. **254.** $\frac{3\pi a^2}{2}$. **255.** $\frac{\pi a^2}{4}$. **256.** $\frac{p^2}{6}(3+4\sqrt{2})$.

257. $\frac{\pi p^2}{(1-\varepsilon^2)^{3/2}}$. **258.** 11π. **259.** $\frac{1}{\pi}$. **260.** $(\pi-1)\frac{a^2}{4}$. **261.** $\frac{1}{2}\left(1-\ln 2+\frac{\pi}{\sqrt{3}}\right)$. **262.** $\frac{2}{3}$.

263. $\frac{1}{\pi}$. **264.** $4\frac{4}{15}$. **265.** $\pi\left(1+\frac{\pi^2}{6}\right)$. **266.** $\pi\left(1-\frac{\pi}{4}\right)a^2$. **267.** $\frac{3}{2}a^2$. **268.** $\pi a^2\sqrt{2}$.

269. a^2. **270.** $\frac{3}{8}\pi a^2$. **271.** $\frac{\pi a^2}{8\sqrt{2}}$. **272.** $\frac{8}{27}(10\sqrt{10}-1)$. **273.** $2\sqrt{x_0\left(x_0+\frac{p}{2}\right)}+p\times$ $\times\ln\frac{\sqrt{x_0}+\sqrt{x_0+p/2}}{\sqrt{p/2}}$. **274.** $\sqrt{h^2-a^2}$. **275.** $x_0-\sqrt{2}+\sqrt{1+e^{2x_0}}-\ln\frac{1+\sqrt{1+e^{2x_0}}}{1+\sqrt{2}}$.

276. $\frac{e^2+1}{4}$. **277.** $a\ln\frac{a+b}{a-b}-b$. **278.** $\ln\operatorname{tg}\left(\frac{\pi}{4}+\frac{a}{2}\right)$. **279.** $a\ln\frac{a}{b}$. **280.** $4a(1+\sqrt{3}\times$ $\times\ln\frac{1+\sqrt{3}}{\sqrt{2}}\Big)$. **281.** $6a$. **282.** $\frac{4(a^3-b^3)}{ab}$. **283.** $1+\frac{\ln(1+\sqrt{2})}{\sqrt{2}}$. **284.** $8a$. **285.** $2\pi^2 a$.

286. $2\left(\operatorname{ch}\frac{T}{2}\sqrt{\operatorname{ch}T}-1\right)-\sqrt{2}\ln\frac{\sqrt{2}\operatorname{ch}(T/2)+\sqrt{\operatorname{ch}T}}{1+\sqrt{2}}$. **287.** $\frac{1}{2}\left(\operatorname{ch}^{3/2}2T-1\right)$. **288.** $\pi a\times$ $\times\sqrt{1+4\pi^2}+\frac{a}{2}\ln\left(2\pi+\sqrt{1+4\pi^2}\right)$. **289.** $\frac{\sqrt{1+m^2}}{m}a$. **290.** $8a$. **291.** $p[\sqrt{2}+\ln(1+\sqrt{2})]$.

292. $3\pi a/2$. **293.** $a(2\pi-\operatorname{th}\pi)$. **294.** $2+\frac{1}{2}\ln 3$. **295.** $6\frac{1}{3}$. **296.** $\operatorname{sh}R$. **297.** T. **300.** $\frac{2\pi}{5\sqrt{3}}\approx 0{,}73$.

301. $\frac{bh}{6}(2a+c)$. **302.** $\frac{h}{6}[(2A+a)B+(A+2a)b]$. **303.** $\frac{\pi h}{6}[(2A+a)B+(A+2a)b]$.

304. $\frac{1}{2}SH$. **307.** $\frac{2}{3}abc$. **308.** $\frac{4}{3}\pi abc$. **309.** $\frac{8\pi abc}{3}$. **310.** $\frac{16}{3}a^3$. **311.** $\frac{2}{3}a^3\left(\pi-\frac{4}{3}\right)$.

312. $\frac{16}{15}a^2\sqrt{ab}$. **313.** $\frac{\pi a^3}{2}$. **314.** $\frac{4}{15}$. **315.** $\frac{4\pi\sqrt{2}}{3}a^3$. **317.** $\frac{3}{7}\pi ab^2$. **318.** a) $\frac{16\pi}{15}$, b) $\frac{8\pi}{3}$.

319. a) $\frac{\pi^2}{2}$, b) $2\pi^2$. **320.** a) $\frac{4}{15}\pi ab^2$, b) $\frac{\pi a^2 b}{6}$. **321.** a) $\frac{\pi}{2}$, b) 2π. **322.** $2\pi^2a^2b$. **323.** $\frac{8\pi a^3}{3}$.

324. $\frac{\pi}{5(1-e^{-2\pi})}$. **325.** a) $5\pi^2a^3$, b) $6\pi^3a^3$, ç) $7\pi^2a^3$. **326.** a) $\frac{32}{105}\pi ab^2$, b) $\frac{32}{105}\pi a^2 b$.

327. $V_x=\frac{64}{35}\pi$, $V_y=\frac{64}{105}\pi$. **329.** a) $\frac{8}{3}\pi a^3$, b) $\frac{13}{4}\pi^2a^3$. **330.** a) $\frac{\pi a^3}{4}\left[\sqrt{2}\ln\left(1+\sqrt{2}\right)-\frac{2}{3}\right]$, b) $\frac{\pi^2a^3}{4\sqrt{2}}$, ç) $\frac{\pi^2a^3}{4}$. **331.** $\frac{2}{3}(\pi^4-6\pi^2)a^3$. **332.** $\frac{2}{3}\pi$. **333.** $\frac{\pi^2a^3}{2\sqrt{2}}$. **334.** $\frac{4\pi a^2}{243}(21\sqrt{13}+$ $+2\ln\frac{3+\sqrt{13}}{2}\Big)$. **335.** $2a\sqrt{\pi^2a^2+4b^2}+\frac{8b^2}{\pi}\ln\left(\frac{\pi a}{2b}+\frac{\sqrt{\pi^2a^2+4b^2}}{2b}\right)$. **336.** $\pi\Big[(\sqrt{5}-$ $-\sqrt{2})+\ln\frac{(\sqrt{2}+1)(\sqrt{5}-1)}{2}\Big]$. **337.** a) $\frac{2\pi}{3}\left[(2x_0+p)\sqrt{2px_0+p^2}-p^2\right]$, b) $\frac{\pi}{4}[(p+4x_0)\times$ $\times\sqrt{2x_0(p+2x_0)}-p^2\ln\frac{\sqrt{2x_0}+\sqrt{p+2x_0}}{\sqrt{p}}\Big]$. **338.** a) $2\pi b^2+2\pi ab\frac{\arcsin\varepsilon}{\varepsilon}$, b) $2\pi a^2+$ $+\frac{2\pi b^2}{\varepsilon}\ln\left[\frac{a}{b}(1+\varepsilon)\right]$, bu ýerde $\varepsilon=\frac{\sqrt{a^2-b^2}}{a}$ ellipsiň ekssentrisiteti. **339.** $4\pi^2ab$.

340. $\frac{12}{5}\pi a^2$. **341.** a) $\pi a\left(2b+a\operatorname{sh}\frac{2b}{a}\right)$, b) $2\pi a\left(a+b\operatorname{sh}\frac{b}{a}-a\operatorname{ch}\frac{b}{a}\right)$. **342.** $4\pi a^2$.

343. a) $\frac{64}{3}\pi a^2$, b) $16\pi^2a^2$, ç) $\frac{32}{3}\pi a^2$. **344.** $\frac{3\pi}{5}a^2(4\sqrt{2}-1)$. **345.** $\frac{32}{5}\pi a^2$. **346.** a) $2\pi a^2\times$ $\times\left(2-\sqrt{2}\right)$; b) $2\pi a^2\sqrt{2}$; ç) $4\pi a^2$. **347.** $\frac{5}{128\sqrt[3]{10}}[14\sqrt{5}+17\ln(2+\sqrt{5})]\approx 1{,}013$.

348. $V=\frac{4\pi}{3}p^2$; $P=2\pi p^2[(2+\sqrt{2})+\ln(1+\sqrt{2})]$.

§5. Kesgitli integrallaryň fizikada ulanylyşlary

349. $M_1=2a^2$; $M_2=\frac{\pi a^3}{2}$. **350.** $\frac{p^2}{8}[\sqrt{2}+5\ln(1+\sqrt{2})]$. **351.** $M_1=bh^2/6$; $M_2=bh^3/12$. **352.** $I_x=8a^4/35$; $I_y=8a^4/5$; $r_x=a\sqrt{6/35}$; $r_y=a\sqrt{6/5}$. **353.** $M_2^{(x)}=\pi ab^3/4$; $M_2^{(y)}=\pi a^3b/4$. **354.** $M_1=\pi r^2h^2/12$, $M_2=\pi r^2h^3/30$. **355.** $I=2MR^2/5$. **358.** $x_0=a\sin\alpha/\alpha$, $y_0=0$. **359.** $(9a/20,\ 9a/20)$. **360.** $(4a/3\pi,\ 4b/3\pi)$. **361.** $(0,\ 0,\ 3a/8)$. **362.** $\varphi_0=\varphi-\alpha$, $\alpha=\mathrm{arctg}(1/2m)$ we $r_0=mr/\sqrt{1+4m^2}$; bu ýerde $r_0=ame^{m(\varphi_0+\alpha)}/\sqrt{1+4m^2}$ logarifmik spiral. **363.** $\varphi_0=0$, $r_0=5a/6$. **364.** $x_0=\pi a$, $y_0=5a/6$. **365.** $x_0=2a/3$, $y_0=0$. **366.** $(0,\ 0,\ a/2)$. **367.** 75 *kg*. **368.** $A_h=mg(Rh/(R+h))$, $A_\infty=mgR$; bu ýerde R – ýeriň radiusy. **369.** 0,5 $kg\cdot m$. **370.** 1740 $kg\cdot m$. **371.** $2a^3/3$. **372.** $2125T/3$. **373.** $\vartheta_0T+aT^2/2$. **374.** $4\pi\delta\omega^2R^5/15$. **375.** Dartylma güýjüniň koordinatalar oklaryna bolan proýeksiýasy $x=0$, $y=-2km\mu_0/a$; bu ýerde k – dartylma hemişeligi. **376.** $2\pi km\delta_0\left(1-b/\sqrt{a^2+b^2}\right)$; bu ýerde k – dartylma hemişeligi. **377.** Takmynan üç sagat. **378.** Gap Oy okunyň daşyndan $y=Cx^4$ egri çyzygyň aýlanmagy bilen emele gelýär. **377.** $Q=Q_02^{-t/1600}$. **380.** 99,92%. **381.** $\gamma H^2/6E$.

§6. Kesgitli integrallaryň takmyny hasaplanylyşy

Jogaplarda takmyny integraly hasaplamakda tablisadaky bahalary alynýar. **382.** – 6,2832. **383.** 0,69315. **384.** 0,83566. **385.** 1,4675. **386.** 17,333. **387.** 5,4024. **388.** 1,37039. **289.** 0,2288. **390.** 0,915966. **391.** 3,14159. **392.** 1,463. **393.** 0,3179. **394.** 0,8862. **395.** 51,04. **396.**

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	0	0,99	1,65	1,85	1,72	1,52	1,42

VIII. §1. Köp üýtgeýänli funksiýalaryň predeli we üznüksizligi

1. $x\geq0$ ýarym tekizlik. **2.** $|x|\leq1$; $|y|\geq1$. **3.** $x^2+y^2\leq1$ tegelek. **4.** $x^2+y^2>1$ tegelegiň daşy. **5.** $1\leq x^2+y^2\leq4$ halka. **6.** $x\leq x^2+y^2<2x$ çukurjyk. **7.** $-1\leq x^2+y\leq1$. **8.** $x+y<0$ ýarymüst. **9.** $|y|\leq|x|$ $(x\neq0)$ wertikal burçlaryň jübüti. **10.** $y=0$ we $y=-2x$ göni çyzyklar bilen çäklenen wertikal kütek burçlaryň jübüti, $O(0,\ 0)$ umumy depesiz çägi girizip. **11.** $y^2=x$, $y^2=-x$ parabolalar we $y=2$ göni çyzyk bilen çäklenen, $O(0,\ 0)$ depesiz egri çyzykly üçburçluk. **12.** $2\pi k\leq x^2+y^2\leq\pi(2k+1)$ $(k=0,\ 1,\ 2...)$ konsentrik halkalary. **13.** Araçägi girizilen we depesi aýrylan $x^2+y^2-z^2=0$ konusyň daşky tarapy. **14.** Giňişligiň dört oktantlarynyň toplumy. **15.** Iki boşlukly $x^2+y^2-z^2=-1$ giperboloidiň içi. **16.** Parallel göni çyzyklar. **17.** Konsentrik töwerekler. **18.** $y=\pm x$ umumy asimptotaly deňtaraply giperbolalar. **19.** Parallel göni çyzyklar. **20.** Depeleri koordinatalar başlangyjynda ýerleşýän göni çyzyklaryň dessesi, depäni aýyrmak bilen. **21.** Meňzeş ellipsler. **22.** Koordinatalar oklaryna asimptotik ýakynlaşýan hem-de I we III kwadrantlarda ýerleşýän deňtaraply giperbolalar. **23.** Depeleri Oy okunda ýerleşýän iki zynjyrly döwük çyzyklar. **24.** $z=0$ bolanda I we III kwadrantlar; $z>0$ bolanda depeleri $x+y=0$ göni çyzykda ýerleşýän, zynjyrlary koordinatalar oklaryna parallel iki zynjyrly döwük çyzyklar. **25.** Depeleri $y=x$ göni çyzykda, Ox we Oy oklarynyň položitel ugurlaryna parallel bolan çyzyk derejeleri – burçuň taraplary. **26.** $z>0$ bolanda umumy $O(0, 0)$ depeli, taraplary Ox we Oy koordinatalar oklaryna parallel kwadratlaryň konturlary; $z=0$ bolanda $O(0, 0)$ nokat. **27.** $z<0$ bolanda Ox okuna parallel göni çyzyklar; $z>0$ bolanda depeleri $y=x^2$ parabolada, taraplary Ox okuna we položitel Oy ýarym okuna parallel burçlaryň taraplary; $z=0$ bolanda položitel Oy ýarym oky. **28.** Koordinatalar oklarynyň başlangyjyndan geçýän (başyny girizmezden) we Ox okuna ortogonal töwerekleriň dessesi. **29.** $y=C/\ln x$ egri çyzyklar. **30.** $y=(C+x)/\ln x$ egri çyzyklar. **31.** $x^2+y^2=a^2$ töwerege ortogonal, merkezli Ox okunda ýerleşýän töwerekler. **32.** $(-a, 0)$, $(a, 0)$ nokatlardan geçýän, şol nokatlary aýyrmak bilen, Oy okuna ortogonal töwerekler. **33.** $z=0$ bolanda, $x=m\pi$ we $y=n\pi$ (m, $n=0, \pm1, \pm2, ...$), göni çyzyklar; $z=-1$ ýa-da $z=1$ bolanda $m\pi<x<(m+1)\pi$, $n\pi<y<(n+1)\pi$ kwadratlar sistemasy, bu ýerde $(-1)^{m+n}=z$. **34.** Parallel tekizlikler. **35.** Merkezli koordinatalar oklarynyň başlangyjynda ýerleşýän konsentrik sferalar. **36.** $u<0$ bolanda, iki boşlukly giperboloidler; $u>0$ bolanda, bir boşlukly giperboloidler, $u=0$ bolanda, konus. **37.** Umumy oky $x+y=0$, $z=0$ göni çyzyk bolan elliptiki silindrler. **38.** $u=0$ bolanda, $x^2+y^2+z^2=\pi n$ ($n=0, 1, 2$ ...) konsentrik sferalar; $u=-1$ ýa-da $u=1$ bolanda, $\pi n<x^2+y^2+z^2<\pi(n+1)$ sferik gatlaklaryň maşgalasy, bu ýerde $(-1)^n=u$. **39.** Emele getirijileri $y=ax$, $z=0$ göni çyzyklara parallel, $z=f(y)$, $x=0$ ugrukdyryjyly silindr şekilli üst. **40.** Oz okunyň daşyndan $z=f(x)$, $y=0$ egri çyzygyň aýlanmagyndan alynýan üst. **41.** Depesi koordinatalar başlangyjynda ýerleşýän we ugrukdyryjysy $x=1$, $z=f(y)$ bolan koniki üst. **42.** $x=1$, $z=f(y)$ ugrukduryjyly konoid, onuň emele getirijileri Oxy tekizlige parallel. **44.** $f(1, y/x)=f(x, y)$. **45.** $\sqrt{1+x^2}$. **46.** $f(t)=2t+t^2$; $z=x-1+\sqrt{y}$ $(x>0)$. **47.** $f(x)=x^2-x$; $z=2y+(x-y)^2$. **48.** $f(x, y)=x^2(1-y)/(1+y)$. **52.** Ýok. **53.** 0. **54.** a) 0, 1; b) 1/2, 1; ç) 0, 1; d) 0, 1; e) 1, ∞. **55.** 0. **56.** 0. **57.** a. **58.** 0. **59.** 0. **60.** 1. **61.** e. **62.** ln2. **63.** a) $\pi/2\le\varphi\le3\pi/2$; b) $\pi/4<\varphi<3\pi/4$ we $5\pi/4<\varphi<7\pi/4$. **64.** Üzülme nokady: $x=0$, $y=0$. **65.** $x+y=0$ göni çyzygyň ähli nokatlary. **66.** $O(0, 0)$ tükeniksiz üzülme nokady; $x+y=0$ $(x\neq0)$ göni çyzygyň nokatlary – aýrylýan üzülme nokatlary. **67.** Koordinatalar oklarynda ýerleşýän nokatlar. **68.** $x=m\pi$ we $y=n\pi$ (m, $n=0, \pm1, \pm2, ...$) göni çyzyklaryň nokatlarynyň toplumy. **69.** $x^2+y^2=1$ töweregiň nokatlary. **70.** $x=0$, $y=0$ we $z=0$ koordinatalar tekizlikleriniň nokatlary. **71.** (a, b, c). **74.** Deňölçegli üznüksiz. **75.** Deňölçegli üznüksiz. **76.** Deňölçegsiz üznüksiz. **77.** Funksiýa B köplükde üznüksiz, ýöne deňölçegli däl.

§2. Köp üýtgeýänli funksiýalaryň hususy önümleri we differensiallary

86. $f_x'(x, 1)=1$. **87.** $f_x'(0, 0)=0$; $f_y'(0, 0)=0$; funksiýa $O(0, 0)$ nokatda differensirlenmeýär. **88.** Funksiýa $O(0, 0)$ nokatda differensirlenmeýär. **89.** Funksiýa $O(0, 0)$ nokatda differensirlenýär. **90.** $\frac{\partial u}{\partial x}=4x^3-8xy^2$, $\frac{\partial u}{\partial y}=4y^3-8x^2y$, $\frac{\partial^2 u}{\partial x^2}=12x^2-8y^2$, $\frac{\partial^2 u}{\partial x\partial y}=-16xy$, $\frac{\partial^2 u}{\partial y^2}=12y^2-8x^2$. **91.** $\frac{\partial u}{\partial x}=y+\frac{1}{y}$, $\frac{\partial u}{\partial y}=x-\frac{x}{y^2}$, $\frac{\partial^2 u}{\partial x^2}=0$, $\frac{\partial^2 u}{\partial x\partial y}=1-\frac{1}{y^2}$,

$\frac{\partial^2 u}{\partial y^2}=\frac{2x}{y^3}$. **92.** $\frac{\partial u}{\partial x}=\frac{1}{y^2}$, $\frac{\partial u}{\partial y}=-\frac{2x}{y^3}$, $\frac{\partial^2 u}{\partial x^2}=0$, $\frac{\partial^2 u}{\partial x\partial y}=-\frac{2}{y^3}$, $\frac{\partial^2 u}{\partial y^2}=\frac{6x}{y^4}$. **93.** $\frac{\partial u}{\partial x}=\frac{y^2}{(x^2+y^2)^{3/2}}$, $\frac{\partial u}{\partial y}=-\frac{xy}{(x^2+y^2)^{3/2}}$, $\frac{\partial^2 u}{\partial x^2}=-\frac{3xy^2}{(x^2+y^2)^{5/2}}$, $\frac{\partial^2 u}{\partial x\partial y}=\frac{y(2x^2-y^2)}{(x^2+y^2)^{5/2}}$, $\frac{\partial^2 u}{\partial y^2}=-\frac{x(x^2-2y^2)}{(x^2+y^2)^{5/2}}$. **94.** $\frac{\partial u}{\partial x}=\sin(x+y)+x\cos(x+y)$, $\frac{\partial u}{\partial y}=x\cos(x+y)$, $\frac{\partial^2 u}{\partial x^2}=2\cos(x+y)-x\sin(x+y)$, $\frac{\partial^2 u}{\partial x\partial y}=\cos(x+y)-x\sin(x+y)$, $\frac{\partial^2 u}{\partial y^2}=-x\times$ $\times\sin(x+y)$. **95.** $\frac{\partial u}{\partial x}=-\frac{2x\sin x^2}{y}$, $\frac{\partial u}{\partial y}=-\frac{\cos x^2}{y^2}$, $\frac{\partial^2 u}{\partial x^2}=-\frac{2\sin x^2+4x^2\cos x^2}{y}$, $\frac{\partial^2 u}{\partial x\partial y}=\frac{2x\sin x^2}{y^2}$, $\frac{\partial^2 u}{\partial y^2}=\frac{2\cos x^2}{y^3}$. **96.** $\frac{\partial u}{\partial x}=\frac{2x}{y}\sec^2\frac{x^2}{y}$, $\frac{\partial u}{\partial y}=-\frac{x^2}{y^2}\sec^2\frac{x^2}{y}$, $\frac{\partial^2 u}{\partial x^2}=$ $=\frac{2}{y}\sec^2\frac{x^2}{y}+\frac{8x^2}{y^2}\sin\frac{x^2}{y}\sec^3\frac{x^2}{y}$, $\frac{\partial^2 u}{\partial x\partial y}=-\frac{2x}{y^2}\sec^2\frac{x^2}{y}-\frac{4x^3}{y^3}\sin\frac{x^2}{y}\sec^3\frac{x^2}{y}$, $\frac{\partial^2 u}{\partial y^2}=$ $=\frac{2x^2}{y^3}\sec^2\frac{x^2}{y}+\frac{2x^4}{y^4}\sin\frac{x^2}{y}\sec^3\frac{x^2}{y}$. **97.** $\frac{\partial u}{\partial x}=yx^{y-1}$, $\frac{\partial u}{\partial y}=x^y\ln x$, $\frac{\partial^2 u}{\partial x^2}=y(y-1)x^{y-2}$, $\frac{\partial^2 u}{\partial x\partial y}=x^{y-1}(1+y\ln x)$, $\frac{\partial^2 u}{\partial y^2}=x^y\ln^2 x$ $(x>0)$. **98.** $\frac{\partial u}{\partial x}=\frac{1}{x+y^2}$, $\frac{\partial u}{\partial y}=\frac{2y}{x+y^2}$, $\frac{\partial^2 u}{\partial x^2}=-\frac{1}{(x+y^2)^2}$, $\frac{\partial^2 u}{\partial x\partial y}=-\frac{2y}{(x+y^2)^2}$, $\frac{\partial^2 u}{\partial y^2}=\frac{2(x-y^2)}{(x+y^2)^2}$. **99.** $\frac{\partial u}{\partial x}=-\frac{y}{x^2+y^2}$, $\frac{\partial u}{\partial y}=\frac{x}{x^2+y^2}$, $\frac{\partial^2 u}{\partial x^2}=\frac{2xy}{(x^2+y^2)^2}$, $\frac{\partial^2 u}{\partial x\partial y}=-\frac{x^2-y^2}{(x^2+y^2)^2}$, $\frac{\partial^2 u}{\partial y^2}=-\frac{2xy}{(x^2+y^2)^2}$. **100.** $\frac{\partial u}{\partial x}=\frac{1}{1+x^2}$, $\frac{\partial u}{\partial y}=\frac{1}{1+y^2}$, $\frac{\partial^2 u}{\partial x^2}=-\frac{2x}{(1+x^2)^2}$, $\frac{\partial^2 u}{\partial x\partial y}=0$, $\frac{\partial^2 u}{\partial y^2}=-\frac{2y}{(1+y^2)^2}$ $(xy\neq 1)$. **101.** $\frac{\partial u}{\partial x}=\frac{|y|}{x^2+y^2}$, $\frac{\partial u}{\partial y}=-\frac{x\,\mathrm{sgn}\,y}{x^2+y^2}$, $\frac{\partial^2 u}{\partial x^2}=-\frac{2x|y|}{(x^2+y^2)^2}$, $\frac{\partial^2 u}{\partial x\partial y}=\frac{(x^2-y)^2\,\mathrm{sgn}\,y}{(x^2+y^2)^2}$, $\frac{\partial^2 u}{\partial y^2}=\frac{2x|y|}{(x^2+y^2)^2}$ $(y\neq 0)$. **102.** $\frac{\partial u}{\partial x}=-\frac{x}{(x^2+y^2+z^2)^{3/2}}$, $\frac{\partial^2 u}{\partial x^2}=\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$, $\frac{\partial^2 u}{\partial x\partial y}=$ $=\frac{3xy}{\left(x^2+y^2+z^2\right)^{5/2}}$. **103.** $\frac{\partial u}{\partial x}=\frac{z}{x}\left(\frac{x}{y}\right)^z$, $\frac{\partial u}{\partial y}=-\frac{z}{y}\left(\frac{x}{y}\right)^z$, $\frac{\partial u}{\partial z}=\left(\frac{x}{y}\right)^z\ln\frac{x}{y}$, $\frac{\partial^2 u}{\partial x^2}=$ $=\frac{z(z-1)}{x^2}\left(\frac{x}{y}\right)^z$, $\frac{\partial^2 u}{\partial y^2}=\frac{z(z+1)}{y^2}\left(\frac{x}{y}\right)^z$, $\frac{\partial^2 u}{\partial z^2}=\left(\frac{x}{y}\right)^z\ln^2\frac{x}{y}$, $\frac{\partial^2 u}{\partial x\partial y}=-\frac{z^2}{xy}\left(\frac{x}{y}\right)^z$, $\frac{\partial^2 u}{\partial x\partial z}=\frac{1}{x}\left(\frac{x}{y}\right)^z\left(1+z\ln\frac{x}{y}\right)$, $\frac{\partial^2 u}{\partial y\partial z}=-\frac{1}{y}\left(\frac{x}{y}\right)^z\left(1+z\ln\frac{x}{y}\right)$ $\left(\frac{x}{y}>0\right)$. **104.** $\frac{\partial u}{\partial x}=$ $=\frac{yu}{xz}$, $\frac{\partial u}{\partial y}=\frac{u\ln x}{z}$, $\frac{\partial u}{\partial z}=-\frac{yu}{z^2}\ln x$, $\frac{\partial^2 u}{\partial x^2}=\frac{y(y-z)u}{x^2z^2}$, $\frac{\partial^2 u}{\partial y^2}=\frac{u\ln^2 x}{z^2}$, $\frac{\partial^2 u}{\partial z^2}=$ $=\frac{yu\ln x}{z^4}(2z+y\ln x)$, $\frac{\partial^2 u}{\partial x\partial y}=\frac{(z+y\ln x)u}{xz^2}$, $\frac{\partial^2 u}{\partial x\partial z}=-\frac{yu(z+y\ln x)}{xz^3}$, $\frac{\partial^2 u}{\partial y\partial z}=$

$= -\dfrac{u\ln x(z+y\ln x)}{z^3}$ $(xz \neq 0)$. **105.** $\dfrac{\partial u}{\partial x} = \dfrac{y^z}{x}u$, $\dfrac{\partial u}{\partial y} = zy^{z-1}u\ln x$, $\dfrac{\partial u}{\partial z} = y^z u\ln x\ln y$, $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{y^z(y^z-1)}{x^2}u$, $\dfrac{\partial^2 u}{\partial y^2} = zy^{z-2}u(z-1+zy^z\ln x)\ln x$, $\dfrac{\partial^2 u}{\partial z^2} = y^z u(1+y^z\ln x)\ln x\ln^2 y$, $\dfrac{\partial^2 u}{\partial x\partial y} = \dfrac{zy^{z-1}u}{x}(1+y^z\ln x)$, $\dfrac{\partial^2 u}{\partial x\partial z} = \dfrac{y^z u\ln y}{x}(1+y^z\ln x)$, $\dfrac{\partial^2 u}{\partial y\partial z} = y^{z-1}u\ln x[1+z\ln y\times$ $\times(1+y^z\ln x)]$ $(x>0,\ y>0)$. **108.** $f_{xy}''(0,\ 0)$ ýok. **113.** $du = x^{m-1}y^{n-1}(mydx+nxdy)$, $d^2u =$ $= x^{m-2}y^{n-2}[m(m-1)y^2dx^2+2mnxydxdy+n(n-1)x^2dy^2]$. **114.** $du = \dfrac{ydx-xdy}{y^2}$, $d^2u = -\dfrac{2}{y^3}dy\ \times$ $\times\left(ydx-xdy\right)$. **115.** $du = \dfrac{xdx+ydy}{\sqrt{x^2+y^2}}$, $d^2u = \dfrac{(ydx-xdy)^2}{(x^2+y^2)^{3/2}}$. **116.** $du = \dfrac{xdx+ydy}{x^2+y^2}$, $d^2u = \dfrac{(y^2-x^2)(dx^2-dy^2)-4xydxdy}{(x^2+y^2)^2}$. **117.** $du = e^{xy}(ydx+xdy)$; $d^2u = e^{xy}[y^2dx^2+2(1+$ $+xy)dxdy+x^2dy^2]$. **118.** $du=(y+z)dx+(z+x)dy+(x+y)dz$, $d^2u = 2(dxdy+dydz+$ $+dzdx)$. **119.** $du = \dfrac{(x^2+y^2)dz-2z(xdx+ydy)}{(x^2+y^2)^2}$, $d^2u = -\dfrac{4(x^2+y^2)(xdx+ydy)dz}{(x^2+y^2)^3}+$ $+\dfrac{2z[(3x^2-y^2)dx^2+8xydxdy+(3y^2-x^2)dy^2]}{(x^2+y^2)^3}$. **120.** $dx-dy$; $-2(dx-dy)(dy+dz)$. **122.** a) $1+$ $+mx+ny$; b) xy; ç) $x+y$. **123.** a) 108,972; b) 1,055; ç) 2,95; d) 0,502; e) 0,97. **124.** Diagonal takmynan 3 *mm* kiçeler; meýdan takmynan 140 sm^2 kiçeler. **125.** 1,7 *mm* kiçeltmeli. **127.** $\Delta \approx 10{,}2\ m^3$; $\delta \approx 13\%$. **128.** $\Delta \approx 7{,}6\ m$. **129.** $f_x'(x, y)$ we $f_y'(x, y)$ $(0, 0)$ nokadyň golaý töwereginde çäksizdir. **134.** $\dfrac{\partial^4 u}{\partial x^4} = 24$, $\dfrac{\partial^4 u}{\partial x^3\partial y} = 0$, $\dfrac{\partial^4 u}{\partial x^2\partial y^2} = -16$. **135.** $\dfrac{\partial^3 u}{\partial x^2\partial y} = 0$. **136.** $\dfrac{\partial^6 u}{\partial x^3\partial y^3} = -6(\cos x+\cos y)$. **137.** $\dfrac{\partial^3 u}{\partial x\partial y\partial z} = 0$. **138.** $\dfrac{\partial^3 u}{\partial x\partial y\partial z} = e^{xyz}(1+3xyz+$ $+x^2y^2z^2)$. **139.** $\dfrac{\partial^4 u}{\partial x\partial y\partial\xi\partial\eta} = -\dfrac{6}{r^4}+\dfrac{48(x-\xi)^2(y-\eta)^2}{r^8}$, bu ýerde $r = \sqrt{(x-\xi)^2+(y-\eta)^2}$. **140.** $\dfrac{\partial^{p+q}u}{\partial x^p\partial y^q} = p!q!$. **141.** $\dfrac{2(-1)^m(m+n-1)!(nx+my)}{(x+y)^{m+n+1}}$. **142.** $e^{x+y}[x^2+y^2+2(mx+ny)+$ $+m(m-1)+n(n-1)]$. **143.** $(x+p)(y+q)(z+r)e^{x+y+z}$. **144.** $\sin\dfrac{n\pi}{2}$. **145.** $F(t)=f'(t)+3tf''(t)+$ $+t^2f'''(t)$. **146.** $d^4u = 24(dx^4-2dx^3dy-2dxdy^3+dy^4)$, $\dfrac{\partial^4 u}{\partial x^4} = 24$, $\dfrac{\partial^4 u}{\partial x^3\partial y} = -12$, $\dfrac{\partial^4 u}{\partial x^2\partial y^2} = 0$, $\dfrac{\partial^4 u}{\partial x\partial y^3} = -12$, $\dfrac{\partial^4 u}{\partial y^4} = 24$. **147.** $d^3u = 6(dx^3-3dx^2dy+3dxdy^2+dy^3)$. **148.** $d^3u = -8(xdx+$ $+ydy)^3\cos(x^2+y^2)-12(xdx+ydy)(dx^2+dy^2)\sin(x^2+y^2)$. **149.** $d^{10}u = -\dfrac{9!(dx+dy)^{10}}{(x+y)^{10}}$.

150. $d^6u=-(dx^6-15dx^4dy^2+15dx^2dy^4-dy^6)\cos x\,\mathrm{ch}\,y-2dxdy(3dx^4-10dx^2dy^2+3dy^4)\sin x\,\mathrm{sh}\,y$.
151. $d^3u=6dxdydz$. **152.** $d^4u=2\left(\frac{dx^4}{x^3}+\frac{dy^4}{y^3}+\frac{dz^4}{z^3}\right)$. **153.** $d^nu=e^{ax+by}(adx+bdy)^n$.
154. $d^nu=\sum_{k=0}^{n}C_n^kX^{(n-k)}(x)Y^{(k)}(y)dx^{n-k}dy^k$. **155.** $d^nu=f^{(n)}(x+y+z)(dx+dy+dz)^n$.
156. $d^nu=e^{ax+by+cz}(adx+bdy+cdz)^n$. **158.** a) $Au=-u$, $A^2u=u$; b) $Au=1$, $A^2u=0$.
159. a) $\Delta u=0$; b) $\Delta u=0$. **160.** a) $\Delta_1u=9[(x^2-yz)^2+(y^2-xz)^2+(z^2-xy)^2]$, $\Delta_2u=6(x+y+z)$; b) $\Delta_1u=\frac{1}{r^4}$, bu ýerde $r=\sqrt{x^2+y^2+z^2}$, $\Delta_2u=0$. **161.** $\frac{\partial u}{\partial x}=2xf'(x^2+y^2+z^2)$; $\frac{\partial^2u}{\partial x^2}=2f'(x^2+y^2+z^2)+4x^2f''(x^2+y^2+z^2)$; $\frac{\partial^2u}{\partial x\partial y}=4xyf''(x^2+y^2+z^2)$. **162.** $\frac{\partial u}{\partial x}=$ $=f_1'\left(x,\frac{x}{y}\right)+\frac{1}{y}f_2'\left(x,\frac{x}{y}\right)$; $\frac{\partial u}{\partial y}=-\frac{x}{y^2}f_2'\left(x,\frac{x}{y}\right)$; $\frac{\partial^2u}{\partial x^2}=f_{11}''\left(x,\frac{x}{y}\right)+\frac{2}{y}f_{12}''\left(x,\frac{x}{y}\right)+\frac{1}{y^2}f_{22}''\times$ $\times\left(x,\frac{x}{y}\right)$; $\frac{\partial^2u}{\partial x\partial y}=-\frac{x}{y^2}f_{12}''\left(x,\frac{x}{y}\right)-\frac{x}{y^3}f_{22}''\left(x,\frac{x}{y}\right)-\frac{1}{y^2}f_2'\left(x,\frac{x}{y}\right)$; $\frac{\partial^2u}{\partial y^2}=\frac{x^2}{y^4}f_{22}''\left(x,\frac{x}{y}\right)+\frac{2x}{y^3}\times$ $\times f_2'\left(x,\frac{x}{y}\right)$. **163.** $\frac{\partial u}{\partial x}=f_1'+yf_2'+yzf_3'$; $\frac{\partial u}{\partial y}=xf_2'+xzf_3'$; $\frac{\partial u}{\partial z}=xyf_3'$; $\frac{\partial^2u}{\partial x^2}=f_{11}''+y^2f_{22}''+$ $+y^2z^2f_{33}''+2yf_{12}''+2yzf_{13}''+2y^2zf_{23}''$; $\frac{\partial^2u}{dy^2}=x^2f_{22}''+2x^2zf_{23}''+x^2z^2f_{33}''$; $\frac{\partial^2u}{\partial z^2}=x^2y^2f_{33}''$; $\frac{\partial^2u}{\partial x\partial y}=xyf_{22}''+xyz^2f_{33}''+xf_{12}''+xzf_{13}''+2xyzf_{23}''+f_2'+zf_3'$; $\frac{\partial^2u}{\partial x\partial z}=xyf_{13}''+xy^2f_{23}''+xy^2zf_{33}''+$ $+yf_3'$; $\frac{\partial^2u}{\partial y\partial z}=x^2yf_{23}''+x^2yzf_{33}''+xf_3'$. **164.** $\frac{\partial^2u}{\partial x\partial y}=f_{11}''+(x+y)f_{12}''+xyf_{22}''+f_2'$.
165. $\Delta u=3f_{11}''+4(x+y+z)f_{12}''+4(x^2+y^2+z^2)f_{22}''+6f_2'$. **166.** $du=f'(t)(dx+dy)$; $d^2u=f''(t)\times$ $\times(dx+dy)^2$. **167.** $du=f'(t)\frac{xdy-ydx}{x^2}$; $d^2u=f''(t)\frac{(xdy-ydx)^2}{x^4}-2f'(t)\frac{dx(xdy-ydx)}{x^3}$.
168. $du=f'\frac{xdx+ydy}{\sqrt{x^2+y^2}}$; $d^2u=f''\frac{(xdx+ydy)^2}{x^2+y^2}+f'\frac{(xdy-ydx)^2}{(x^2+y^2)^{3/2}}$. **169.** $du=f'(t)dt$, $d^2u=$ $=f''(t)dt^2+f'(t)d^2t$, bu ýerde $dt=yzdx+zxdy+xydz$ we $d^2t=2(zdxdy+ydxdz+xdydz)$.
170. $du=2f'(xdx+ydy+zdz)$, $d^2u=4f''(xdx+ydy+zdz)^2+2f'(dx^2+dy^2+dz^2)$. **171.** $du=af_1'\times$ $\times dx+bf_2'dy$; $d^2u=a^2f_{11}''dx^2+2abf_{12}''dxdy+b^2f_{22}''dy^2$. **172.** $du=f_1'(dx+dy)+f_2'(dx-dy)$; $d^2u=f_{11}''(dx+dy)^2+2f_{12}''(dx^2-dy^2)+f_{22}''(dx-dy)^2$. **173.** $du=f_1'(ydx+xdy)+f_2'\times$ $\times\frac{ydx-xdy}{y^2}$; $d^2u=f_{11}''(ydx+xdy)^2+2f_{12}''\frac{y^2dx^2-x^2dy^2}{y^2}+f_{22}''\frac{(ydx-xdy)^2}{y^4}+2f_1'dxdy-2f_2'\times$ $\times\frac{(ydx-xdy)dy}{y^3}$. **174.** $du=f_1'(dx+dy)+f_2'dz$; $d^2u=f_{11}''(dx+dy)^2+2f_{12}''(dx+dy)dz+$

$+ f_{22}''dz^2$. **175.** $du = f_1'(dx + dy + dz) + 2f_2'(xdx + ydy + zdz)$; $d^2u = f_{11}''(dx + dy + dz)^2 + 4f_{12}''(dx + dy + dz)(xdx + ydy + zdz) + 4f_{22}''(xdx + ydy + zdz)^2 + 2f_2'(dx^2 + dy^2 + dz^2)$.
176. $du = f_1' \frac{ydx - xdy}{y^2} + f_2' \frac{zdy - ydz}{z^2}$; $d^2u = f_{11}'' \frac{(ydx - xdy)^2}{y^4} + 2f_{12}'' \frac{(ydx - xdy)(zdy - ydz)}{y^2 z^2} + f_{22}'' \frac{(zdy - ydz)^2}{z^4} - 2f_1' \frac{(ydx - xdy)dy}{y^3} - 2f_2' \frac{(zdy - ydz)dz}{z^3}$. **177.** $du = (f_1' + 2tf_2' + 3t^2 f_3')dt$, $d^2u = (f_{11}'' + 4tf_{12}'' + 4t^2 f_{22}'' + 6t^2 f_{13}'' + 12t^3 f_{23}'' + 9t^4 f_{33}'' + 2f_2' + 6tf_3')dt^2$. **178.** $du = af_1'dx + bf_2'dy + cf_3'dz$; $d^2u = a^2 f_{11}''dx^2 + b^2 f_{22}''dy^2 + c^2 f_{33}''dz^2 + 2abf_{12}''dxdy + 2acf_{13}''dxdz + 2bcf_{23}''dydz$.
179. $du = 2f_1'(xdx + ydy) + 2f_2'(xdx - ydy) + 2f_3'(ydx + xdy)$; $d^2u = 4f_{11}''(xdx + ydy)^2 + 4f_{22}''(xdx - ydy)^2 + 4f_{33}''(ydx + xdy)^2 + 8f_{12}''(x^2dx^2 - y^2dy^2) + 8f_{13}''(xdx + ydy)(ydx + xdy) + 8f_{23}''(xdx - ydy) \times (ydx + xdy) + 2f_1'(dx^2 + dy^2) + 2f_2'(dx^2 - dy^2) + 4f_3'dxdy$. **180.** $d^n u = f^{(n)}(ax + by + cz)(adx + bdy + cdz)^n$. **181.** $d^n u = \left(adx\frac{\partial}{\partial \xi} + bdy\frac{\partial}{\partial \eta} + cdz\frac{\partial}{\partial \zeta}\right)^n f(\xi, \eta, \zeta)$, bu ýerde $\xi = ax$, $\eta = by$, $\zeta = cz$.

182. $d^n u = \left[dx\left(a_1\frac{\partial}{\partial \xi} + a_2\frac{\partial}{\partial \eta} + a_3\frac{\partial}{\partial \zeta}\right) + dy\left(b_1\frac{\partial}{\partial \xi} + b_2\frac{\partial}{\partial \eta} + b_3\frac{\partial}{\partial \zeta}\right) + dz\left(c_1\frac{\partial}{\partial \xi} + c_2 \times \frac{\partial}{\partial \eta} + c_3\frac{\partial}{\partial \zeta}\right)\right]^n f(\xi, \eta, \zeta)$. **183.** $F(r) = f''(r) + \frac{2}{r}f'(r)$. **194.** 1. **197.** xyz. **209.** $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = x$. **210.** $2x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$. **211.** $y\frac{\partial z}{\partial y} - x\frac{\partial z}{\partial y} = 0$. **212.** $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.
213. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$. **214.** $\frac{\partial^2 z}{\partial x \partial y} = 0$. **215.** $z\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$. **216.** $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$. **217.** $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$. **218.** $x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$. **219.** $1 - \sqrt{3}$.
220. $\frac{\partial z}{\partial l} = \cos\alpha + \sin\alpha$, a) $\alpha = \frac{\pi}{4}$; b) $\alpha = \frac{5\pi}{4}$; ç) $\alpha = \frac{3\pi}{4}$ we $\alpha = \frac{7\pi}{4}$.
221. $\frac{2}{\sqrt{x_0^2 + y_0^2}}$. **222.** $\frac{1}{ab}\sqrt{2(a^2 + b^2)}$. **223.** $\frac{\partial u}{\partial l} = \cos\alpha + \cos\beta + \cos\gamma$; $|\mathrm{grad}\, u| = \sqrt{3}$.
224. $|\mathrm{grad}\, u| = \frac{1}{r_0^2}$; $\cos\left(\mathrm{grad}\, \widehat{u, x}\right) = -\frac{x_0}{r_0}$, $\cos\left(\mathrm{grad}\, \widehat{u, y}\right) = -\frac{y_0}{r_0}$, $\cos\left(\mathrm{grad}\, \widehat{u, z}\right) = -\frac{z_0}{r_0}$, $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$. **225.** $\frac{\pi}{2}$. **226.** ≈ 3142. **228.** $\frac{\partial^2 u}{\partial l^2} = \frac{\partial^2 u}{\partial x^2}\cos^2\alpha + \frac{\partial^2 u}{\partial y^2}\cos^2\beta + \frac{\partial^2 u}{\partial z^2} \times \cos^2\gamma + 2\frac{\partial^2 u}{\partial x \partial y}\cos\alpha\cos\beta + 2\frac{\partial^2 u}{\partial x \partial z}\cos\alpha\cos\gamma + \frac{\partial^2 u}{\partial y \partial z}\cos\beta\cos\gamma$. **230.** $\frac{\partial u}{\partial y} = -0,5$.
231. $u_{xx}''(x, 2x) = u_{yy}''(x, 2x) = -4/3x$, $u_{xy}''(x, 2x) = 5/3x$. **232.** $z = x\varphi(y) + \psi(y)$. **233.** $z = \varphi(x) + \psi(y)$.
234. $z = \varphi_0(x) + y\varphi_1(x) + \ldots + y^{n-1}\varphi_{n-1}(x)$. **235.** $u = \varphi(x, y) + \psi(x, z) + \chi(y, z)$. **236.** $u = 1 + x^2y + y^2 - 2x^4$. **237.** $z = 1 + xy + y^2$. **238.** $z = x + y^2 + 0,5xy(x + y)$.

§3. Anyk däl funksiýalaryň barlygy we differensirlenmegi

240. $f(x)$ funksiýanyň nollary hiç bir $(\alpha, \beta) \subset (a, b)$ interwaly tutuş doldurmaly däldir. **241.** $f(x)$ funksiýanyň nollarynyň köplügi (α, β) interwalda hiç ýerde dykyz bolmaly däl, şeýle-de, $f(x)$ funksiýanyň her bir ξ noly şol bir wagtda $g(x)$ funksiýanyň hem noludyr we ondan daşgary tükenikli $\lim\limits_{x\to\xi}[g(x)/f(x)]$ predel hem bardyr. **242.** 1) tükeniksiz köp; 2) iki; 3) a) bir; b) iki. **243.** 1) tükeniksiz köp; 2) dört: $y=x$; $y=-x$; $y=|x|$ we $y=-|x|$; 3) iki; 4) a) iki; b) dört; 5) bir. **244.** 1) hiç ýerde; 2) $0<|x|<1$, $|x| = \sqrt{(1+\sqrt{2})/2}$; 3) $x=0$, $|x|=1$; 4) $1 < |x| < \sqrt{\dfrac{1+\sqrt{2}}{2}}$; bir bahaly şahalary: $y = \varepsilon\sqrt{\dfrac{1}{2} + \sqrt{\dfrac{1}{4} + x^2 - x^4}}$ $\left(|x| \le \sqrt{\dfrac{1+\sqrt{2}}{2}}\right)$; $y = \varepsilon\sqrt{\dfrac{1}{2} - \sqrt{\dfrac{1}{4} + x^2 - x^4}}$ $\left(1 \le |x| \le \sqrt{\dfrac{1+\sqrt{2}}{2}}\right)$, bu ýerde $\varepsilon=-1,1$. **245.** Şahalanma nokatlary: $(-1,\ 0)$, $(0,\ 0)$, $(1,\ 0)$; $y = \varepsilon(x)\sqrt{\dfrac{\sqrt{8x^2+1} - (2x^2+1)}{2}}$ $(|x|\le 1)$, $\varepsilon(x)=-1,1$, $\operatorname{sgn} x$ we $-\operatorname{sgn} x$. **246.** $\varphi(y)$ funksiýanyň bahalar köplüginiň $f(x)$ funksiýanyň bahalar köplügi bilen umumy nokady bolmalydyr. **249.** $y' = -\dfrac{x+y}{x-y}$; $y'' = \dfrac{2a^2}{(x-y)^3}$. **250.** $y' = \dfrac{x+y}{x-y}$; $y'' = \dfrac{2(x^2+y^2)}{(x-y)^3}$. **251.** $y' = \dfrac{1}{1-\varepsilon\cos y}$; $y'' = \dfrac{-\varepsilon\sin y}{(1-\varepsilon\cos y)^3}$. **252.** $y' = \dfrac{y^2(1-\ln x)}{x^2(1-\ln y)}$; $y'' = \dfrac{y^2[y(1-\ln x)^2 - 2(x-y)(1-\ln x)(1-\ln y) - x(1-\ln y)^2]}{x^4(1-\ln y)^3}$. **253.** $y' = \dfrac{y}{x}$; $y''=0$.
256. $y_1'(0)=-1$; $y_2'(0)=1$. **257.** $y_1'(0)=0$; $y_2'(0) = -\sqrt{33}$; $y_3'(0) = \sqrt{3}$. **258.** $y' = -\dfrac{2x+y}{x+2y}$; $y'' = -\dfrac{18}{(x+2y)^3}$; $y''' = -\dfrac{162x}{(x+2y)^5}$. **259.** $y'=0$; $y'' = -\dfrac{2}{3}$; $y''' = -\dfrac{2}{3}$. **261.** $\dfrac{\partial z}{\partial x} = -\dfrac{x}{z}$; $\dfrac{\partial z}{\partial y} = -\dfrac{y}{z}$; $\dfrac{\partial^2 z}{\partial x^2} = -\dfrac{x^2+z^2}{z^3}$; $\dfrac{\partial^2 z}{\partial x\partial y} = -\dfrac{xy}{z^3}$; $\dfrac{\partial^2 z}{\partial y^2} = -\dfrac{y^2+z^2}{z^3}$. **262.** $\dfrac{\partial z}{\partial x} = \dfrac{yz}{z^2-xy}$; $\dfrac{\partial z}{\partial y} = \dfrac{xz}{z^2-xy}$; $\dfrac{\partial^2 z}{\partial x^2} = -\dfrac{2xy^3z}{(z^2-xy)^3}$; $\dfrac{\partial^2 z}{\partial y^2} = -\dfrac{2x^3yz}{(z^2-xy)^3}$; $\dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{z(z^4-2xyz^2-x^2y^2)}{(z^2-xy)^3}$.
263. $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = \dfrac{1}{x+y+z-1}$; $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial x\partial y} = -\dfrac{x+y+z}{(x+y+z-1)^3}$. **264.** $\dfrac{\partial z}{\partial x} = \dfrac{xz}{x^2-y^2}$; $\dfrac{\partial z}{\partial y} = -\dfrac{yz}{x^2-y^2}$; $\dfrac{\partial^2 z}{\partial x^2} = -\dfrac{y^2z}{(x^2-y^2)^2}$; $\dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{xyz}{(x^2-y^2)^2}$; $\dfrac{\partial^2 z}{\partial y^2} = -\dfrac{x^2z}{(x^2-y^2)^2}$.
265. $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = -1$; $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{\partial^2 z}{\partial y^2} = 0$. **266.** a) -2; b) -1. **267.** $\dfrac{\partial^2 z}{\partial x^2} = -\dfrac{2}{5}$; $\dfrac{\partial^2 z}{\partial x\partial y} = -\dfrac{1}{5}$; $\dfrac{\partial^2 z}{\partial y^2} = -\dfrac{394}{125}$. **268.** $dz = -\dfrac{c^2}{z}\left(\dfrac{xdx}{a^2} + \dfrac{ydy}{b^2}\right)$; $d^2z = -\dfrac{c^4}{z^3}\left[\left(\dfrac{x^2}{a^2} + \dfrac{z^2}{c^2}\right)\times\right.$ $\left.\times\dfrac{dx^2}{a^2} + \dfrac{2xy}{a^2b^2}dxdy + \left(\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}\right)\dfrac{dy^2}{b^2}\right]$. **269.** $dz = -\dfrac{(1-yz)dx + (1-xz)dy}{1-xy}$; $d^2z =$

$= -\dfrac{2\{y(1-yz)dx^2 + [x+y-z(1+xy)]dxdy + x(1-xz)dy^2\}}{(1-xy)^2}$. **270.** $dz = \dfrac{z(ydx+zdy)}{y(x+z)}$; $d^2z = -\dfrac{z^2(ydx-xdy)^2}{y^2(x+z)^3}$. **271.** $dz = dx - \dfrac{(x-z)dy}{(x-z)^2+y(y+1)}$; $d^2z = \dfrac{2(x-z)(y+1)[(x-z)^2+y^2]}{[(x-z)^2+y(y+1)]^3}dy^2$.

272. $du = -\dfrac{u^2(dx+dy)-z^2dz}{u[2(x+y)-u]}$. **273.** $\dfrac{\partial^2 z}{\partial x\partial y} = -\dfrac{4(x-z)(y-z)}{(F_1'+2zF_2')^3}[F_2^{2\prime}F_{11}''-2F_1'F_2'F_{12}''+ F_1^{2\prime}F_{22}''] - \dfrac{2(F_1'+2xF_2')(F_1'+2yF_2')F_2'}{(F_1'+2zF_2')^3}$. **274.** $\dfrac{\partial z}{\partial x} = \dfrac{F_1'-F_3'}{F_2'-F_3'}$; $\dfrac{\partial z}{\partial y} = \dfrac{F_2'-F_1'}{F_2'-F_3'}$. **275.** $\dfrac{\partial z}{\partial x} = -\left(1+\dfrac{F_1'+F_2'}{F_3'}\right)$; $\dfrac{\partial z}{\partial y} = -\left(1+\dfrac{F_2'}{F_3'}\right)$; $\dfrac{\partial^2 z}{\partial x^2} = -F_3'^{-3}[F_3'^2(F_{11}''+2F_{12}''+F_{22}'') - 2(F_1'+F_2')\times F_3'(F_{13}''+F_{23}'') + (F_1'+F_2')^2F_{33}'']$. **276.** $\dfrac{\partial^2 z}{\partial x^2} = -(xF_1'+yF_2')^{-3}[y^2z^2(F_2'^2F_{11}''-2F_1'F_2'F_{12}''+ F_1'^2F_{22}'') - 2z(xF_1'+yF_2')F_1'^2]$. **277.** a) $d^2z = -\dfrac{F_2'^2F_{11}''-2F_1'F_2'F_{12}''+F_1'^2F_{22}''}{(F_1'+F_2')^3}(dx-dy)^2$;

b) $d^2z = -\dfrac{F_2'^2F_{11}''-2F_1'F_2'F_{12}''+F_1'^2F_{22}''}{(xF_1'+yF_2')^3}(ydx-xdy)^2$. **278.** $dz = \dfrac{1}{9}(2dx-dy)$; $d^2z = -\dfrac{2}{243}\left(2dx^2-5dxdy+2dy^2\right)$. **280.** $\dfrac{dx}{dz} = \dfrac{y-z}{x-y}$; $\dfrac{dy}{dz} = \dfrac{z-x}{x-y}$. **281.** $\dfrac{dx}{dz} = 0$, $\dfrac{dy}{dz} = -1$, $\dfrac{d^2x}{dz^2} = -\dfrac{d^2y}{dz^2} = -\dfrac{1}{4}$. **282.** $\dfrac{du}{dx} = -\dfrac{xu+yv}{x^2+y^2}$; $\dfrac{dv}{dx} = \dfrac{yu-xv}{x^2+y^2}$; $\dfrac{du}{dy} = \dfrac{xv-yu}{x^2+y^2}$; $\dfrac{dv}{dy} = -\dfrac{xu+yv}{x^2+y^2}$ $(x^2+y^2>0)$. **283.** $du = -\dfrac{1}{3}dy$; $dv = -dx+\dfrac{1}{3}dy$. **284.** $du = \dfrac{(\sin v + x\cos v)dx - (\sin u - x\cos v)dy}{x\cos v + y\cos u}$; $dv = \dfrac{-(\sin v - y\cos u)dx}{x\cos v + y\cos u} + \dfrac{(\sin u + y\cos u)dy}{x\cos v + y\cos u}$;

$d^2u = -d^2v = \dfrac{(2dx\cos v - xdv\sin v)dv}{x\cos v + y\cos u} - \dfrac{(2dy\cos u - ydu\sin u)du}{x\cos v + y\cos u}$. **285.** $du = \dfrac{1}{2}\times\left(dx+dy\right)$; $dv = \dfrac{\pi}{4}dy - \dfrac{1}{2}(dx-dy)$; $d^2u = dx^2$; $d^2v = \dfrac{1}{2}(dx-dy)^2$. **286.** $\dfrac{dy}{dx} = 2\left(t+\dfrac{1}{t}\right)$; $\dfrac{dz}{dx} = 3\left(t^2+\dfrac{1}{t^2}+1\right)$; $\dfrac{d^2y}{dx^2} = 2$; $\dfrac{d^2z}{dx^2} = 6\left(t+\dfrac{1}{t}\right)$. **287.** $y \geq \dfrac{x^2}{2}$; $\dfrac{\partial z}{\partial x} = -3uv$; $\dfrac{\partial z}{\partial y} = \dfrac{3}{2}(u+v)$ $(u \neq v)$. **288.** $\dfrac{\partial z}{\partial x} = \dfrac{3}{2}$; $\dfrac{\partial z}{\partial y} = -\dfrac{1}{2}$. **289.** $\dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{26}{121}$. **290.** $\dfrac{\partial^2 z}{\partial x^2} = -\dfrac{\sin^2\varphi + \cos^2\varphi\cos^2\psi}{\sin^3\varphi}$. **291.** $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\sin 2v}{u^2}$; $\dfrac{\partial^2 z}{\partial x\partial y} = -\dfrac{\cos 2v}{u^2}$, $\dfrac{\partial^2 z}{\partial y^2} = -\dfrac{\sin 2v}{u^2}$. **292.** $dz=0$; $d^2z = \dfrac{1}{2}\left(dx^2-dy^2\right)$. **293.** $\dfrac{\partial z}{\partial x} = \dfrac{2(x^2-y^2)}{x-2y}$; $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{4x-2y}{x-2y} + \dfrac{6x}{(x-2y)^3}$. **294.** $\dfrac{\partial u}{\partial x} = \dfrac{1}{y+z} + \dfrac{(x+1)(y-x)}{(z+1)(y+z)^2}e^{x-z}$; $\dfrac{\partial u}{\partial y} = -\dfrac{x+z}{(y+z)^2} + \dfrac{(y+1)(y-x)}{(z+1)(y+z)^2}e^{y-z}$. **295.** $\dfrac{\partial z}{\partial x} = -\dfrac{1}{I}\left(\dfrac{\partial\psi}{\partial u}\dfrac{\partial\chi}{\partial v} -\right.$

$-\frac{\partial\psi}{\partial\upsilon}\frac{\partial\chi}{\partial u}\Big);\ \frac{\partial z}{\partial y}=-\frac{1}{I}\Big(\frac{\partial\chi}{\partial u}\frac{\partial\varphi}{\partial\upsilon}-\frac{\partial\chi}{\partial\upsilon}\frac{\partial\varphi}{\partial u}\Big),\ I=\frac{\partial\varphi}{\partial u}\frac{\partial\psi}{\partial\upsilon}-\frac{\partial\psi}{\partial u}\frac{\partial\varphi}{\partial\upsilon}$. **296.** $\frac{\partial u}{\partial x}=\frac{1}{I}\frac{\partial\psi}{\partial\upsilon};\ \frac{\partial u}{\partial y}=$ $=-\frac{1}{I}\frac{\partial\varphi}{\partial\upsilon};\ \frac{\partial^2u}{\partial x^2}=-\frac{1}{I^3}\Big\{\Big(\frac{\partial\psi}{\partial\upsilon}\frac{\partial^2\varphi}{\partial u^2}-\frac{\partial\varphi}{\partial\upsilon}\frac{\partial^2\psi}{\partial u^2}\Big)\Big(\frac{\partial\psi}{\partial\upsilon}\Big)^2-2\Big(\frac{\partial\psi}{\partial\upsilon}\frac{\partial^2\varphi}{\partial u\partial\upsilon}-\frac{\partial\varphi}{\partial\upsilon}\frac{\partial^2\psi}{\partial u\partial\upsilon}\Big)\frac{\partial\psi}{\partial u}\frac{\partial\psi}{\partial\upsilon}+$ $+\Big(\frac{\partial\psi}{\partial\upsilon}\frac{\partial^2\varphi}{\partial\upsilon^2}-\frac{\partial\varphi}{\partial\upsilon}\frac{\partial^2\psi}{\partial\upsilon^2}\Big)\Big(\frac{\partial\psi}{\partial u}\Big)^2\Big\};\ \frac{\partial^2u}{\partial x\partial y}=\frac{1}{I^3}\Big\{\Big(\frac{\partial\psi}{\partial\upsilon}\frac{\partial^2\varphi}{\partial u^2}-\frac{\partial\varphi}{\partial\upsilon}\frac{\partial^2\psi}{\partial u^2}\Big)\frac{\partial\varphi}{\partial\upsilon}\frac{\partial\psi}{\partial\upsilon}-\Big(\frac{\partial\psi}{\partial\upsilon}\frac{\partial^2\varphi}{\partial u\partial\upsilon}-$ $-\frac{\partial\varphi}{\partial\upsilon}\frac{\partial^2\psi}{\partial u\partial\upsilon}\Big)\Big(\frac{\partial\varphi}{\partial u}\frac{\partial\psi}{\partial\upsilon}+\frac{\partial\varphi}{\partial\upsilon}\frac{\partial\psi}{\partial u}\Big)+\Big(\frac{\partial\psi}{\partial\upsilon}\frac{\partial^2\varphi}{\partial\upsilon^2}-\frac{\partial\varphi}{\partial\upsilon}\frac{\partial^2\psi}{\partial\upsilon^2}\Big)\frac{\partial\varphi}{\partial u}\frac{\partial\psi}{\partial u}\Big\};\ \frac{\partial^2u}{\partial y^2}=-\frac{1}{I^3}\Big\{\Big(\frac{\partial\psi}{\partial u}\frac{\partial^2\varphi}{\partial u^2}-$ $-\frac{\partial\varphi}{\partial\upsilon}\frac{\partial^2\psi}{\partial u^2}\Big)\Big(\frac{\partial\varphi}{\partial\upsilon}\Big)^2-2\Big(\frac{\partial\psi}{\partial\upsilon}\frac{\partial^2\varphi}{\partial u\partial\upsilon}-\frac{\partial\varphi}{\partial\upsilon}\frac{\partial^2\psi}{\partial u\partial\upsilon}\Big)\frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial\upsilon}+\Big(\frac{\partial\psi}{\partial\upsilon}\frac{\partial^2\varphi}{\partial\upsilon^2}-\frac{\partial\varphi}{\partial\upsilon}\frac{\partial^2\psi}{\partial\upsilon^2}\Big)\Big(\frac{\partial\varphi}{\partial u}\Big)^2\Big\}$, bu ýerde $I=\frac{\partial\varphi}{\partial u}\frac{\partial\psi}{\partial\upsilon}-\frac{\partial\varphi}{\partial\upsilon}\frac{\partial\psi}{\partial u}$. **297.** a) $\frac{\partial u}{\partial x}=\cos\frac{\upsilon}{u};\ \frac{\partial u}{\partial y}=\sin\frac{\upsilon}{u};\ \frac{\partial\upsilon}{\partial x}=-\Big(\sin\frac{\upsilon}{u}-\frac{\upsilon}{u}\cos\frac{\upsilon}{u}\Big);$ $\frac{\partial\upsilon}{\partial y}=\cos\frac{\upsilon}{u}+\frac{\upsilon}{u}\sin\frac{\upsilon}{u};$ b) $\frac{\partial u}{\partial x}=\frac{\sin\upsilon}{e^u(\sin\upsilon-\cos\upsilon)+1};\ \frac{\partial u}{\partial y}=\frac{-\cos\upsilon}{e^u(\sin\upsilon-\cos\upsilon)+1};$ $\frac{\partial\upsilon}{\partial x}=\frac{-(e^u-\cos\upsilon)}{u[e^u(\sin\upsilon-\cos\upsilon)+1]};\ \frac{\partial\upsilon}{\partial y}=\frac{e^u+\sin\upsilon}{u[e^u(\sin\upsilon-\cos\upsilon)+1]}$. **298.** $\frac{\partial u}{\partial x}=\frac{I}{I_1};\ \frac{\partial^2u}{\partial x^2}=$ $=\frac{1}{I_1^3}\Big\{\frac{\partial(g,h)}{\partial(y,z)}\Big(I_1\frac{\partial}{\partial x}+I_2\frac{\partial}{\partial y}+I_3\frac{\partial}{\partial z}\Big)^2f+\frac{\partial(h,f)}{\partial(y,z)}\Big(I_1\frac{\partial}{\partial x}+I_2\frac{\partial}{\partial y}+I_3\frac{\partial}{\partial z}\Big)^2g+\frac{\partial(f,g)}{\partial(y,z)}\Big(I_1\frac{\partial}{\partial x}+$ $+I_2\frac{\partial}{\partial y}+I_3\frac{\partial}{\partial z}\Big)^2h\Big\}$, bu ýerde $I_1=\frac{\partial(g,h)}{\partial(y,z)}$, $I_2=\frac{\partial(g,h)}{\partial(z,x)}$, $I_3=\frac{\partial(g,h)}{\partial(x,y)}$ we $I=\frac{D(f,g,h)}{D(x,y,z)}$.
299. $\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x};\ \frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}+\frac{I_2\partial g}{I_1\partial y}$, bu ýerde $I_1=\frac{\partial(g,h)}{\partial(z,t)}$, $I_2=\frac{\partial(h,f)}{\partial(z,t)}$. **300.** $\frac{\partial u}{\partial x}=\frac{I_1}{I};$ $\frac{\partial u}{\partial y}=\frac{I_2}{I};\ \frac{\partial u}{\partial z}=\frac{I_3}{I}$, bu ýerde $I_1=\frac{\partial(g,h)}{\partial(\upsilon,\omega)}$, $I_2=\frac{\partial(h,f)}{\partial(\upsilon,\omega)}$, $I_3=\frac{\partial(f,g)}{\partial(\upsilon,\omega)}$ we $I=\frac{D(f,g,h)}{D(u,\upsilon,\omega)}$.
301. $dz=-\frac{I_1dx+I_2dy}{I_3}$, bu ýerde $I_1=\frac{\partial(f,g)}{\partial(x,t)}$, $I_2=\frac{\partial(f,g)}{\partial(y,t)}$, $I_3=\frac{\partial(f,g)}{\partial(z,t)}$.

§4. Üýtgeýän ululyklary çalşyrmak

313. $x'''+xx'^5=0$. **314.** $x^{IV}=0$. **315.** $\frac{d^2x}{dt^2}-t\Big(\frac{dx}{dt}\Big)^3=0$. **316.** $\frac{d^2y}{dt^2}+y=0$. **317.** $\frac{d^3y}{dt^3}-$ $-3\frac{d^2y}{dt^2}+2\frac{dy}{dt}-6y=0$. **318.** $\frac{d^2y}{dt^2}+n^2y=0$. **319.** $\frac{d^2y}{dt^2}+m^2y=0$. **320.** $u''+\Big[q(x)-$ $-\frac{1}{4}p^2(x)-\frac{1}{2}p'(x)\Big]u=0$. **321.** $\frac{d^2u}{dt^2}+(u+3)\frac{du}{dt}+2u=0$. **322.** $\frac{d^2u}{dt^2}=0$. **323.** $\frac{d^2u}{dt^2}=$ $=0$. **324.** $\frac{d^2u}{dt^2}+8u\Big(\frac{du}{dt}\Big)^3=0$. **325.** $t^5\frac{d^3u}{dt^3}+(3t^4+1)\frac{d^2u}{dt^2}+\frac{du}{dt}=0$. **326.** $u''-u'=$

$= \dfrac{A}{(a-b)^2}u$. **328.** $F(1,\ u,\ u'+u^2)=0$. **329.** $F(xu'+u^2-u, u, 1)=0$. **332.** $\dfrac{dr}{d\varphi}=r$.

333. $r'^2=\dfrac{1-\sin 2\varphi}{\sin 2\varphi}r^2$. **334.** $r(r^2+2r'^2-rr'')=r'^3$. **335.** $\dfrac{r'}{r}$. **336.** $K=\dfrac{|r^2+2r'^2-rr''|}{(r^2+r'^2)^{3/2}}$.

337. $\dfrac{dr}{dt}=kr^3$; $\dfrac{d\varphi}{dt}=-1$. **338.** $\omega=\dfrac{d}{dt}\left(r^2\dfrac{d\varphi}{dt}\right)$. **339.** $Y'=x$; $Y''=\dfrac{1}{y''}$; $Y'''=-\dfrac{y'''}{y''^3}$.

340. $z=\varphi(x+y)$, bu ýerde φ erkin differensirlenýän funksiýa. **341.** $z=\varphi(x^2+y^2)$.

342. $z=\dfrac{x}{a}+\varphi(y-bz)$. **343.** $z=x\varphi\left(\dfrac{y}{x}\right)$. **344.** $\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial \upsilon}=e^u\operatorname{sh}\upsilon$. **345.** $\dfrac{\partial z}{\partial u}=\dfrac{\partial z}{\partial \upsilon}$.

346. $\dfrac{\partial z}{\partial \upsilon}=\dfrac{1}{2}$. **347.** $\dfrac{\partial z}{\partial \upsilon}=\dfrac{z}{\upsilon}\cdot\dfrac{z^2+u}{z^2-u}$. **348.** $(2u+\upsilon-z)\dfrac{\partial z}{\partial u}+(u+2\upsilon-z)\dfrac{\partial z}{\partial \upsilon}=u+$

$+\upsilon-z$. **349.** $\dfrac{e^{x+y}-z^2}{1-e^{-x}\dfrac{\partial z}{\partial \xi}-e^{-y}\dfrac{\partial z}{\partial \eta}}$. **350.** $\dfrac{\left(\dfrac{\partial z}{\partial u}\right)^2+\left(\dfrac{\partial z}{\partial \upsilon}\right)^2}{u^2+\upsilon^2}$. **351.** $\dfrac{\partial u}{\partial \xi}=0$. **352.** $\dfrac{\partial x}{\partial y}=\dfrac{x-z}{y}$.

353. $\dfrac{\partial x}{\partial u}+\dfrac{\partial x}{\partial \upsilon}=\dfrac{u}{\upsilon}$. **354.** $A=\dfrac{x^2-2xu+u^2\left[\left(\dfrac{\partial x}{\partial u}\right)^2+\left(\dfrac{\partial x}{\partial \upsilon}\right)^2\right]}{x^4\left(u\dfrac{\partial x}{\partial u}+\upsilon\dfrac{\partial x}{\partial \upsilon}\right)^2}$. **355.** $\dfrac{\partial u}{\partial \xi}+\dfrac{\partial u}{\partial \eta}+\dfrac{\partial u}{\partial \zeta}+$

$+3u+(e^{\xi}+e^{\eta}+e^{\zeta})=0$. **356.** $\dfrac{\partial \omega}{\partial \upsilon}=0$. **357.** $\dfrac{\partial \omega}{\partial u}=0$. **358.** $\dfrac{\partial \omega}{\partial \upsilon}=0$. **359.** $u^2\left(\dfrac{\partial \omega}{\partial u}\right)^2+$

$+\upsilon^2\left(\dfrac{\partial \omega}{\partial \upsilon}\right)^2=\omega^2\dfrac{\partial \omega}{\partial u}\dfrac{\partial \omega}{\partial \upsilon}$. **360.** $\dfrac{e^{2u}\left(1-\dfrac{\partial \omega}{\partial \upsilon}\cos^2\upsilon\right)}{\dfrac{\partial \omega}{\partial u}}$. **361.** $A=\dfrac{\partial \omega}{\partial u}:\dfrac{\partial \omega}{\partial \upsilon}$. **362.** $\dfrac{\partial \omega}{\partial \zeta}=\dfrac{\xi\eta}{\zeta}$.

363. $\omega=\dfrac{\partial u}{\partial \varphi}$. **364.** $\omega=r\dfrac{\partial u}{\partial r}$. **365.** $\omega=\left(\dfrac{\partial u}{\partial r}\right)^2+\dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial \varphi}\right)^2$. **366.** $\omega=\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+$

$+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \varphi^2}$. **367.** $\omega=r^2\dfrac{\partial^2 u}{\partial r^2}$. **368.** $\omega=\dfrac{\partial^2 u}{\partial \varphi^2}$. **369.** $I=\dfrac{1}{r}\left(\dfrac{\partial u}{\partial r}\dfrac{\partial \upsilon}{\partial \varphi}-\dfrac{\partial u}{\partial \varphi}\dfrac{\partial \upsilon}{\partial r}\right)$. **370.** $u=\varphi(x-$

$-at)+\psi(x+at)$, bu ýerde φ we ψ erkin funksiýalar. **371.** $3\dfrac{\partial^2 z}{\partial u\partial \upsilon}+\dfrac{\partial z}{\partial u}=0$. **372.** $\dfrac{\partial^2 z}{\partial u^2}+$

$+\dfrac{\partial^2 z}{\partial \upsilon^2}=0$. **373.** $a\left(\dfrac{\partial^2 z}{\partial u^2}-\dfrac{\partial z}{\partial u}\right)+2b\dfrac{\partial^2 z}{\partial u\partial \upsilon}+c\left(\dfrac{\partial^2 z}{\partial \upsilon^2}-\dfrac{\partial z}{\partial \upsilon}\right)=0$. **374.** $\dfrac{\partial^2 z}{\partial u^2}+\dfrac{\partial^2 z}{\partial \upsilon^2}=0$.

375. $\dfrac{\partial^2 z}{\partial u^2}+\dfrac{\partial^2 z}{\partial \upsilon^2}+m^2e^{2u}z=0$. **376.** $\dfrac{\partial^2 z}{\partial u\partial \upsilon}=0$. **377.** $\dfrac{\partial^2 z}{\partial u\partial \upsilon}=\dfrac{1}{2u}\dfrac{\partial z}{\partial \upsilon}$. **378.** $\dfrac{\partial^2 z}{\partial u\partial \upsilon}=$

$=\dfrac{2}{u(4-u\upsilon)}\dfrac{\partial z}{\partial \upsilon}$. **379.** $(u^2-\upsilon^2)\dfrac{\partial^2 z}{\partial u\partial \upsilon}=\upsilon\dfrac{\partial z}{\partial u}$. **380.** $\dfrac{\partial^2 z}{\partial \upsilon^2}=\dfrac{2u}{u^2+\upsilon^2}\dfrac{\partial z}{\partial u}$. **381.** $\dfrac{\partial^2 z}{\partial u\partial \upsilon}+$

$+\dfrac{1}{u^2-\upsilon^2}\left(\upsilon\dfrac{\partial z}{\partial u}-u\dfrac{\partial z}{\partial \upsilon}\right)=0$. **382.** $\left(1-\dfrac{\partial z}{\partial \upsilon}\right)\dfrac{\partial^2 z}{\partial u\partial \upsilon}+\dfrac{\partial z}{\partial \upsilon}\dfrac{\partial^2 z}{\partial \upsilon^2}=1$. **383.** $u=\varphi(x+\lambda_1 y)+$

$+\psi(x+\lambda_2 y)$, bu ýerde λ_1 we λ_2 sanlar $A+2B\lambda+C\lambda^2=0$ deňlemäniň kökleri. **385.** a) $\Delta u=$

$= \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr}$; b) $\Delta(\Delta u) = \frac{d^4u}{dr^4} + \frac{2}{r}\frac{d^3u}{dr^3} - \frac{1}{r^2}\frac{d^2u}{dr^2} + \frac{1}{r^3}\frac{du}{dr}$. **386.** $u\frac{d^2\omega}{du^2} + \frac{d\omega}{du} + c\omega = 0$.

387. $A = X\frac{\partial^2 u}{\partial X^2} - Y\frac{\partial^2 u}{\partial X\partial Y} + \frac{\partial u}{\partial X}$. **390.** $\xi\frac{\partial}{\partial\xi}\left(\xi\frac{\partial u}{\partial\xi}\right) + \eta\frac{\partial}{\partial\eta}\left(\eta\frac{\partial u}{\partial\eta}\right) + \zeta\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial u}{\partial\zeta}\right) =$

$= 2\left(\xi\eta\frac{\partial^2 u}{\partial\xi\partial\eta} + \xi\zeta\frac{\partial^2 u}{\partial\xi\partial\zeta} + \zeta\eta\frac{\partial^2 u}{\partial\eta\partial\zeta}\right)$. **391.** $\frac{\partial^2 z}{\partial y_1^2} + \frac{\partial^2 z}{\partial y_2^2} + \frac{\partial^2 z}{\partial y_3^2} = 0$. **392.** $\frac{\partial^2 u}{\partial\zeta^2} = 0$.

393. $\Delta_1 u = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial u}{\partial\varphi}\right)^2$; $\Delta_2 u = \frac{1}{r^2}\Big[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\times$

$\times\frac{\partial u}{\partial\theta}\Big) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}\Big]$. **394.** $\omega\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right) = \left(\frac{\partial\omega}{\partial x}\right)^2 + \left(\frac{\partial\omega}{\partial y}\right)^2$. **395.** $\frac{\partial^2\omega}{\partial u^2} = 0$. **396.** $\frac{\partial^2\omega}{\partial\upsilon^2} = 0$.

397. $\frac{\partial^2\omega}{\partial u^2} = \frac{1}{2}$. **398.** $\frac{\partial^2\omega}{\partial u^2} + \frac{\partial^2\omega}{\partial u\partial\upsilon} = 2\omega$. **399.** $\frac{\partial^2\omega}{\partial u^2} + \left(\frac{\upsilon}{u} - 1\right)\frac{\partial^2\omega}{\partial\upsilon^2} = 0$. **400.** $\frac{\partial^2\omega}{\partial u^2} +$

$+\frac{\partial^2\omega}{\partial\upsilon^2} + \left(\frac{\partial\omega}{\partial u}\right)^2 + \left(\frac{\partial\omega}{\partial\upsilon}\right)^2 = 0$. **401.** $\frac{\partial^2\omega}{\partial u\partial\upsilon} = \frac{\omega}{4\sin^2(u-\upsilon)}$. **402.** $\frac{\partial^2\omega}{\partial u^2} + \frac{\partial^2\omega}{\partial\upsilon^2} = 0$.

405. $\frac{\partial^2\omega}{\partial u\partial\upsilon} = 0$. **406.** $\frac{\partial^2\omega}{\partial\xi^2} + \frac{\partial^2\omega}{\partial\eta^2} + \frac{\partial^2\omega}{\partial\zeta^2} = \frac{\partial\omega}{\partial\xi} + \frac{\partial\omega}{\partial\eta} + \frac{\partial\omega}{\partial\zeta} + (e^{\omega} - 1)\Big[\left(\frac{\partial\omega}{\partial\xi}\right)^2 + \left(\frac{\partial\omega}{\partial\eta}\right)^2 +$

$+\left(\frac{\partial\omega}{\partial\zeta}\right)^2\Big]$. **408.** $x = y\varphi(z) + \psi(z)$. **409.** $A(X,Y)\frac{\partial^2 Z}{\partial Y^2} - 2B(X,Y)\frac{\partial^2 Z}{\partial X\partial Y} + C(X,Y)\frac{\partial^2 Z}{\partial X^2} = 0$.

§5. Geometrik goşundylar

410. $\frac{x - x_0}{-\cos\alpha\sin t_0} = \frac{y - y_0}{-\sin\alpha\sin t_0} = \frac{z - z_0}{\cos t_0}$; $z - z_0 = (x - x_0)\cos\alpha\,\mathrm{tg}t_0 + (y - y_0)\sin\alpha\,\mathrm{tg}t_0$,

bu ýerde $x_0 = a\cos\alpha\cos t_0$, $y_0 = a\sin\alpha\cos t_0$, $z_0 = a\sin t_0$. **411.** $\frac{x}{a} + \frac{z}{c} = 1$, $y = \frac{b}{2}$; $ax - cz =$

$= \frac{1}{2}(a^2 - c^2)$. **412.** $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{2}$; $x + y + 2z = 4$. **413.** $\frac{x-1}{3} = \frac{y-1}{3} = \frac{z-3}{-1}$;

$3x + 3y - z = 3$. **414.** $x + z = 2$, $y + 2 = 0$; $x - z = 0$. **415.** $M_1 = (-1, 1, -1)$; $M_2 = (-1/3, 1/9, -1/2)$.

419. $\mathrm{tg}\varphi = f_x'(x_0, y_0)\cos\alpha + f_y'(x_0, y_0)\sin\alpha$. **420.** $\frac{\partial u}{\partial l} = -\frac{16}{243}$. **421.** $2x + 4y - z -$

$-5 = 0$; $\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-5}{-1}$. **422.** $3x + 4y + 12z = 169$; $\frac{x}{3} = \frac{y}{4} = \frac{z}{12}$. **423.** $z = \frac{\pi}{4} -$

$-\frac{1}{2}(x - y)$; $\frac{x-1}{1} = \frac{y-1}{-1} = \frac{z - \pi/4}{2}$. **424.** $ax_0x + by_0y + cz_0z = 1$; $\frac{x - x_0}{ax_0} = \frac{y - y_0}{by_0} =$

$= \frac{z - z_0}{cz_0}$. **425.** $x + y - 2z = 0$; $\frac{x-1}{-1} = \frac{y-1}{-1} = \frac{z-1}{2}$. **426.** $x + y - 4z = 0$; $\frac{x-2}{1} = \frac{y-2}{1} =$

$= \frac{z-1}{-4}$. **427.** $\frac{x}{a}\cos\psi_0\cos\varphi_0 + \frac{y}{b}\cos\psi_0\sin\varphi_0 + \frac{z}{c}\sin\psi_0 = 1$; $\frac{x\sec\psi_0\sec\varphi_0 - a}{bc} =$

$= \dfrac{y \sec \psi_0 \operatorname{cosec} \varphi_0 - b}{ac} = \dfrac{z \operatorname{cosec} \psi_0 - c}{ab}$. **428.** $x\cos\varphi_0 + y\sin\varphi_0 - z\operatorname{tg}\alpha = 0$; $\dfrac{x - r_0 \cos \varphi_0}{\cos \varphi_0} =$ $= \dfrac{y - r_0 \sin \varphi_0}{\sin \varphi_0} = \dfrac{z - r_0 \operatorname{ctg}\alpha}{-\operatorname{tg}\alpha}$. **429.** $ax\sin\upsilon_0 - ay\cos\upsilon_0 + u_0 z = au_0\upsilon_0$; $\dfrac{x - u_0 \cos \upsilon_0}{a \sin \upsilon_0} =$ $= \dfrac{y - u_0 \sin \upsilon_0}{-a \cos \upsilon_0} = \dfrac{z - a\upsilon_0}{u_0}$. **430.** $\dfrac{3x}{u_0} - \dfrac{3y}{u_0^2} + \dfrac{z}{u_0^3} = 2$. **431.** $A(0, \pm 2\sqrt{2}, \mp 2\sqrt{2})$; $B(\pm 2, \mp 4, \pm 2)$; $C(\pm 4, \mp 2, 0)$. **432.** $x = \pm\dfrac{a^2}{d}$, $y = \pm\dfrac{b^2}{d}$, $z = \pm\dfrac{c^2}{d}$, bu ýerde $d = \sqrt{a^2 + b^2 + c^2}$. **433.** $x + 4y + 6z = \pm 21$. **438.** $x^2 + y^2 - xy = 1$, $z = 0$; $3y^2 + 4z^2 = 4$, $x = 0$; $3x^2 + 4z^2 = 4$, $y = 0$. **439.** $\delta < 0{,}003$. **441.** $\cos\varphi = \dfrac{2bz_0}{a\sqrt{a^2 + b^2}}$. **445.** $\dfrac{\partial u}{\partial n} = x_0 + y_0 + z_0$; a) $x_0 = y_0 = z_0 = \dfrac{1}{\sqrt{3}}$; b) $x_0 = y_0 = z_0 = -\dfrac{1}{\sqrt{3}}$; ç) $x + y + z = 0$, $x^2 + y^2 + z^2 = 1$ töwerekde. **446.** $\dfrac{\partial u}{\partial n} =$ $= \dfrac{2}{\sqrt{x_0^2/a^4 + y_0^2/b^4 + z_0^2/c^4}}$. **448.** $x^2 + y^2 = p^2$. **449.** $y = \pm x$. **450.** $y^2 = 4ax$. **451.** Egreldiji çyzyk ýok. **452.** $x^{2/3} + y^{2/3} = l^{2/3}$. **453.** $|xy| = \dfrac{S}{2\pi}$. **454.** $y = \dfrac{\vartheta_0^2}{2g} - \dfrac{gx^2}{2\vartheta_0^2}$. **456.** a) $y = 0$ – egreldiji çyzyk (epin nokatlarynyň geometrik orny); b) $y = 0$ – egreldiji çyzyk; ç) $y = 0$ – aýratyn nokatlaryň (gaýdyş nokatlarynyň) geometrik orny; d) $x = 0$ – iki gat nokatlaryň geometrik orny, $x = a$ – egreldiji çyzyk. **457.** $(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2$ tory. **458.** $x^2\sin^2\alpha +$ $+ y^2\sin^2\beta + z^2\sin^2\gamma - 2xy\cos\alpha\cos\beta - 2xz\cos\alpha\cos\gamma - 2yz\cos\beta\cos\gamma = 1$. **459.** $|xyz| = \dfrac{V}{4\pi\sqrt{3}}$.

460. $|z \pm \sqrt{x^2 + y^2}| = \rho\sqrt{2}$. **461.** $\begin{vmatrix} x & y \\ x_0 & y_0 \end{vmatrix}^2 + \begin{vmatrix} y & z \\ y_0 & z_0 \end{vmatrix}^2 + \begin{vmatrix} z & x \\ z_0 & x_0 \end{vmatrix}^2 \le R^2(x^2 + y^2 + z^2)$.

462. $(x - x_0)^2 + (y - y_0)^2 = (z - z_0)^2$.

§6. Teýloryň formulasy

463. $f(x, y) = 5 + 2(x-1)^2 - (y+2)^2 - (x-1)(y+2)$. **464.** $f(x, y, z) = 3[(x-1)^2 + (y-1)^2 +$ $+ (z-1)^2 - (x-1)(y-1) - (x-1)(z-1) - (y-1)(z-1)] + (x-1)^3 + (y-1)^3 + (z-1)^3 - 3(x-1)\times$ $\times(y-1)(z-1)$. **465.** $\Delta f(1, -1) = h - 3k + (-h^2 - 2hk + k^2) + (h^2k + hk^2)$. **466.** $f(x+h, y+k,$ $z+l) = f(x, y, z) + 2[h(Ax + Dy + E) + k(Dx + By + F) + l(Ex + Fy + Cz)] + f(h, k, l)$. **467.** $x^y = 1 +$ $+ (x-1) + (x-1)(y-1) + R_2(1 + \theta(x-1), 1 + \theta(y-1))$ $(0 < \theta < 1)$, bu ýerde $R_2(x, y) = \dfrac{1}{6}x^y \times$ $\times\left[\left(\dfrac{y}{x}dx + \ln x \cdot dy\right)^3 + 3\left(\dfrac{y}{x}dx + \ln x \cdot dy\right)\left(-\dfrac{y}{x^2}dx^2 + \dfrac{2}{x}dxdy\right) + \left(\dfrac{2y}{x^3}dx^3 - \dfrac{3}{x^2}dx^2dy\right)\right]$ we $dx = x - 1$, $dy = y - 1$. **468.** $1 - \dfrac{1}{2}(x^2 + y^2) - \dfrac{1}{8}(x^2 + y^2)^2$. **469.** a) $1 - \dfrac{1}{2}(x^2 - y^2)$; b) $\dfrac{\pi}{4} + x - xy$. **470.** $-(xy + xz + yz)$. **471.** $F(x, y) = \dfrac{h^2}{4}(f''_{xx} + f''_{yy}) + \dfrac{h^4}{48}(f^{\mathrm{IV}}_{xxxx} + f^{\mathrm{IV}}_{yyyy}) + \ldots$.

472. $F(\rho)=f(x,y)+\frac{\rho^2}{4}[f_{xx}''(x,y)+f_{yy}''(x,y)]$. **473.** $\Delta_{xy}f(x,y)=hk\left[\frac{\partial^2 f}{\partial x\partial y}+\sum_{n=3}^{\infty}\sum_{m=1}^{n-1}\left(\frac{1}{m!}\times\right.\right.$ $\left.\left.\times\frac{h^{m-1}k^{n-m-1}}{(n-m)!}\frac{\partial^n f(x,y)}{\partial x^m\partial y^{n-m}}\right)\right]$. **474.** $F(\rho)=f(x,y)+\sum_{n=1}^{\infty}\frac{1}{(n!)^2}\left(\frac{\rho}{2}\right)^{2n}\Delta^n f(x,y)$, bu ýerde $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$. **475.** $1+mx+ny+\frac{m(m-1)}{2!}x^2+mnxy+\frac{n(n-1)}{2!}y^2+\ldots$ ($|x|<1$, $|y|<1$). **476.** $\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n-1}(m+n-1)!}{m!n!}x^m y^n$ ($|x|+|y|<1$). **477.** $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\frac{x^m y^{2n+1}}{m!(2n+1)!}$ ($|x|<+\infty$, $|y|<+\infty$). **478.** $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\frac{x^m y^{2n}}{m!(2n)!}$ ($|x|<+\infty$, $|y|<+\infty$). **479.** $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left((-1)^m\times\right.$ $\left.\times\frac{x^{2m+1}y^{2n+1}}{(2m+1)!(2n+1)!}\right)$ ($|x|<+\infty$, $|y|<+\infty$). **480.** $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m\frac{x^{2m}y^{2n}}{(2m)!(2n)!}$ ($|x|<+\infty$, $|y|<+\infty$). **481.** $\sum_{n=0}^{\infty}(-1)^n\frac{(x^2+y^2)^{2n+1}}{(2n+1)!}$ ($x^2+y^2<+\infty$). **482.** $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+n}\frac{x^m y^n}{mn}$ ($|x|<1$, $|y|<1$). **483.** $f(x,y)=1+\frac{1}{3}\left(x-\frac{x^2}{2}\right)y$. **484.** $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(x-1)^m(y+1)^n}{m!n!}$ ($|x|<+\infty$, $|y|<+\infty$).

485. $\sum_{n=0}^{\infty}(-1)^n[1+(x-1)](y-1)^n$ ($-\infty<x<+\infty$, $0<y<2$). **486.** $z=1+[2(x-1)-(y-1)]-$ $-[8(x-1)^2-10(x-1)(y-1)+3(y-1)^2]+\ldots$. **487.** Eger $a<0$ bolsa, onda (0, 0) – üzňe nokady; eger $a=0$ bolsa, onda (0, 0) – gaýdyş nokady; eger-de $a>0$ bolsa, iki gat nokady. **488.** (0, 0) – iki gat nokady. **489.** (0, 0) – üzňe nokady. **490.** (0, 0) – üzňe nokady. **491.** (0, 0) – iki gat nokady. **492.** (0, 0) – gaýdyş nokady (ikinji görnüşli). **493.** (0, 0) – iki gat nokady. **494.** Eger $a<b<c$ bolsa, onda egri çyzyk owaldan we tükeniksiz şahadan durýar; eger-de $a=b<c$ bolsa, onda $A(a,0)$ – üzňe nokady; eger-de $a<b=c$ bolsa, onda $B(b,0)$ – iki gat nokady; eger-de $a=b=c$ bolsa , onda $A(a,0)$ – gaýdyş nokady. **495.** (0, 0) – iki gat nokady. **496.** (0, 0) – gaýdyş nokady. **497.** (0, 0) – gutardyş nokady. **498.** (0, 0) – burç nokady; **499.** $x=k\pi$ ($k=0,\pm1,\pm2,\ldots$) – 1-nji görnüşli üzülme nokatlary. **500.** $x=0$ – 2-nji görnüşli üzülme nokady. **501.** $x=0$ – iki gat nokady. **502.** $x=k\pi$ ($k=0,\pm1,\pm2,\ldots$) – gaýdyş nokatlary.

§7. Köp üýtgeýänli funksiýanyň ekstremumy

503. $x=0$, $y=1$ bolanda, $z_{\min}=0$. **504.** Ekstremum nokatlary ýok. **505.** $x-y+1=0$ göni çyzygyň nokatlarynda, $z=0$ – minimum. **506.** $x=1$, $y=0$ bolanda, $z_{\min}=-1$. **507.** $x=2$, $y=3$ bolanda, $z_{\min}=108$; $x=0$, $0<y<6$ bolanda, $z=0$ minimum, $x=0$, $-\infty<y<0$ we $6<y<+\infty$ bolanda, $z=0$ – maksimum. **508.** $x=1$, $y=1$ bolanda $z_{\min}=-1$. **509.** $x_1=-1$, $y_1=-1$ we $x_2=1$, $y_2=1$ bolanda, $z_{\min}=-2$; $x=0$, $y=0$ bolanda, ekstremum ýok. **510.** $x=0$, $y=0$ bolanda, $z=0$

– maksimum; $x=\pm\frac{1}{2}$, $y=\pm1$ bolanda, $z=-1\frac{1}{8}$ minimum; $x=0$, $y=\pm1$ bolanda, $z=-1$ eýer; $x=\pm\frac{1}{2}$, $y=0$ bolanda, $z=-\frac{1}{8}$ eýer. **511.** $x=5$, $y=2$ bolanda, $z=30$ minimum. **512.** $\frac{x}{a}=-\frac{y}{b}=\pm\frac{1}{\sqrt{3}}$ bolanda, $z_{\min}=-\frac{ab}{3\sqrt{3}}$; $\frac{x}{a}=\frac{y}{b}=\pm\frac{1}{\sqrt{3}}$ bolanda, $z_{\max}=\frac{ab}{3\sqrt{3}}$. **513.** $x=\frac{a}{c}$, $y=\frac{b}{c}$ hem-de $c>0$ bolanda, $z_{\max}=\sqrt{a^2+b^2+c^2}$; $x=\frac{a}{c}$, $y=\frac{b}{c}$ hem-de $c<0$ bolanda, $z_{\min}=-\sqrt{a^2+b^2+c^2}$; $a^2+b^2\neq0$, $c=0$ bolanda, ekstremumy ýok. **514.** $x=0$ we $y=0$ bolanda, $z_{\max}=1$. **515.** $x=0$, $y=0$ bolanda, $z=0$ minimum; $x=-1/4$, $y=-1/2$ bolanda, $z=e^{-2}/2$ eýer. **516.** $x=1$, $y=-2$ bolanda, $z=e^3$ eýer. **517.** $x=1$, $y=3$ bolanda, $z=e^{-13}\approx2{,}26\cdot10^{-6}$ maksimum; $x=-1/26$, $y=-3/26$ bolanda, $z=-26\cdot e^{-1/52}\approx-25{,}51$ minimum. **518.** $x=1$, $y=2$ bolanda, $z=7-10\ln2\approx0{,}0685$ minimum. **519.** $x=\pi/3$, $y=\pi/6$ bolanda, $z_{\max}=3\sqrt{3}/2$. **520.** $x=y=2\pi/3$ bolanda, $z_{\min}=-\frac{3\sqrt{3}}{8}$; $x=y=\frac{\pi}{3}$ bolanda, $z_{\max}=\frac{3\sqrt{3}}{8}$. **521.** $x=1$, $y=1$ bolanda, $z=-1+\frac{1}{2}\ln2+\frac{3}{4}\pi\approx1{,}70$ eýer. **522.** $x=y=\pm1/\sqrt{2e}\approx\pm0{,}43$ bolanda, $z=-1/2e\approx-0{,}184$ minimum; $x=-y=$ $=\pm1/\sqrt{2e}$ bolanda, $z=1/2e$ maksimum; $x=0$, $y=\pm1$ we $x=\pm1$, $y=0$ stasionar nokatlarda ekstremum ýok. **523.** $x=\frac{\pi}{12}(-1)^{m+1}+(m+n)\frac{\pi}{2}$, $y=\frac{\pi}{12}(-1)^{m+1}+(m-n)\times$ $\times\frac{\pi}{2}$ (m, $n=0$, ±1, ±2, ...) – stasionar nokatlar. m we n dürli jübütlikde bolanda, $z=m\pi+(\pi/6+\sqrt{3})(-1)^{m+1}+2\cdot(-1)^n$ ekstremum (m täk we n jübüt bolanda – maksimum, m jübüt we n täk bolanda – minimum); m we n bir jübütlikde bolanda, ekstremum ýok. **524.** $x=0$ we $y=0$ bolanda, $z_{\min}=0$; $x^2+y^2=1$ bolanda, $z=e^{-1}$ maksimum. **525.** $x=-1$, $y=-2$, $z=3$ bolanda, $u_{\min}=-14$. **526.** $x=24$, $y=-144$, $z=-1$ bolanda, $u=-6913$ minimum. **527.** $x=1/2$, $y=1$, $z=1$ bolanda, $u=4$ minimum. **528.** $x=y=z=a/7$ bolanda, $u_{\max}=a^7/7^7$; $y=0, x\neq0, z\neq0, x+2y+3z\neq a$ bolanda, $u=0$ ekstremum. **529.** $x=\frac{1}{2}\sqrt[15]{16a^{14}b}$, $y=\frac{1}{4}\sqrt[5]{16a^4b}$, $z=\frac{1}{2}\sqrt{\frac{a^8b^7}{4}}$ bolanda, $u=\frac{15a}{4}\sqrt[15]{\frac{a}{16b}}$ minimum. **530.** $x=y=z=\frac{\pi}{2}$ bolanda, $u=4$ maksimum; $x=y=z=0$ we $x=y=z=\pi$ bolanda, $u=0$ gyraky minimum. **531.** $x_1=x_2=...=x_n=\frac{2}{n^2+n+2}$ bolanda, $u_{\max}=\left(\frac{2}{n^2+n+2}\right)^{\frac{n^2+n+2}{2}}$. **532.** $x_1=2^{\frac{1}{n+1}}$, $x_2=x_1^2$, ..., $x_n=x_1^n$ bolanda, $u=(n+1)2^{\frac{1}{n+1}}$ minimum. **533.** a, x_1, x_2, ..., x_n, b sanlar $q=\sqrt[n+1]{b/a}$ maýdalawjyly geometrik progressiýany düzýärler. **534.** $x=1$, $y=-1$ bolanda, $z_1=-2$ minimum we $z_2=6$ maksimum. **535.** $x=y=-(3+\sqrt{6})$ bolanda, $z_{\min}=-(4+2\sqrt{6})$; $x=y=-(3-\sqrt{6})$ bolanda, $z_{\max}=2\sqrt{6}-4$. **536.** $x^2+y^2=\frac{3a^2}{8}$,

$z<0$ bolanda, $z=-\frac{a}{2\sqrt{2}}$ minimum; $x^2+y^2=\frac{3a^2}{8}$, $z>0$ bolanda, $z=\frac{a}{2\sqrt{2}}$ maksimum.

537. $x=\frac{1}{2}$, $y=\frac{1}{2}$ bolanda, $z_{\max}=\frac{1}{4}$. **538.** $x=-\frac{b\varepsilon}{\sqrt{a^2+b^2}}$, $y=-\frac{a\varepsilon}{\sqrt{a^2+b^2}}$ bolanda, $z_{\min}=-\frac{\sqrt{a^2+b^2}}{|ab|}$; $x=\frac{b\varepsilon}{\sqrt{a^2+b^2}}$, $y=\frac{a\varepsilon}{\sqrt{a^2+b^2}}$ bolanda, $z_{\max}=\frac{\sqrt{a^2+b^2}}{|ab|}$, bu ýerde $\varepsilon=\mathrm{sgn}\,ab\neq 0$. **539.** $x=\frac{ab^2}{a^2+b^2}$, $y=\frac{a^2b}{a^2+b^2}$ bolanda, $z_{\max}=\frac{a^2b^2}{a^2+b^2}$. **540.** $z_{\min}=\lambda_1$, $z_{\max}=\lambda_2$, bu ýerde λ_1 we λ_2 sanlar $(A-\lambda)(C-\lambda)-B^2=0$ deňlemäniň kökleri we $\lambda_1<\lambda_2$. **541.** $x=\pm 1\frac{1}{2}$, $y=\pm 4$ bolanda, $z=106\frac{1}{4}$ maksimum; $x=\pm 2$, $y=\mp 3$ bolanda, $z=-50$ minimum. **542.** $x=\frac{\pi}{8}+\frac{\pi k}{2}$, $y=-\frac{\pi}{8}+\frac{\pi k}{2}$ $(k=0, \pm 1, \pm 2, ...)$ bolanda $z=1+\frac{(-1)^k}{\sqrt{2}}$ ekstremum (k – jübüt san bolanda maksimum we k – täk san bolanda minimum). **543.** $x=-\frac{1}{3}$, $y=\frac{2}{3}$, $z=-\frac{2}{3}$ bolanda, $u_{\min}=-3$; $x=\frac{1}{3}$, $y=-\frac{2}{3}$, $z=\frac{2}{3}$ bolanda, $u_{\max}=3$. **544.** $\frac{x}{m}=\frac{y}{n}=\frac{z}{p}=\frac{a}{m+n+p}$ bolanda, $u_{\max}=\frac{a^{m+n+p}m^m n^n p^p}{(m+n+p)^{m+n+p}}$. **545.** $x=0$, $y=0$, $z=\pm c$ bolanda, $u_{\min}=c^2$; $x=\pm a$, $y=0$, $z=0$ bolanda, $u_{\max}=a^2$. **546.** $x=y=z=\frac{a}{6}$ bolanda, $u_{\max}=\left(\frac{a}{b}\right)^6$. **547.** $x=y=\frac{1}{\sqrt{6}}$ we $z=-\frac{2}{\sqrt{6}}$, $x=z=\frac{1}{\sqrt{6}}$, we $y=-\frac{2}{\sqrt{6}}$, $y=z=\frac{1}{\sqrt{6}}$ we $x=-\frac{2}{\sqrt{6}}$ bolanda $u_{\min}=-\frac{1}{3\sqrt{6}}$; $x=y=-\frac{1}{\sqrt{6}}$ we $z=\frac{2}{\sqrt{6}}$, $x=z=-\frac{1}{\sqrt{6}}$ we $y=\frac{2}{\sqrt{6}}$, $y=z=-\frac{1}{\sqrt{6}}$ we $x=\frac{2}{\sqrt{6}}$ bolanda $u_{\max}=\frac{1}{3\sqrt{6}}$. **548.** $x=y=z=1$ bolanda, $u=2$ şertli maksimum. **549.** $x=y=z=\frac{\pi}{6}$ bolanda, $u_{\max}=1/8$. **550.** $u_{\min}=\lambda_1$ we $u_{\max}=\lambda_2$, bu ýerde λ_1 we λ_2 sanlar $\lambda^2-\left(\frac{\sin^2\alpha}{a^2}+\frac{\sin^2\beta}{b^2}+\right.$ $\left.+\frac{\sin^2\gamma}{c^2}\right)\lambda+\left(\frac{\cos^2\alpha}{b^2c^2}+\frac{\cos^2\beta}{a^2c^2}+\frac{\cos^2\gamma}{a^2b^2}\right)=0$ deňlemäniň kökleri $(\lambda_1<\lambda_2)$. **551.** $u_{\max}=R^2$; $u_{\min}=\frac{R^2(A\cos\alpha+B\cos\beta+C\cos\gamma)^2}{A^2+B^2+C^2}$. **552.** $x_i=\frac{1}{a_i}\left(\sum_{j=1}^{n}\frac{1}{a_j^2}\right)^{-1}$ $(i=1, 2, ..., n)$ bolanda, $u_{\min}=\left(\sum_{j=1}^{n}\frac{1}{a_j^2}\right)^{-1}$. **553.** $x_i=\frac{a}{n}$ $(i=1, 2, ..., n)$ bolanda, $u_{\min}=\frac{a^p}{n^{p-1}}$. **554.** $x_i=\sqrt{\frac{\alpha_i}{\beta_i}}\times$ $\times\left(\sum_{j=1}^{n}\sqrt{\alpha_j\beta_j}\right)^{-1}$ $(i=1, 2, ..., n)$ bolanda, $u_{\min}=\left(\sum_{j=1}^{n}\sqrt{\alpha_j\beta_j}\right)^2$. **555.** $\frac{x_1}{\alpha_1}=\frac{x_2}{\alpha_2}=...=\frac{x_n}{\alpha_n}=$

$=\dfrac{a}{\alpha_1+\alpha_2+...+\alpha_n}$ bolanda, $u_{\max}=\left(\dfrac{a}{\alpha_1+\alpha_2+...+\alpha_n}\right)^{\alpha_1+\alpha_2+...+\alpha_n}\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}...\alpha_n^{\alpha_n}$. **556.** $u=\lambda$ ekstremumlar $|a_{ij}-\lambda\delta_{ij}|=0$ deňlemeden kesgitlenilýär, bu ýerde $i\neq j$ bolanda $\delta_{ij}=0$ we $\delta_{ii}=1$. **560.** $\inf z=-5$; $\sup z=-2$. **561.** $\inf z=-75$; $\sup z=125$. **562.** $\inf z=0$; $\sup z=1$. **563.** $\inf u=0$; $\sup u=300$. **564.** $\inf u=-1/2$; $\sup u=1+\sqrt{2}$. **565.** $\inf u=0$; $\sup u=e^{-1}\approx 0{,}37$. **567.** Ýok. **568.** Minimum $n/\sqrt[n]{a}$ deňdir. **569.** Goşulyjylar deňdir. **570.** Köpeldijiler $x_i=\dfrac{\left(a\alpha_1^{\frac{1}{\alpha_1}}\alpha_2^{\frac{1}{\alpha_2}}...\alpha_n^{\frac{1}{\alpha_n}}\right)^{\frac{1/\alpha_i}{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+...+\frac{1}{\alpha_n}}}}{(\alpha_i)^{\frac{1}{\alpha_i}}}$ $(i=1, 2, ..., n)$ deňdir, bu ýerde α_i $(i=1, 2, ..., n)$ $-$degişli derejeleriň görkezijileri; $\left(\dfrac{1}{\alpha_1}+\dfrac{1}{\alpha_2}+...+\dfrac{1}{\alpha_n}\right)\left(a\alpha_1^{\frac{1}{\alpha_1}}\alpha_2^{\frac{1}{\alpha_2}}...\alpha_n^{\frac{1}{\alpha_n}}\right)^{\frac{1}{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+...+\frac{1}{\alpha_n}}}$ jemiň iň kiçi bahasy. **571.** $x=\dfrac{1}{M}\sum_{i=1}^{n}m_ix_i$, $y=\dfrac{1}{M}\sum_{i=1}^{n}m_iy_i$, bu ýerde $M=\sum_{i=1}^{n}m_i$. **572.** Wannanyň ölçegleri $\sqrt[3]{2V}$, $\sqrt[3]{2V}$, $\sqrt[3]{2V}/2$. **573.** $H=2R=2\sqrt{S/3\pi}$, bu ýerde R – silindrik üstüň radiusy we H – onuň emele getirijisi. **574.** $x=\dfrac{1}{N}\sum_{i=1}^{n}x_i$, $y=\dfrac{1}{N}\sum_{i=1}^{n}y_i$, $z=\dfrac{1}{N}\sum_{i=1}^{n}z_i$, bu ýerde $N=\sqrt{\left(\sum_{i=1}^{n}x_i\right)^2+\left(\sum_{i=1}^{n}y_i\right)^2+\left(\sum_{i=1}^{n}z_i\right)^2}$. Uzaklyklaryň kwadratlarynyň iň kiçi jemi $n-2N+\sum_{i=1}^{n}(x_i^2+y_i^2+z_i^2)$ deňdir. **575.** Konusyň emele getirijileriniň esasy bilen emele getirýän burçy arcsin(2/3)-ä deňdir. **576.** Piramidalaryň gapdal granlarynyň esasy bilen emele getirýän burçy arcsin(2/3)-ä deňdir. **577.** Gönüburçlugyň taraplary $2p/3$ we $p/3$. **578.** Üçburçlugyň taraplary $p/2$, $3p/4$ we $3p/4$. **579.** Parallelepipediň ölçegleri $2R/\sqrt{3}$, $2R/\sqrt{3}$ we $R/\sqrt{3}$. **580.** Parallelepipediň beýikligi konusyň beýikliginiň 1/3-ne deňdir. **581.** Parallelepipediň ölçegleri $2a/\sqrt{3}$, $2b/\sqrt{3}$ we $2c/\sqrt{3}$. **582.** Parallelepipediň beýikligi $h=l\sin\alpha\cdot\dfrac{\operatorname{tg}\alpha-\sqrt{2}}{2\operatorname{tg}\alpha-\sqrt{2}}$, eger $\alpha\geq\operatorname{arctg}\sqrt{2}$ bolsa we $h=0$, eger $0<\alpha<\operatorname{arctg}\sqrt{2}$ bolsa. **583.** Parallelepipediň ölçegleri a, b we $c/2$. **584.** $\dfrac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$. **585.** $d=\dfrac{1}{\pm\Delta}\begin{vmatrix}x_1-x_2 & y_1-y_2 & z_1-z_2\\ m_1 & n_1 & p_1\\ m_2 & n_2 & p_2\end{vmatrix}$, $\Delta=\sqrt{\begin{vmatrix}m_1 & n_1\\ m_2 & n_2\end{vmatrix}^2+\begin{vmatrix}n_1 & p_1\\ n_2 & p_2\end{vmatrix}^2+\begin{vmatrix}p_1 & m_1\\ p_2 & m_2\end{vmatrix}^2}$. **586.** $\dfrac{7}{4\sqrt{2}}$. **587.** Ýarym oklaryň $a^2=\lambda_1$ we $b^2=\lambda_2$ kwadratlary $(1-\lambda A)(1-\lambda C)-\lambda^2B^2=0$ deňlemäniň kökleri. **588.** Ýarym oklaryň

$a^2=\lambda_1$, $b^2=\lambda_2$ we $c^2=\lambda_3$ kwadratlary $\begin{vmatrix} A\lambda-1 & D\lambda & F\lambda \\ D\lambda & B\lambda-1 & E\lambda \\ F\lambda & E\lambda & C\lambda-1 \end{vmatrix}=0$ deňlemäniň kökleri.

589. $\frac{\pi ab}{|C|}\sqrt{A^2+B^2+C^2}$. **590.** $\frac{\pi abc}{\sqrt{a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma}}$. **592.** Düşme burçy $\arcsin(n\sin(\alpha/2))$ deňdir; şöhläniň gyşarmasy $2\arcsin\left(n\sin\frac{\alpha}{2}\right)-\alpha$ deňdir. **593.** Gözlenýän a we b koeffisiýentler $a[xx]+b[x1]=[xy]$, $a[x1]+bn=[y1]$ deňlemeler sistemasyndan kesgitlenilýär, bu ýerde $[xy]=\sum_{i=1}^{n}x_iy_i$ we ş.m. Eger $\sum_{i\neq j}(x_i-x_j)^2\neq 0$ bolsa, onda meseläniň kesgitli çözüwi bar. **594.** $\operatorname{tg}2\alpha=\frac{2(\overline{x}\cdot\overline{y}-\overline{xy})}{[\overline{x^2}-(\overline{x})^2]-[\overline{y^2}-(\overline{y})^2]}$, $p=\overline{x}\cos\alpha+\overline{y}\sin\alpha$, bu ýerde $\overline{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$, $\overline{xy}=\frac{1}{n}\sum_{i=1}^{n}x_iy_i$ we ş.m. orta bahalar. **595.** $4x-7/2$; $\Delta_{\min}=1/2$.

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MAZMUNY

VII. Kesgitli integral

VIII. Köp üýtgeýänli funksiýalar

Orazmuhammet Aşyrow, Hajymämmet Soltanow

MATEMATIKI ANALIZ BOÝUNÇA MESELELER WE GÖNÜKMELER

(Bir üýtgeýänli funksiýalaryň differensialy we integraly.
Köp üýtgeýänli funksiýalaryň differensialy)

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Redaktor	*H. Sapargulyýew*
Surat redaktory	*O. Çerkezowa*
Teh. redaktor	*O. Nurýagdyýewa*
Kompýuter bezegi	*M. Mullikowa,*
	B. Mämmetgurbanow
Neşir üçin jogapkärler	*G. Hekimow, A. Karýagdyýew*

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