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ÝOKARY MATEMATIKADAN BÄSLEŞIK MESELELERI
I

Ýokary okuw mekdepleri üçin okuw gollanmasy

Türkmenistanyň Bilim ministrligi tarapyndan hödürlenildi

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Ö 73 **Ýokary matematikadan bäsleşik meseleleri. I.** *Ýokary okuw mekdepleri üçin okuw gollanmasy*. – A.: Ylym, 2016. – 104 sah.

Gollanma ýokary okuw mekdepleriň talyplaryna, mugallymlaryna niýetlenip, onda ýokary matematika degişli çylşyrymly meseleler çözgüleri bilen getirilýär.

Bu gollanmadan döwlet we Halkara bäsleşiklerine taýýarlyk görýänler we şu ugurdan ylmy-derňew işleri alyp barýanlar peýdalanyp biler.

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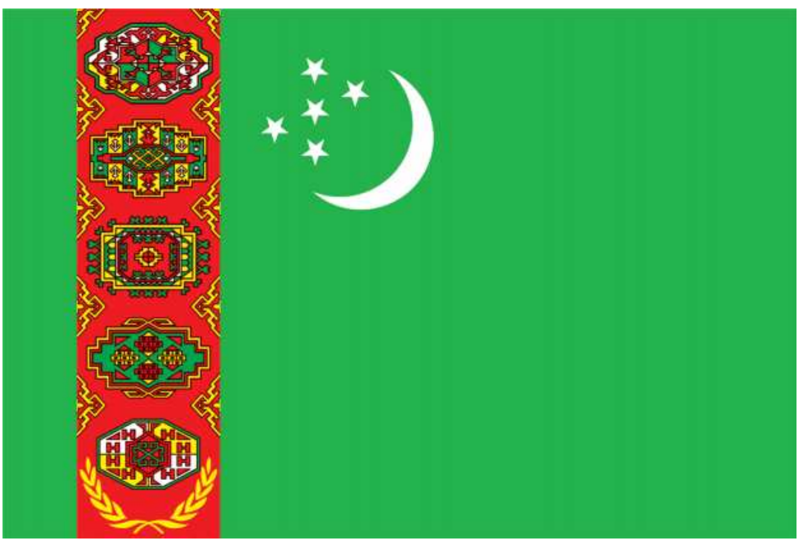
TÜRKMENISTANYŇ PREZIDENTI
GURBANGULY BERDIMUHAMEDOW

TÜRKMENISTANYŇ DÖWLET TUGRASY



TÜRKMENISTANYŇ DÖWLET BAÝDAGY

TÜRKMENISTANYŇ DÖWLET SENASY

Janym gurban saňa, erkana ýurdum,
Mert pederleň ruhy bardyr köňülde.
Bitarap, garaşsyz topragyň nurdur,
Baýdagyň belentdir dünýäň öňünde.

Gaýtalama:
Halkyň guran Baky beýik binasy,
Berkarar döwletim, jigerim-janym.
Başlaryň täji sen, diller senasy,
Dünýä dursun, sen dur, Türkmenistanym!

Gardaşdyr tireler, amandyr iller,
Owal-ahyr birdir biziň ganymyz.
Harasatlar almaz, syndyrmaz siller,
Nesiller döş gerip gorar şanymyz.

Gaýtalama:
Halkyň guran Baky beýik binasy,
Berkarar döwletim, jigerim-janym.
Başlaryň täji sen, diller senasy,
Dünýä dursun, sen dur, Türkmenistanym!

SÖZBAŞY

Hormatly Prezidentimiz Gurbanguly Berdimuhamedowyň tagallasy bilen bilim ulgamynyň düýpli özgerdilmegi halkymyzda uly kanagatlanma duýgusyny döretdi we goldaw tapdy. Gysga wagtyň içinde şeýle uly özgertmeleri durmuşa geçirmek bilim ulgamynyň işgärleriniň öňünde hem gaýragoýulmasyz ýerine ýetirilmeli işleri ýüze çykardy.

Şeýle meseleleriň esasylarynyň biri-de täze düzülen okuw maksatnamalaryna laýyk gelýän okuw kitaplaryny, gollanmalary, mesele ýygyndylaryny taýýarlamak we çap etmekden ybaratdyr.

Bilim ulgamyny ösdürmek we kämilleşdirmek döwlet syýasatynyň gaýragoýulmasyz meseleleriniň biri diýip yglan edilenden soň, mekdep okuwçylarynyň, talyplaryň arasynda dürli bäsleşikleri we okuw dersleri boýunça olimpiadalary yzygiderli geçirmeklige hem üns güýçlendirildi.

Ýokary okuw mekdepleriniň talyplarynyň arasynda matematika dersi boýunça olimpiadalary geçirmek, matematiki bilimleri giňden mahabatlandyrmagyň esasy görnüşleriniň biri bolup durýar.

Bu gollanma ýokary okuw mekdepleriniň talyplarynyň arasynda matematika dersi boýunça Döwlet olimpiadalaryny geçirmekde we talyplary Halkara bäsleşiklerine taýýarlamakda gerekli kitaplaryň biri bolar diýip umyt edýäris. Ýygyndyda, esasan hem, [3]-[4] kitaplarda çözüwleri görkezilmedik meseleleriň belli bir bölegi alyndy.

Ýygyndydaky meseleleriň çözüwleriniň beýan edilişi ýokary okuw mekdepleriniň matematika we onuň bilen ugurdaş hünärlerinde okaýan talyplar üçin doly güýçýeterli bolup, ýygyndy olimpiada meseleleri bilen gyzyklanýan okyjylaryň giň köpçüligine niýetlenendir.

§1. KÖPAGZALAR BILEN BAGLANYŞYKLY MESELELER

1-nji mysal. $p(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^n}{n!}$ köpagzanyň kratny köklere eýe bolup bilmejekdigini subut ediň.

Çözülişi. Goý, $p(x)$ köpagzanyň iň bolmanda deň bolan iki kratny köki bar bolsun. Belli bolşy ýaly, $p(x)=\left(x-a\right)^2 q(x)$ ýa-da $q'(x)=(x-a)r(x)$.

Bu ýerde $q(x)$ we $r(x)$ derejesi $(n-2)$ bolan köpagzalar. Görnüşi ýaly, $p'(x)=0$. Bu deňligi ulanyp alarys:

$$0=p(a)=1+\frac{a}{1!}+\frac{a^2}{2!}+...+\frac{a^n}{n!};$$

$$0=p'(a)=1+\frac{a}{1!}+\frac{a^2}{2!}+...+\frac{a^{n-1}}{(n-1)!}.$$

Soňky iki deňlikleri deňläp, $\frac{a^n}{n!}=0$ bolýandygyny alarys. Emma $a\neq 0$ bolany sebäpli, bu deňlik ýerine ýetmeýär. Diýmek, $p(x)$ köpagzanyň kratny kökleri bolup bilmez.

2-nji mysal. x argumentiň dürli üç hakyky bahalarynda x^3+px+q üçagzanyň nola deň bolmagy üçin p we q sanlar haýsy şertleri kanagatlandyrmaly.

Çözülişi. Eger $r(x)=x^3+px+q$ üçagzanyň kökleri bar bolsa, onda Kardanonyň formulasyna görä:

$$x_1=A+B,\quad x_{2,3}=-\frac{A+B}{2}\pm i\frac{A-B}{2}\sqrt{3}$$

bolar. Bu ýerde,

$$A=\sqrt[3]{-\frac{q}{2}+\sqrt{C}},\ \ B=\sqrt[3]{-\frac{q}{2}-\sqrt{C}},\ \ C=\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2.$$

$r(x)$ üçagzanyň ähli kökleriniň hakyky sanlar bolmagy üçin

$$A=a+ib,\ \ B=a-ib\ \ \left(a\neq 0, b\neq 0\right)$$

bolar we $AB=a^2+b^2=-p/3$. Bu ýerden $p<0$ bolmalydygy gelip çykýar. Soňky deňlikleri göz öňünde tutup, alarys:

$$x_1=A+B=2a,$$

$$x_{2,3}=-\frac{A+B}{2}\pm i\frac{A-B}{2}\sqrt{3}=-a\mp b\sqrt{3},$$

$a\neq 0, b\neq 0$ bolany üçin kökler dürli hakyky sanlar bolýar. Wiýetiň teoremasyna görä:

$$p=x_1x_2+x_2x_3+x_3x_1=-3(a^2+b^2),$$
$$q=x_1x_2x_3=2a(a^2-3b^2).$$

Jogaby: $p=-3\left(a^2+b^2\right)$ we $q=2a\left(a^2-3b^2\right)$.

3-nji mysal. Derejesi $n\geq 3$ bolan täk derejeli köpagzanyň iň bolmanda bir epin nokadynyň bardygyny subut ediň.

Çözülişi. Şerte görä $p(x)=a_1x^n+a_2x^{n-1}+...+a_n$.

Bu ýerde $n=2k+1$, $k\in N$. Onda:

$$p''(x)=\sum_{i=1}^{n}a_i(n-i+1)(n-i)x^{n-i-1}.$$

Ahyrky $p''(x)$ köpagzanyň täk derejelidigini göz öňünde tutsak, onuň hökman iň bolmanda bir hakyky çözüwi bolmaly. Bu ýerden hem berlen köpagzanyň iň bolmanda bir epin nokada eýedigi gelip çykýar.

4-nji mysal. $p(x)=x^n+a_1x^{n-1}+...+a_n$ bitin koeffisiýentli köpagzada $p(0)$ we $p(1)$ täk sanlar. Bu köpagzanyň bitin köküniň bolup bilmejekdigini subut ediň.

Çözülişi. Goý, $p(x)$ köpagzanyň $x = l$ bitin köki bar diýip guman edeliň.

1-nji ýagdaý. l-jübüt san bolsun. Ýagny, $l = 2m,\ m \in \mathrm{Z}$.

$$0 = p(l) = p(2m) = (2m)^n + a_1(2m)^{n-1} + ... + a_{n-1}(2m) + a_n =$$

$$= 2m \cdot R(m) + a_n = 2m \cdot R(m) + p(0)$$

ýa-da

$$0 = 2m \cdot R(m) + p(0).$$

Ahyrky deňligiň çep tarapy jübüt, emma sag tarapy täk. Diýmek, l jübüt san bolup bilmez.

2-nji ýagdaý. l-täk san bolsun. Ýagny, $l = 2m + 1,\ m \in \mathrm{Z}$.

$$0 = p(l) = p(2m+1) =$$

$$= (2m+1)^n + a_1(2m+1)^{n-1} + ... + a_{n-1}(2m+1) + a_n =$$

$$= 2m \cdot Q(m) + (1 + a_1 + ... + a_{n-1} + a_n) = 2m \cdot Q(m) + p(1)$$

ýa-da

$$0 = 2m \cdot Q(m) + p(1).$$

Ahyrky deňligiň çep tarapy jübüt, emma sag tarapy täk. Diýmek, l täk san hem bolup bilmez. 1-nji we 2-nji ýagdaýdan alnan netijeler berlen köpagzanyň bitin köküniň bolup bilmejekdigini görkezýär.

5-nji mysal. Eger köpagzanyň koeffisiýentleri umumy bölüjä eýe bolmadyk bitin sanlar bolsalar, bu köpagza ýönekeý köpagza diýilýär. Ýönekeý köpagzalaryň köpeltmek hasylynyň hem ýönekeý köpagza bolýandygyny subut ediň.

Çözülişi. Goý, $p(x)$ we $q(x)$ köpagzalar ýönekeý köpagzalar bolsun we olaryň $p(x)q(x)$ köpeltmek hasyly ýönekeý köpagza däl diýip güman edeliň we aşakdaky bellenişikleri geçireliň:

$$p(x) = p_1x^n + p_2x^{n-1} + ... + p_{n-1}x + p_n,\ (p_1, p_2, ..., p_n) = 1;$$

$$q(x) = q_1x^m + q_2x^{m-1} + ... + q_{n-1}x + q_m,\ (q_1, q_2, ..., q_m) = 1;$$

$$r(x) = p(x)q(x) = r_1x^n + r_2x^{n-1} + ... + r_n,\ (r_1, r_2, ..., r_n) = d.$$

Bu ýerde $p_i, q_i, r_i, \in \mathrm{Z},\ i \in N,\ d \neq 1$. Görnüşi ýaly, $\frac{r(x)}{d}$ köpagza bitin koeffisiýentli köpagza. d sany ýönekeý köpeldijilere dagydyp, $d = a_1^{\beta_1} a_2^{\beta_2} a_3^{\beta_3} ... a_k^{\beta_k}$ görnüşde ýazalyň, bu ýerde $\alpha_1, \alpha_2, ..., \alpha_k$ – ýönekeý, $\beta_1, \beta_2, ... \beta_n$ – natural sanlar.

$$\frac{r(x)}{d} = \frac{p(x)q(x)}{a_1^{\beta_1} a_2^{\beta_2} a_3^{\beta_3} ... a_k^{\beta_k}}$$

köpagzanyň bitin koeffisiýentli bolmagy üçin $p(x)$ ýa-da $q(x)$ köpagzalaryň iň bolmanda biri α_1 sana galyndysyz bölünmeli, ýagny $(p_1, p_2, ..., p_n) = \alpha_1$ ýa-da $(q_1, q_2, ..., q_m) = \alpha_1$ bolmaly. Emma meseläniň şertine görä bu deňlik ýerine ýetip bilmez. Alnan garşylyk $p(x)q(x)$ köpagzanyň primitiw köpagza bolýandygyna güwä geçýär. Diýmek, iki primitiw köpagzanyň köpeltmek hasyly primitiw köpagza bolýar. Matematiki induksiýa usuly arkaly n sany primitiw köpagzalaryň köpeltmek hasylynyň hem primitiw köpagza bolýandygyny subut etmek bolar.

6-njy mysal. Her bir p köpagza aşakdaky şertleri kanagatlandyrýan $D(p)$ san degişli edilýär:

1) $D(\alpha_1 p_1 + \alpha_2 p_2) = \alpha_1 D(p_1) + \alpha_2 D(p_2)$;

2) $D(p_1 p_2) = D(p_1) p_2\left(\frac{1}{2}\right) + D(p_2) p_1\left(\frac{1}{2}\right);\ \left(\alpha_1, \alpha_2 \in R\right)$.

a) $D(p) = cp'\left(\frac{1}{2}\right)$ bolýandygyny subut ediň;

b) Eger D sanyň kesgitlemesinde p köpagzany, $[0,1]$ aralykda üznüksiz bolan islendik f funksiýa bilen çalyşsak, onda ähli f funksiýalar üçin $D(f) = 0$ boljakdygyny subut ediň.

Çözülişi.

a) $D(p)$ sanyň kesgitlemesindäki 2)-nji şertinde ilki $p_1 = p_2 = 1$ bahany, soňra $p_1 = 1$, $p_2 = x$ bahany goýup, degişlilikde

$$D(1) = 0,\ \ D(x) = \text{const} = c$$

deňlikleri alarys. 2-nji şerti ulanyp,

$$D(x^n) = D(x^{n-1}x) = D(x^{n-1}) \cdot \frac{1}{2} + D(x) \cdot \left(\frac{1}{2}\right)^{n-1} =$$

$$= D(x^{n-2}) \cdot \left(\frac{1}{2}\right)^{n-2} + D(x) \cdot \left(\frac{1}{2}\right)^{n-1} + D(x) \cdot \left(\frac{1}{2}\right)^{n-1} = ... = \frac{nD(x)}{2^{n-1}}$$

ýa-da

$$D(x^n) = \frac{nD(x)}{2^{n-1}}.$$

Alnan netijeleri ulanyp, $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$ köpagza üçin $D(p)$ sany tapalyň:

$$D(p) = D\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i D(x^i) = \sum_{i=1}^{n} a_i \frac{i}{2^{i-1}} D(x) =$$

$$= D(x)\sum_{i=1}^{n} ia_i \left(\frac{1}{2}\right)^{i-1} = D(x)p'\left(\frac{1}{2}\right) = cp'\left(\frac{1}{2}\right)$$

ýa-da

$$D(p) = cp'\left(\frac{1}{2}\right).$$

b) Geliň, $[0,1]$ kesimde kesgitlenen iki sany funksiýa seredeliň:

$$f_1(x) = \sqrt{x} + \sqrt{1-x}, \quad f_2(x) = \sqrt{x} - \sqrt{1-x};$$

$$f_1 f_2 = x - (1-x) = 2x - 1;$$

$$f_1^2 + f_2^2 = 2 .$$

Ahyrky deňlikde meseläniň şertindäki amaly geçirip alarys:

$$0 = D(2) = D(f_1^2 + f_2^2) = D(f_1^2) + D(f_1^2) =$$

$$= 2D(f_1)f_1\left(\frac{1}{2}\right) + 2D(f_2)f_2\left(\frac{1}{2}\right) = 2\sqrt{2}D(f_1).$$

Diýmek,

$$D(f_1) = 0, \quad D(\sqrt{x} + \sqrt{x-1}) = 0$$

ýa-da

$$D(\sqrt{x}) = -D(\sqrt{1-x});$$

$$D\left(f_1 f_2\right) = D\left(2x-1\right) = 2D(x) = 4\sqrt{2}D\left(\sqrt{x}\right).$$

Emma başga bir tarapdan

$$D(f_1 f_2) = D(f_1) f_2\left(\frac{1}{2}\right) + D(f_2) f_1\left(\frac{1}{2}\right) = D(f_2) f_1\left(\frac{1}{2}\right) = D(f_2)\sqrt{2} =$$

$$= D(\sqrt{x} - \sqrt{x-1})\sqrt{2} = \left(D(\sqrt{x}) - D(\sqrt{1-x})\right)\sqrt{2} = 2\sqrt{2}D(\sqrt{x}).$$

Ahyrky deňliklerden

$$2\sqrt{2}D(\sqrt{x}) = 4\sqrt{2}D(\sqrt{x}),$$

ýa-da $D(\sqrt{x}) = 0,$ bu ýerde hem $D(x) = 0$ deňligi gelip çykýar.

Bu ýerden islendik $x \in [0,1]$ san üçin $D(x) = 0$ bolýar diýen netijäni alarys. Bu netijäni ulanyp islendik $[0,1]$ kesimde kesgitlenen $f(x)$ funksiýa üçin hem $D(f) = 0$ bolýandygyny subut edeliň.

Biziň bilşimiz ýaly, kesimde kesgitlenen üznüksiz funksiýa bu kesimde çäklenendir. Oňa görä-de, aşakdaky amallar ýerine ýetirmek mümkindir.

$$|f(x)| \leq M, \;\; M \geq 0;$$

$$f(x) = \left(\frac{f(x)}{M}\right) M;$$

$$D(f(x)) = \pm M \cdot D\left(\frac{|f(x)|}{M}\right) = M \cdot 0 = 0.$$

Bu ýerde $0 \leq \dfrac{|f(x)|}{M} \leq 1$ bolýanlygy göz öňünde tutuldy. Ahyrky deňlikden $D(f) = 0$ bolýandygy gelip çykýar.

7-nji mysal. Goý, $p(x)$ köpagza diňe hakyky köklere eýe bolsun. Eger $a -$ san $p'(x)$ köpagzanyň kratny köki bolsa, onda $p(a) = 0$ bolýandygyny subut ediň.

Subudy. Goý, $x_1, x_2, ..., x_n$ sanlar $p(x)$ deňlemäniň kökleri we $p(a) \neq 0$ bolsun. Eger $a -$ san $p'(x)$ köpagzanyň kratny köki bolsa, onda a-san $p''(x)$ köpagzanyň hem köki bolar. Öňden mälim bolan,

$$p'(x) = p(x)\left(\frac{1}{x-x_1} + \frac{1}{x-x_2} + ... + \frac{1}{x-x_n}\right);$$

$$p''(x) = p'(x)\left(\frac{1}{x-x_1} + \frac{1}{x-x_2} + ... + \frac{1}{x-x_n}\right) -$$

$$-p(x)\left(\frac{1}{(x-x_1)^2} + \frac{1}{(x-x_2)^2} + ... + \frac{1}{(x-x_n)^2}\right);$$

deňliklerde $x = a$ goýsak,

$$0 = 0 - p(a)\left(\frac{1}{(a-x_1)^2} + \frac{1}{(a-x_2)^2} + ... + \frac{1}{(a-x_n)^2}\right);$$

$$0 = p(a)\left(\frac{1}{(a-x_1)^2} + \frac{1}{(a-x_2)^2} + ... + \frac{1}{(a-x_n)^2}\right)$$

bolar. Eger, $p(a) \neq 0$ bolsa,

$$\frac{1}{(a-x_1)^2} + \frac{1}{(a-x_2)^2} + ... + \frac{1}{(a-x_n)^2} = 0$$

deňlik alnar. Emma, bu deňlik dogry däldir, çünki, deňligiň çep tarapy n sany položitel sanlaryň jeminden ybarat. Diýmek, $p(a) = 0$ bolmaly.

8-nji mysal. Hakyky sanlaryň her bir $a_0, a_1, ..., a_n$ toplumy we islendik $x = x_1$ nokat üçin

$$p^{(s)}(x_0) = a_s, \ (s = 0, 1, ..., n)$$

şertleri kanagatlandyrýan n derejeli köpagzanyň barlygyny subut ediň. Bu köpagzanyň koeffisiýentlerini a_s sanlar arkaly aňladyň.

Çözülişi. Käbir $p(x) = p_1 x^{n-1} + ... + p_{n-1}x + p_n$ köpagza alyp, onuň meseläniň şertini kanagatlandyrmagyny talap edeliň, şunlukda bu köpagzanyň koeffisiýentlerini $a_0, a_1, ..., a_n$ sanlar arkaly taparys. Teýloryň formulasyna görä:

$$p(x) = p(x_0) + \frac{p'(x_0)}{1!}(x - x_0) + ... + \frac{p^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Berlen $p^{(s)}(x_0) = a_s$ şerti peýdalanyp,

$$p(x) = a_0 + \frac{a_1}{1!}(x - x_0) + ... + \frac{a_n}{n!}(x - x_0)^n$$

alarys. Belli bolşy ýaly, $p_s = \frac{p^s_{(0)}}{s!}$. Bu deňlikde $s = 0, 1, ..., n$ bahalary goýup, alarys:

$$p_0 = a_0 + \frac{a_1}{1!}(-x_0) + ... + \frac{a_n}{n!}(-x_0)^n;$$

$$p_1 = a_1 + \frac{a_2}{1!}(-x_0) + ... + \frac{a_n}{(n-1)!}(-x_0)^{n-1};$$

$$p_2 = a_2 + \frac{a_3}{1!}(-x_0) + ... + \frac{a_n}{(n-1)!}(-x_0)^{n-2};$$

....

$$p_n = a_n$$

Diýmek, meseläniň şertini kanagatlandyrýan köpagza bar we ol köpagzanyň koeffisiýentleri ýokardaky getirilip çykarylan deňlikler arkaly tapylýar.

9-njy mysal. $p_1(x), p_2(x), ..., p_n(x)$ derejeleri $(n-1)$ -den uly bolmadyk köpagzalar. Bu köpagzalar üçin Wronskiniň kesgitleýjisiniň hemişelik ululykdygyny subut ediň.

Subudy. Belli bolşy ýaly, $y_1, y_2, ..., y_n$ funksiýalar üçin düzülen

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

kesgitleýjä Wronskiniň kesgitleýjisi diýilýär.

Bu kesgitleýjini $p_1(x), p_2(x), ..., p_n(x)$ köpagzalar üçin ýazalyň:

$$W = \begin{vmatrix} p_1 & p_2 & \cdots & p_n \\ p_1^{'} & p_2^{'} & \cdots & p_n^{'} \\ \vdots & \vdots & \ddots & \vdots \\ p_1^{(n-1)} & p_2^{(n-1)} & \cdots & p_n^{(n-1)} \end{vmatrix}.$$

Goý, $p_i(x) = a_{1i}x^{n-1} + a_{2i}x^{n-2} + ... + a_{ni}$ bolsun. Onda W kesgitleýjiniň elementlerini aşakdaky ýaly tapmak bolar:

$$p_i(x) = a_{1i}x^{n-1} + ... + a_{ni};$$

$$p_i'(x) = \frac{(n-1)!}{1!}a_{1i}x^{n-2} + ... + \frac{1!}{0!}a_{n-1,i};$$

....

$$p_i^{(k)}(x) = \frac{(n-1)!}{k!}a_{1i}x^{k} + ... + \frac{(k+1)!}{1!}a_{k+1,i}x + \frac{k!}{0!}a_{ki};$$

$$p_i^{(k+1)}(x) = \frac{(n-1)!}{(k-1)!}a_{1i}x^{k-1} + ... + \frac{(k+1)!}{0!}a_{k+1,i};$$

....

$$p_i^{(n-3)}(x) = \frac{(n-1)!}{2!}a_{1i}x^{2} + \frac{(n-2)!}{1!}a_{2i}x + \frac{(n-3)!}{0!}a_{3i};$$

$$p_i^{(n-2)}(x) = \frac{(n-1)!}{1!}a_{1i}x + \frac{(n-2)!}{0!}a_{2i};$$

$$p_i^{(n-1)}(x) = \frac{(n-1)!}{0!}a_{1i}.$$

W kesgitleýjininiň setirleriniň üstünde aşakdaky çyzykly özgertmeleri geçireliň:

$$p_i^{(n-1)}(x) = \frac{(n-1)!}{0!}a_{1i} = \Delta_{i1};$$

$$p_i^{(n-2)}(x) - p_i^{(n-1)}(x)\cdot\frac{x}{1} = \frac{(n-2)!}{0!}a_{2i} = \Delta_{i2};$$

$$p_i^{(n-3)}(x) - \Delta_{i1}\cdot\frac{x^2}{2!} - \Delta_{i2}\cdot\frac{x}{1!} = \frac{(n-3)!}{0!}a_{3i} = \Delta_{i3};$$

$$\frac{0!}{0!}a_{ni} = \Delta_{in}.$$

Onda,

$$W = \begin{vmatrix} p_1 & p_2 & \cdots & p_n \\ p_1^{'} & p_2^{'} & \cdots & p_n^{'} \\ \vdots & \vdots & \ddots & \vdots \\ p_1^{(n-1)} & p_2^{(n-1)} & \cdots & p_n^{(n-1)} \end{vmatrix} = \begin{vmatrix} \Delta_{11} & \Delta_{21} & \cdots & \Delta_{n1} \\ \Delta_{12} & \Delta_{22} & \cdots & \Delta_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{1n} & \Delta_{2n} & ... & \Delta_{nn} \end{vmatrix};$$

$$W = \begin{vmatrix} \frac{(n-1)!}{0!}a_{11} & \frac{(n-1)!}{0!}a_{12} & \cdots & \frac{(n-1)!}{0!}a_{1n} \\ \frac{(n-2)!}{0!}a_{21} & \frac{(n-2)!}{0!}a_{22} & \cdots & \frac{(n-2)!}{0!}a_{2n} \\ \vdots & \vdots & & \vdots \\ \frac{(n-n)!}{0!}a_{n1} & \frac{(n-n)!}{0!}a_{n2} & \cdots & \frac{(n-n)!}{0!}a_{nn} \end{vmatrix} = \prod_{k=1}^{n-1} k!\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & ... & a_{nn} \end{vmatrix}$$

bolar.

Diýmek,

$$W = \begin{vmatrix} p_1 & p_2 & \cdots & p_n \\ p_1^{'} & p_2^{'} & \cdots & p_n^{'} \\ \vdots & \vdots & \ddots & \vdots \\ p_1^{(n-1)} & p_2^{(n-1)} & \cdots & p_n^{(n-1)} \end{vmatrix} = \prod_{k=1}^{n-1} k!\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & ... & a_{nn} \end{vmatrix},$$

ýagny, berlen köpagzalar üçin düzülen kesgitleýji hemişelik ululykdyr.

10-njy mysal. Eger $\lim\limits_{x\to\infty}\varphi(x)=\infty$ we $\lim\limits_{x\to\infty}\varphi'(x)=0$ bolsa, $\varphi(x)$ funksiýany iki köpagzanyň gatnaşygy görnüşinde ýazyp bolmaýandygyny subut ediň.

Subudy. Tersine güman edeliň, goý, $\varphi(x)$ – funksiýa iki köpagzanyň gatnaşygyna deň bolsun. Ýagny,

$$\varphi(x)=\frac{P_m(m)}{R_n(x)}.$$

Meseläniň şertinde berlen birinji deňlige görä $m>n$ bolmaly, ýagny, $m=n+k.$ Bu ýerden,

$$\varphi'(x)=\left(\frac{P_m(m)}{R_n(x)}\right)'=\frac{P'_{n+k}(x)R_n(x)-P_{n+k}(x)R'_n(x)}{\left(R_n(x)\right)^2}=\frac{p_{2n+k-1}(x)}{r_{2n}(x)}$$

alarys.

Ahyrky deňlikde predele geçip, $(2n+k-1)\geq 2n$ bolany üçin,

$$\lim_{x\to\infty}\varphi'(x)=\lim_{x\to\infty}\frac{p_{2n+k-1}(x)}{r_{2n}(x)}\neq 0$$

garşylyk alarys. Alnan garşylyk biziň eden gümanymyzyň nädogrudygyny görkezýär.

§2. YZYGIDERLILIKLER WE PREDELLER BILEN BAGLANYŞYKLY MESELELER

1-nji mysal. Eger

$$x_1 = \sqrt{a},\ \ x_2 = \sqrt{a+\sqrt{a}},\ \ x_2 = \sqrt{a+\sqrt{a+\sqrt{a,...}}}$$

bolsa, onda bu yzygiderliligiñ predeleni tapmaly.

Çözülişi. Bu yzygiderlik üçin $x_n = \sqrt{a + x_{n-1}}$, $(n = 2, 3, ...)$ bolýandygyny görmek kyn däldir. Matematiki induksiýa usulyny peýdalanyp (x_n) yzygiderliligiñ monoton ösýändigini we onuñ ýokardan çäklenendigini görkezmek bolar. Oña görä-de (x_n) yzygiderliligiñ tükenikli $l \geq 0$ predeli bardyr.

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{a + x_{n-1}};$$

$$l = \sqrt{a+l},\ \ l^2 - l - a = 0,\ \ l = \frac{\sqrt{4a+1}+1}{2}.$$

Jogaby: $l = \dfrac{\sqrt{4a+1}+1}{2}.$

2-nji mysal. Predeli hasaplaň

$$\lim_{n\to\infty}\left(\cos\frac{x}{2}\cos\frac{x}{4}....\cos\frac{x}{2^n}\right).$$

Çözülişi. Aşakdaky bellenişigi girizeliň

$$\cos\frac{x}{2}\cos\frac{x}{4}...\cos\frac{x}{2^n} = a_n\,.$$

Bu aňlatmany $2^n \sin\dfrac{x}{2^n}$ köpeldip, hem bölüp alalyň we aňlatmany ýönekeýleşdireliň:

$$\cos\frac{x}{2}\cos\frac{x}{4}\ldots.\cos\frac{x}{2^n}\cdot\frac{2^n\sin\frac{x}{2^n}}{2^n\sin\frac{x}{2^n}}=\cos\frac{x}{2}\cos\frac{x}{4}\ldots\cos\frac{x}{2^{n-1}}\cdot\frac{2^{n-1}\sin\frac{x}{2^{n-1}}}{2^n\sin\frac{x}{2^n}}=$$

$$=\ldots=\frac{\sin x}{2^n\sin\frac{x}{2^n}}.$$

Onda bu aňlatmanyň predeli aşakdaky ýaly bolar:

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{\sin x}{2^n\sin\frac{x}{2^n}}=\lim_{n\to\infty}\frac{\sin x}{x\cdot\frac{\sin\frac{x}{2^n}}{\frac{x}{2^n}}}=\frac{\sin x}{x}.$$

Jogaby: $\frac{\sin x}{x}$.

3-nji mysal. Predeli hasaplaň

$$\lim_{n\to\infty}\ \sin^2\left(\pi\sqrt{n^2+n}\right).$$

Çözülişi. Aşakdaky deňlik dogrudyr.

$$\sin^2\left(\pi\sqrt{n^2+n}+\pi n-\pi n\right)=\sin^2\left(\pi\sqrt{n^2+n}-\pi n\right).$$

Bu ýerden alarys:

$$\lim_{n\to\infty}\ \sin^2\left(\pi\sqrt{n^2+n}\right)=\lim_{n\to\infty}\ \sin^2\pi\left(\sqrt{n^2+n}-n\right)=$$

$$=\lim_{n\to\infty}\ \sin^2\pi\,\frac{\left(\sqrt{n^2+n}-n\right)\left(\sqrt{n^2+n}+n\right)}{\sqrt{n^2+n}+n}=\lim_{n\to\infty}\ \sin^2\pi\,\frac{n}{\sqrt{n^2+n}+n}=$$

$$=\lim_{n\to\infty}\ \sin^2\pi\,\frac{1}{\sqrt{1+\frac{1}{n}}+1}=\left(\sin\frac{\pi}{2}\right)^2=1.$$

Jogaby: $\lim_{n\to\infty}\sin^2\left(\pi\sqrt{n^2+n}\right)=1.$

4-nji mysal. $a_0 = 0,\quad a_n = \dfrac{a_{n-1}+3}{4}$ yzygiderliligiň predeliniň barlygyny subut etmeli we ol predeli tapmaly.

Çözülişi. Bu meseläni çözmek üçin yzygiderliligiň ilkinji birnäçe agzalaryny ýazalyň:

$$a_0 = 0,\ a_1 = \frac{3}{4}, a_2 = \frac{\frac{3}{4}+3}{4} = \frac{3}{4^2}+\frac{3}{4}, a_3 = \frac{\frac{3}{4^2}+\frac{3}{4}+3}{4} = \frac{3}{4^3}+\frac{3}{4^2}+\frac{3}{4},$$

$$a_4 = \frac{\frac{3}{4^3}+\frac{3}{4^2}+\frac{3}{4}+3}{4} = \frac{3}{4^4}+\frac{3}{4^3}+\frac{3}{4^2}+\frac{3}{4}, ..., a_n = \frac{3}{4^n}+\frac{3}{4^{n-1}}+...+\frac{3}{4},$$

$$a_n = 3\left(\frac{1}{4^n}+\frac{1}{4^{n-1}}+...+\frac{1}{4}\right) = 3\frac{\frac{1}{4}-\frac{1}{4^{n+1}}}{1-\frac{1}{4}} = 1-\frac{1}{4^n}.$$

Netijede, $\{a_n\}$ yzygiderliligiň predeli

$$\lim_{n\to\infty} a_n = 1-\frac{1}{\lim\limits_{n\to\infty} 4^n} = 1$$

ýaly bolar. Diýmek, yzygiderliligiň predeli bar we ol 1-e deň.

Jogaby: 1.

5-nji mysal. Yzygiderliligiň predelini tapyň.

$$x_n = \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{k}}.$$

Çözülişi.

$$x_n = \frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+...+\frac{1}{\sqrt{(n+1)^2}}.$$

Bu ýerde $(2n+2)$ sany goşulyjy bar. Yzygiderliligiň umumy agzasyny bahalandyryp alarys:

$$\frac{1}{\sqrt{(n+1)^2}}+\frac{1}{\sqrt{(n+1)^2}}+....+\frac{1}{\sqrt{(n+1)^2}} \le x_n \le \frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2}}+....+\frac{1}{\sqrt{n^2}}$$

ýa-da

$$\frac{1}{\sqrt{(n+1)^2}}\left(2n+2\right)\le x_n \le \frac{1}{\sqrt{n^2}}\left(2n+2\right).$$

Bu deňsizlikden $\lim\limits_{n\to\infty}\dfrac{2n+2}{n+1}\le \lim\limits_{n\to\infty} x_n \le \lim\limits_{n\to\infty}\dfrac{2n+2}{n}$,

$$2\le \lim_{n\to\infty} x_n \le 2 \quad \text{ýa-da} \quad \lim_{n\to\infty} x_n = 2 \text{ alarys.}$$

Jogaby: 2.

6-njy mysal. $xy=1$ giperbolada absissalary degişlilikde $\dfrac{n}{n+1}$ we $\dfrac{n+1}{n}$, $(n=1,2,3,...)$ deň bolan A_n, B_n nokatlaryň üstünden we giperbolanyň depesinden geçýän töweregiň merkezi M_n bilen bellenipdir. $n\to\infty$ bolanda nokatlaryň yzygiderliliginiň predelini tapyň.

Çözülişi. Meseläniň şertine görä ýokarda agzalan töwerek $A_n\left(\dfrac{n}{n+1};\dfrac{n+1}{n}\right)$, $B_n\left(\dfrac{n+1}{n};\dfrac{n}{n+1}\right)$ we $C_n(1;1)$ nokatlaryň üstünden geçýär.

Goý, x_n we y_n sanlar M_n nokadyň koordinatlary bolsun, ýagny $M_n(x_n;y_n)$. Meseläniň şertine görä A_n, B_n we C_n nokatlar töweregiň üstünde ýerleşýärler. Şonuň üçin M_n nokatlardan bu nokatlara çenli aralyklar biri-birine deňdirler. Bu aralyklary hasaplalyň.

$$\left|M_nA_n\right|=\sqrt{\left(x_n-\frac{n}{n+1}\right)^2+\left(y_n-\frac{n}{n+1}\right)^2};$$

$$\left|M_nB_n\right|=\sqrt{\left(x_n-\frac{n+1}{n}\right)^2+\left(y_n-\frac{n}{n+1}\right)^2};$$

$$\left|M_nA_n\right|=\sqrt{(x_n-1)^2+(y_n-1)^2}.$$

Bu ululyklary deňläp alarys:

$$\begin{cases} \sqrt{\left(x_n - \frac{n}{n+1}\right)^2 + \left(y_n - \frac{n+1}{n}\right)^2} = \sqrt{\left(x_n - 1\right)^2 + \left(y_n - 1\right)^2}; \\ \sqrt{\left(x_n - \frac{n+1}{n}\right)^2 + \left(y_n - \frac{n}{n+1}\right)^2} = \sqrt{\left(x_n - 1\right)^2 + \left(y_n - 1\right)^2}. \end{cases}$$

Biz x_n, y_n – näbellilere görä deňlemeler ulgamyny aldyk. Bu deňlemeleri ýönekeýleşdirip $y_n = x_n$ deňligi alarys. Bu deňligi göz öňünde tutup, başdaky deňlemeleriň islendik birinden alarys.

$$2x_n\left(1 - \frac{n+1}{n}\right) + 2x_n\left(1 - \frac{n}{n+1}\right) = 2 - \frac{n^2}{\left(n+1\right)^2} - \frac{\left(n+1\right)^2}{n^2}.$$

Bu deňlemeden x_n näbellini tapalyň:

$$x_n = \frac{2n^2 + 2n + \frac{1}{2}}{n^2 + n}.$$

Onda

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{2n^2 + 2n + \frac{1}{2}}{n^2 + n} = 2.$$

Diýmek, $\left\{M_n\left(x_n, y_n\right)\right\}$ nokatlaryň yzygiderliligi $M\left(2, 2\right)$ nokatda ýygnalýar.

Jogaby: $M\left(2,2\right)$.

7-nji mysal. Predeli hasaplamaly

$$\lim_{n\to\infty} x_n = \left(\frac{0!}{k!} + \frac{1!}{(k+1)!} + \ldots + \frac{(n-1)!}{(n+k-1)!}\right).$$

Çözülişi.

$$\frac{(n-1)!}{(n+k-1)!} = \frac{1}{k-1}\left(\frac{(n-1)!}{(n+k-2)!} - \frac{n!}{(n+k-1)!}\right).$$

$$\sum_{i=1}^{n}\frac{(i-1)!}{(i+k-1)!}=\sum_{i=1}^{n}\frac{1}{k-1}\left(\frac{(i-1)!}{(i+k-2)!}-\frac{i!}{(i+k-1)!}\right)=$$

$$=\frac{1}{k-1}\left(\frac{0!}{(k-1)!}-\frac{n!}{(n+k-1)!}\right).$$

Ahyrky deňlikde predele geçip, alarys:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{(i-1)!}{(i+k-1)!}=\lim_{n\to\infty}\left(\frac{1}{(k-1)(k-1)!}+O\left(\frac{1}{n}\right)\right)=\frac{1}{(k-1)(k-1)!}.$$

Jogaby: $\dfrac{1}{(k-1)(k-1)!}$.

8-nji mysal. Predeli tapyň

$$\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+...+\sin\frac{(n-1)\pi}{n}\right).$$

Çözülişi. Bu predeli tapmak üçin integral jem düzmek usulyndan peýdalanmak amatly bolýar.

Geliň, $f(x)=\sin\pi x$ funksiýa $[0,1]$ kesimde seredeliň. Bu funksiýa $[0,1]$ kesimde üznüksiz, onda görkezilen aralykda onuň kesgitli integraly bar. $[0,1]$ kesimi

$$x_0=0,\ \ x_1=\frac{1}{n},\ \ x_2=\frac{2}{n},...,\ x_{n-1}=\frac{n-1}{n},\ \ x_n=1$$

nokatlaryň kömegi bilen özara deň n bölege böleliň we $f(x)=\sin\pi x$ funksiýa üçin integral jemi düzeliň:

$$\sum_{i=1}^{n-1}f(x_i)\Delta x_i=\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+...+\sin\frac{(n-1)\pi}{n}\right),$$

bu ýerde $\Delta x_i=\dfrac{1}{n}$. Bu integral jemiň predeli biziň tapmaly predelimizdir. Oňa görä-de,

$$\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+...+\sin\frac{(n-1)\pi}{n}\right)=\int_0^1\sin\pi x\,dx=\frac{2}{\pi}.$$

Jogaby: $\dfrac{2}{\pi}$.

9-njy mysal. Goý, $f(x)-$funksiýa $[0,1]$ kesimde položitel we üznüksiz bolsun. Onda

$$\lim_{n\to\infty}\sqrt[n]{f\left(\frac{1}{n}\right)f\left(\frac{2}{n}\right)\ldots f\left(\frac{n}{n}\right)}=\exp\left(\int_0^1 \ln f\left(x\right)dx\right)$$

deňligi subut etmeli.

Çözülişi. Aşakdaky bellenişigi girizeliň:

$$a_n=\sqrt[n]{f\left(\frac{1}{n}\right)f\left(\frac{2}{n}\right)\ldots f\left(\frac{n}{n}\right)}.$$

Bu deňligiň iki tarapyny hem logorifmirläp alarys:

$$\ln a_n=\frac{1}{n}\left[\ln f\left(\frac{1}{n}\right)+\ln f\left(\frac{2}{n}\right)+\ldots+\ln f\left(\frac{n}{n}\right)\right].$$

Soňky deňligiň predeli

$$\lim_{n\to\infty}\frac{1}{n}\left[\ln f\left(\frac{1}{n}\right)+\ln f\left(\frac{2}{n}\right)+\ldots+\ln f\left(\frac{n}{n}\right)\right].$$

$F(x)=\ln f(x)$ funksiýa üçin $[0,1]$ kesimde düzülen integral jemdir. Oňa görä-de kesgitli integralyň kesgitlemesinden

$$\lim_{n\to\infty}\ln a_n=\lim_{n\to\infty}\frac{1}{n}\left[\ln f\left(\frac{1}{n}\right)+\ln f\left(\frac{2}{n}\right)+\ldots+\ln f\left(\frac{n}{n}\right)\right]=$$

$$=\lim_{n\to\infty}\sum_{i-1}^{n}F(\xi_i)\Delta x_i=\int_0^1 \ln f\left(x\right)dx$$

bolýandygy düşnüklidir. Onda

$$\lim_{n\to\infty}\ln a_n=\ln\left(\lim_{n\to\infty}a_n\right)=\int_0^1 \ln f\left(x\right)dx$$

ýa-da

$$\lim_{n\to\infty}a_n=\exp\left(\int_0^1 \ln f\left(x\right)dx\right)$$

bolar.

Jogaby: $\exp\left(\int_0^1 \ln f\left(x\right)dx\right).$

10-njy mysal. Goý, $c>0, q>1$ – käbir fiksirlenen sanlar bolsun. $k(p)$ bilen p-niň natural bahalarynda $(k+c)^p \le qk^p$ deňsizligi kanagatlandyrýan k sanlaryň iň kiçisi bellenen. $\lim\limits_{p\to\infty}\frac{k(p)}{p}$ predeliň bardygyny subut etmeli we ony tapmaly.

Çözülişi. $(k+c)^p \le q\cdot k^p$ deňsizligiň iki tarapyndan p görkezijili kök alyp ýönekeýleşdirsek,

$$(k+c) \ \le \sqrt[p]{q}\cdot k \ \text{ýa-da} \ k \ge \frac{c}{\sqrt[p]{q}-1}$$

deňsizligi alarys.

$$\frac{k(p)}{p} = \left[\frac{c}{\sqrt{q^p}-1}\right]\frac{1}{p},$$

$$\left(\frac{c}{\sqrt{q^p}-1}-1\right)\frac{1}{p} \le \frac{k(p)}{p} \le \frac{c}{\sqrt{q^p}-1}\frac{1}{p}$$

$$\left(\sqrt[p]{q}-1=t, p=\frac{\ln q}{\ln(1+t)}\right),$$

$$\lim_{p\to\infty}\frac{c}{(\sqrt[p]{q}-1)p} = \lim_{t\to0}\frac{c}{t\frac{\ln q}{\ln(1+t)}} = \frac{c}{\ln q}\lim_{t\to0}\ln(1+t)^{\frac{1}{t}} = \frac{c}{\ln q}.$$

Jogaby: $\frac{c}{\ln q}$.

11-nji mysal. $M=\left\{\frac{1}{2}\pm\frac{n}{2n+1}\right\}$ $(n\in N)$ köplügiň diňe 0 we 1 predel nokatlarynyň bardygyny subut etmeli.

Çözülişi. Predel nokadyň kesgitlenişine görä, onuň islendik ýeterlikçe kiçi etrapyny bu köplügiň tükeniksiz köp sany elementleri özünde saklaýar.

M köplügiň *A* we *B* bölek köplüklerine seredeliň.

$$A=\left\{a_n=\frac{1}{2}+\frac{n}{2n+1},(n\in N)\right\}\subset M;$$

$$B = \left\{ b_n = \frac{1}{2} - \frac{n}{2n+1}, (n \in N) \right\} \subset M.$$

Bu ýerde

$$\lim_{n\to\infty} a_n = 1 \text{ we } \lim_{n\to\infty} b_n = 0$$

boljakdygy düşnüklidir.

Diýmek, predeliň kesgitlemesine görä

$$\forall \varepsilon > 0, \exists N_1 \in N \quad \forall n > N_1 : 1 - \varepsilon < a_n < 1 + \varepsilon,$$

$$\forall \varepsilon > 0, \exists N_2 \in N \quad \forall n > N_2 : -\varepsilon < b_n < \varepsilon,$$

deňsizlikler dogrudyr.

Eger $\varepsilon < \frac{1}{2}$ bolsa, onda $(1-\varepsilon, 1+\varepsilon)$ we $(-\varepsilon, \varepsilon)$ etraplaryň umumy nokatlary ýokdur.

Diýmek, diňe 1 we 0 nokatlaryň islendik etrabynda M köplügiň tükeniksiz köp elementleri saklanýar. Şonuň üçin M köplügiň 1 we 0 nokatlaryndan başga predel nokady ýokdur.

12-nji mysal. $x = 0$ nokat $x_n = \sqrt{n} \sin n$ yzygiderliliginiň predel nokady bolup bilermi?

Çözülişi. Goý, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{n} \sin n = 0$ bolsun, onda

$$\lim_{n\to\infty} \sqrt{n} = +\infty$$

bolýanlygy üçin $\lim_{n\to\infty} \sin n = 0$ bolmaly. Emma, bu deňlik dogry bolup bilmez, çünki ol dogry bolsa, onda $\lim_{n\to\infty}(\sin(n+2) - \sin n) = 0$, $\lim_{n\to\infty} 2\sin 1 \cos(n+1) = 0$, $\lim_{n\to\infty} \cos(n+1) = 0$ ýa-da $\lim_{n\to\infty} \cos n = 0$ bolýanlygy gelip çykýar. Netijede,

$$1 = \lim_{n\to\infty}(\sin^2 n + \cos^2 n) = \lim_{n\to\infty} \sin^2 n + \lim_{n\to\infty} \cos^2 n = 0$$

nädogry deňlik alnar. Alnan garşylyk $x=0$ nokadyň $x_n=\sqrt{n}\sin n$ yzygiderliligiň predel nokady bolup bilmeýändigini görkezýär.

Jogaby: bolup bilmez.

13-nji mysal. Goý, $x_1>0$, $x_{n+1}=\frac{1}{2}\left(x_n+\frac{1}{x_n}\right)$, $n\geq 1$ rekurrent formula bilen $\{x_n\}$ san yzygiderliligi berlen bolsun. Bu san yzygiderliginiň ýygnalýandygyny subut ediň we predelini tapyň.

Çözülişi. Orta arifmetik we orta geometrik baha baradaky deňsizlikden peýdalanyp, alarys:

$$x_{n+1}=\frac{x_n+\frac{1}{x_n}}{2}\geq\sqrt{x_n\cdot\frac{1}{x_n}}=1,\ \text{ýagny}\ \forall n\in N: x_{n+1}\geq 1.$$

Bu bolsa $\{x_n\}$ san yzygiderliliginiň aşakdan çäklenenligini aňladýar. Bu deňsizlikden

$$\forall n\in \mathrm{N}: \frac{1}{x_n}\leq 1 \Rightarrow \frac{1}{x_n}\leq x_n$$

bolýanlygy gelip çykýar. Onda

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{1}{x_n}\right)\leq\frac{1}{2}\left(x_n+x_n\right)=x_n;\qquad x_{n+1}\leq x_n$$

bolar. Diýmek, $\{x_n\}$ san yzygiderliligi artmaýar we aşagyndan çäklenen. Oňa görä-de bu yzygiderliligiň $\lim\limits_{n\to\infty}x_n=a$ predeli bardyr. Ýagny,

$$\lim_{n\to\infty}x_{n+1}=\frac{1}{2}\lim_{n\to\infty}\left(x_n+\frac{1}{x_n}\right)$$

ýa-da

$$a=\frac{1}{2}\left(a+\frac{1}{a}\right).$$

Bu ýerden bolsa $a=1$ deňligi alarys. Diýmek, $\lim\limits_{n\to\infty}x_n=1.$

Jogaby: 1.

14-nji mysal. San bahasyny tapyň:

$$\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{17}{16}\cdots\frac{2^{2^n}+1}{2^{2^n}}\cdots.$$

Çözülişi. Bu tükeniksiz köpeltmek hasylyň san bahasyny A – bilen belläliň:

$$A=\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{17}{16}\cdots\frac{2^{2^n}+1}{2^{2^n}}\cdots.$$

Aşakdaky bellenişigi geçireliň

$$P_n=\prod_{k=0}^{n}\frac{2^{2^n}+1}{2^{2^n}}=\prod_{k=0}^{n}\left[1+\left(\frac{1}{2}\right)^{2^k}\right]=$$

$$=\left[1+\frac{1}{2}\right]\cdot\left[1+\left(\frac{1}{2}\right)^2\right]\cdot\left[1+\left(\frac{1}{2}\right)^4\right]\cdots\cdot\left[1+\left(\frac{1}{2}\right)^{2^n}\right]=$$

$$=\frac{\left[1-\frac{1}{2}\right]\cdot\left[1+\frac{1}{2}\right]\cdot\left[1+\left(\frac{1}{2}\right)^2\right]\cdot\left[1+\left(\frac{1}{2}\right)^4\right]\cdots\cdot\left[1+\left(\frac{1}{2}\right)^{2^n}\right]}{\left(1-\frac{1}{2}\right)}=$$

$$=\frac{\left[1-\left(\frac{1}{2}\right)^2\right]\cdot\left[1+\left(\frac{1}{2}\right)^2\right]\cdot\left[1+\left(\frac{1}{2}\right)^4\right]\cdots\cdot\left[1+\left(\frac{1}{2}\right)^{2^n}\right]}{\frac{1}{2}}=$$

$$=2\left[1-\left(\frac{1}{2}\right)^4\right]\cdot\left[1+\left(\frac{1}{2}\right)^4\right]\cdots\cdot\left[1+\left(\frac{1}{2}\right)^{2^n}\right]=\cdots=2\left[1-\left(\frac{1}{2}\right)^{2^{n+1}}\right];$$

$$A=\lim_{n\to\infty}P_n=2\lim_{n\to\infty}\left[1-\left(\frac{1}{2}\right)^{2^{n+1}}\right]=2\,.$$

Jogaby: 2.

15-nji mysal. Goý, $x_{n+1} = \frac{1}{3}\left(2x_n + \frac{a}{x_n^2}\right)$ $\left(a > 0, x_0 > 0\right)$ bolsun. Bu san yzygiderliliginiň predeliniň bardygyny subut ediň we ol predeli tapyň.

Çözülişi. Orta arifmetik we orta geometrik baha baradaky deňsizlikden peýdalanyp, alarys:

$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{a}{x_n^2}\right) = \frac{x_n + x_n + \frac{a}{x_n^2}}{3} \geq \sqrt[3]{x_n \cdot x_n \cdot \frac{a}{x_n^2}} \geq \sqrt[3]{a}.$$

Matematiki induksiýa usuly arkaly $\forall n \in \mathrm{N}.$ üçin $x_{n+1} \leq x_n$ boýandygyny subut etmek bolar. Diýmek, $\{x_n\}$ san yzygiderliligi aşagyndan çäklenen artmaýan yzygiderlilikdir. Şonuň üçin bu san yzygiderliliginiň predeli bardyr, ýagny $\lim\limits_{n\to\infty} x_{n+1} = B$ predel bardyr. Bu predeli tapalyň:

$$B = \frac{1}{3}\left(2B + \frac{a}{B^2}\right) \text{ ýa-da } B^3 = a .$$

Diýmek, $\lim\limits_{n\to\infty} x_n = \sqrt[3]{a}.$

Jogaby: $\sqrt[3]{a}.$

16-njy mysal. Goý, $a > b > 0$ bolsun. Bu sanlar bilen aşakdaky ýaly san yzygiderlilikleri kesgitlenen,

$$a_1 = \frac{a+b}{2}, b_1 = \sqrt{a \cdot b};$$

$$a_2 = \frac{a_1 + b_1}{2}, b_2 = \sqrt{a_1 \cdot b_1};$$

$$a_{n+1} = \frac{a_n + b_n}{2}, b_{n+1} = \sqrt{a_n \cdot b_n}.$$

$\{a_n\}, \{b_n\}$ san yzygiderlikleriniň predelleriniň bardygyny we özara deňdigini subut ediň.

Çözülişi. $a > b > 0$ bolany üçin orta arifmetik we orta geometrik baha baradaky deňsizlikden alarys:

$$a_1 = \frac{a+b}{2} \geq \sqrt{a \cdot b} = b_1 \text{ ýa-da } a_1 \geq b_1 > 0;$$

$$a_2 = \frac{a_1 + b_1}{2} \geq \sqrt{a_1 \cdot b_1} = b_2 \text{ ýa-da } a_2 \geq b_2 > 0;$$

....

$$a_n \geq b_n > 0, \ \forall n \in N.$$

Diýmek, $\{a_n\}, \{b_n\}$ san yzygiderlilikleriň her biri aşagyndan çäklenen. Aşakdaky deňsizlikler dogrudyr:

$$a_2 = \frac{a_1 + b_1}{2} \leq \frac{a_1 + a_1}{2} = a_1;$$

$$a_3 = \frac{a_2 + b_2}{2} \leq \frac{a_2 + a_2}{2} = a_2;$$

....

$$a_{n+1} = \frac{a_n + b_n}{2} \leq \frac{a_n + a_n}{2} = a_n.$$

Diýmek, $\forall n \in N: \ a_{n+1} \leq a_n$. Oňa görä-de $\{a_n\}$ san yzygiderliligi artmaýan yzygiderlilikdir. Şonuň üçin hem $\lim\limits_{n \to \infty} a_n = \alpha$ tükenikli predel bardyr. Ýokardaky usuldan peýdalanyp, aşakdaky deňsizlikleri alarys:

$$b_2 = \sqrt{a_1 \cdot b_1} \geq \sqrt{b_1 \cdot b_1} = b_1;$$

$$b_3 = \sqrt{a_2 \cdot b_2} \geq \sqrt{b_2 \cdot b_2} = b_2;$$

....

$$b_{n+1} = \sqrt{a_n \cdot b_n} \geq \sqrt{b_n \cdot b_n} = b_n.$$

Diýmek, $\forall n \in N: b_{n+1} \geq b_n$ deňsizlik dogrudyr. Oňa görä-de $\{b_n\}$ san yzygiderliligi kemelmeýär:

$$a_1 \geq a_2 \geq .... \geq a_n \geq ...$$

$$b_1 \leq b_2 \leq .... \leq b_n \leq ...$$

deňsizliklerden $\forall n \in N : a_1 \geq b_n$ deňsizlik gelip çykýar. Diýmek, $\{b_n\}$ yzygiderlilik ýokardan çäklenendir. Şonuň üçin hem $\lim\limits_{n\to\infty} b_n = \beta$ predel bardyr. $a_{n+1} = \dfrac{a_n + b_n}{2}$ deňlikde $n \to \infty$ predele geçsek, onda $2a = a + b$ ýa-da $a = b$ deňlik alynýar.

17-nji mysal. Eger $\lim\limits_{n\to\infty} \dfrac{n^{1975}}{n^x - (n-1)^x} = \dfrac{1}{1976}$ bolsa, onda x sany tapyň.

Çözülişi. $|a| < 1$ bolanda

$$(1+a)^x = 1 + xa + \frac{x(x-1)}{2!}a^2 + ... + \frac{x(x-1)...(x-n+1)}{n!}a^n + O(a^{n+1})$$

formula dogrudyr. Ony peýdalanyp, alarys:

$$\frac{n^{1975}}{n^x - (n-1)^x} = \frac{n^{1975-x}}{1 - \left(1 - \frac{1}{n}\right)^x} =$$

$$= \frac{n^{1975-x}}{1 - \left(1 - \frac{x}{n} + ... + \frac{x(x-1)...(x-n+1)}{k!n^k}\left(-\frac{1}{n}\right)^k + O\left(\frac{1}{n^{x+1}}\right)\right)} =$$

$$= \frac{n^{1975-x}}{\frac{x}{n} + O\left(\frac{1}{n^{x+1}}\right)} = \frac{n^{1976-x}}{x + O\left(\frac{1}{n}\right)}.$$

Ahyrky netijäni peýdalanyp, alarys:

$$\lim_{n\to\infty} \frac{n^{1975}}{n^x - (n-1)^x} = \lim_{n\to\infty} \frac{n^{1976-x}}{x + O\left(\frac{1}{n}\right)} = \begin{cases} 0, \text{eger } x>1976 \text{ bolsa;} \\ \dfrac{1}{1976}, \text{eger } x = 1976 \text{ bolsa;} \\ \infty, \text{eger } x<1976 \text{ bolsa.} \end{cases}$$

bolar. Diýmek, mysalyň berlişindäki şertiň kanagatlanmagy üçin $x = 1976$ bolmaly.

Jogaby: $x = 1976$.

Bellik. Bu meseläni umumylaşdyrmak hem bolar. Eger

$$\lim_{n\to\infty}\frac{n^{k-1}}{n^x-(n-1)^x}=\frac{1}{k}$$

predel bar bolsa, onda $\forall k\in N$ üçin $x=k$ bolar.

18-nji mysal. $\{x_n\}$ yzygiderlik $x_0=\frac{1}{3}$, $x_n=0,5x_{n-1}^2-1$ şertler bilen berlipdir. Aşakdaky predeli tapmaly:

$$\lim_{n\to\infty}x_n.$$

Çözülişi. Matematiki induksiýa usulyndan peýdalanyp, aşakdaky deňsizlikleriň dogrudygyny görkezmek bolar:

$$\forall k\in N:\ x_{2k-2}>1-\sqrt{3};\ x_{2k-1}<1-\sqrt{3};$$

$$\forall k\in N:\ x_{2n-2}>x_{2n-4}>...>x_{2k-2}>1-\sqrt{3};$$

$$x_{2n-1}<x_{2n-3}<...<x_{2k-1}<1-\sqrt{3}.$$

Görnüşi ýaly käbir belgiden başlap, bu yzygiderliligiň jübüt indeksli agzalary kemelýär we aşagyndan çäklenen, täk indeksli agzalary bolsa artýar we ýokarsyndan çäklenen. Diýmek, bu yzygiderliligiň predeli bar. Goý, ol predel A bolsun, onda şertde berlen deňlikde predele geçip, alarys:

$$x_n=0,5x_{n-1}^2-1;$$

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}(0,5x_{n-1}^2-1);$$

$$A=0,5A^2-1\ \text{ýa-da}\ A=1-\sqrt{3}.$$

Jogaby: $1-\sqrt{3}$.

19-njy mysal. Goý, $x_1,x_2,...-$ sanlar $\mathrm{tg}x=x$ deňlemäniň artýan tertipde ýerleşdirilen ähli položitel kökleri bolsun. $\lim\limits_{n\to\infty}(x_n-x_{n-1})$. predeli tapmaly.

Çözülişi. $\operatorname{tg} x$ funksiýanyň periodik funksiýadygyny göz öňünde tutup, onuň ähli položitel çözüwleriniň

$$\pi(i-1) \le x_i \le \frac{\pi}{2} + \pi(i-1)$$

deňsizligi kanagatlandyrýandygyny görmek kyn däldir. Bu deňsizligi peýdalanyp,

$$x_n - x_{n-1} \ge \pi(n-1) - \frac{\pi}{2} + \pi(n-2) = \frac{\pi}{2};$$

$$x_n - x_{n-1} \le \frac{\pi}{2} + \pi(n-1) - \pi(n-2) = \frac{3\pi}{2};$$

$$\frac{\pi}{2} \le x_n - x_{n-1} \le \frac{3\pi}{2}$$

alarys. Diýmek, $\{x_n - x_{n-1}\}$ yzygiderlik çäklenendir. Ikinji bir tarapdan

$$\operatorname{tg}(x_n - x_{n-1}) = \frac{x_n - x_{n-1}}{1 + x_n x_{n-1}}.$$

Onda bu deňlikde predele geçip alarys:

$$\lim_{n\to\infty} \operatorname{tg}(x_n - x_{n-1}) = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{1 + x_n x_{n-1}};$$

$$\lim_{n\to\infty} \operatorname{tg}(x_n - x_{n-1}) = \lim_{n\to\infty} \frac{1}{1 + x_n x_{n-1}} \lim_{n\to\infty}(x_n - x_{n-1}) = 0.$$

Diýmek, $\lim\limits_{n\to\infty} \operatorname{tg}(x_n - x_{n-1}) = 0.$ Bu ýerden hem $\lim\limits_{n\to\infty}(x_n - x_{n-1}) = \pi$ deňlik gelip çykýar.

Jogaby: π.

20-nji mysal. $x = \cos x$ deňlemäniň ýeke-täk x_0 köke eýedigini we $x_1 = 20,$ $x_n = \cos x_{n-1}$ ýaly kesgitlenen $\{x_n\}$ yzygiderliligiň predeliniň x_0 nokada ýygnanýandygyny subut ediň.

Çözülişi. $y = x - \cos x$ funksiýa $[0,1]$ kesimde seredeliň. $y(0)<0$ we $y(1)>0$ deňsizliklerden, seredilýän kesimde funksiýa

Ox okuny kesýär. Şol kesişme nokatlaryň birini x_0 bilen belläliň. $y' = 1 + \sin x > 0$ bolýanlygy üçin bu funksiýa monoton artýar. Ýagny,

$$x \neq x_0: \; y(x) \neq y(x_0) = 0$$

bolar. Bu ýerden hem $x = \cos x$ deňlemäniň ýeke-täk köküniň barlygy gelip çykýar.

Ýokardaky yzygiderliligi aşakdaky formada ýazalyň:

$$x_n = \cos x_{n-1} = \cos\cos\ldots\cos x_1.$$

Belli bolşy ýaly, $[0,1]$ aralykda $y = \cos x$ funksiýa kemelýär. Käbir belgiden başlap berlen yzygiderliligiň agzalary bu aralyga düşüp başlaýar. Netijede, yzygiderliligiň täk indeksli agzalary kemelýär, jübüt indeksli agzalary bolsa artýar. Bu yzygiderliligiň çäklenendigini göz öňünde tutup, onuň predeliniň barlygyny aýtmak bolar. $x_n = \cos x_{n-1}$ deňlikde predele geçip,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \cos x_{n-1}$$

$$A = \cos A, \; A = x_0$$

bolýandygyny alarys.

21-nji mysal. $a_n = n\alpha - [n\alpha], \; \alpha \in Q$ yzygiderliligiň ähli predel nokatlaryny tapmaly.

Çözülişi. Belli bolşy ýaly, islendik rasional sany $\alpha = \dfrac{m}{k}$ görnüşde ýazmak mümkin. Bu ýerde $m \in Z, k \in N$ we $IUUB(m,k) = 1$. Başga bir tarapdan $n = kl + r \; (0 \le r \le k-1)$. Oňa görä-de

$$a_n = na - [na] = \frac{(kl+r)m}{k} - \left[\frac{(kl+r)m}{k}\right] =$$

$$= ml + \frac{m}{k}r - \left[ml + \frac{m}{k}r\right] = ml + \frac{m}{k}r - ml - \left[\frac{m}{k}r\right] = \frac{m}{k}r - \left[\frac{m}{k}r\right].$$

Diýmek,

$$a_n = a_{kl+r} = \frac{m}{k}r - \left[\frac{m}{k}r\right]$$

ýa-da

$$a_n = \begin{cases} 0, \quad \text{eger} \quad r = 0 \quad \text{bolsa;} \\ \frac{m}{k} - \left[\frac{m}{k}\right], \text{ eger} \quad r = 1 \quad \text{bolsa;} \\ \vdots \\ \frac{m(k-1)}{k} - \left[\frac{m(k-1)}{k}\right], \text{ eger} \quad r = k-1 \quad \text{bolsa.} \end{cases}$$

Ahyrky alnan netijeden görnüşi ýaly, berlen yzygiderliligiň predel nokatlary

$$0, \ \frac{m}{k} - \left[\frac{m}{k}\right], \ \frac{2m}{k} - \left[\frac{2m}{k}\right], \dots, \ \frac{(k-1)m}{k} - \left[\frac{(k-1)m}{k}\right]$$

bolar.

Jogaby: $\frac{m}{k}r - \left[\frac{m}{k}r\right], \ r = \overline{0, k-1}.$

§3. INTEGRIRLEMEK AMALY BILEN BAGLANYŞYKLY MESELELER

1-nji mysal. Sanlaryň haýsysy uly

$$\int_0^{\pi} e^{\sin^2 x}dx \text{ ýa-da } \frac{2\pi}{2} \text{ ?}$$

Çözülişi. Mälim bolşy ýaly:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\cdots,$$

$$e^{\sin^2 x} = 1 + \sin^2 x + \frac{\sin^4 x}{2!} + \cdots + \frac{\sin^{2n} x}{n!} + \cdots.$$

Bu ýerden $e^{\sin^2 x} > 1 + \sin^2 x$ boljakdygy düşnüklidir. Soňky deňsizligi [0, π] aralykda integrirläp alarys:

$$\int_0^{\pi} e^{\sin^2 x}dx > \int_0^{\pi}\left(1 + \sin^2 x\right)dx = \int_0^{\pi}\frac{3 - \cos 2x}{2}dx = \left(\frac{3}{2}x - \frac{1}{4}\sin 2x\right)\Bigg|_0^{\pi} = \frac{3\pi}{2}.$$

Diýmek,

$$\int_0^{\pi} e^{\sin^2 x}dx > \frac{3\pi}{2}.$$

2-nji mysal. Eger baglanyşyksyz üýtgeýän ululygy $\xi = \int_0^x f(t)dt$ bilen çalşanda $e^{-\xi}$ funksiýa geçýändigi belli bolsa, onda $[0;+\infty)$ aralykda položitel we differensirlenýän $f(x)$ funksiýany tapyň.

Çözülişi. Şerte görä $e^{-\int_0^x f(t)dt} = f(x)$ ýa-da $\int_0^x f(x)dt = -\ln f(x)$.

Soňky deňligi differensirläp alarys:

$$\left(\int_0^x f(t)dt\right)' = \left(-\ln f(x)\right)'; \quad f(x) = -\frac{f'(x)}{f(x)}; \quad f'(x) = -\left(f(x)\right)^2.$$

Ahyrky deńligi integrirläliň:

$$\frac{df(x)}{dx} = -(f(x))^2; \; -\frac{df(x)}{(f(x))^2} = dx; -\int \frac{df(x)}{(f(x))^2} = x + c;$$

$$\frac{1}{f(x)} = x + c; \; f(x) = \frac{1}{x+c}.$$

Emma $f(0) = e^0 = 1$ bolýandygy sebäpli $c = 1$, onda $f(x) = \dfrac{1}{x+1}$.

Jogaby: $f(x) = \dfrac{1}{x+1}$.

3-nji mysal. $\int (-1)^{[x]} dx$ integraly tapmaly.

Çözülişi. Goý, $x > 0$ we $x \in (n, n+1)$ bolsun, onda

$$F(x) = \int_0^x (-1)^{[t]} dt = \begin{cases} -x - n, & \text{eger } \; n - \text{jübüt} \; \text{ bolsa,} \\ x + n, & \text{eger } \; n - \text{täk} \; \text{ bolsa.} \end{cases}$$

Goý, indi $x<0$ we $x \in (-n-1, -n)$ bolsun, onda

$$F(x) = \int_0^x (-1)^{[t]} dt = \begin{cases} -x - n, & \text{eger } \; n - \text{jübüt} \; \text{ bolsa,} \\ x + n + 1, & \text{eger } \; n - \text{täk} \; \text{ bolsa.} \end{cases}$$

Umumy ýagdaýda bolsa

$$F(x) = \pm x + 2k, \quad (k = 0, \pm 1, \pm 2, ...).$$

Diýmek,

$$\int (-1)^{[x]} dx = \pm x + 2n + c, \quad (n = 0, \pm 1, \pm 2, ...).$$

Jogaby: $\pm x + 2n + c, \; n \in Z$.

4-nji mysal. Eger $f(x)$ funksiýa üznüksiz we A-nyň ähli bahalarynda $\int\limits_A^\infty \frac{f(x)}{x}dx$ integral ýygnanýan bolsa, aşakdaky deňligi subut ediň

$$\int\limits_0^\infty \frac{f(ax)-f(bx)}{x}dx = f(0)\ln\frac{b}{a}; \quad (a>0, b>0).$$

Çözülişi. Goý, $F(x)=\int\limits_A^\infty \frac{f(x)}{x}dx$ bolsun we $ax=t$ bellenişik girizip, alarys:

$$\int\limits_A^\infty \frac{f(ax)}{x}dx = \int\limits_{aA}^\infty \frac{f(t)}{t}dt = F(+\infty)-F(aA).$$

Edil şeýle usul bilen $bx=t$ bilen bellesek,

$$\int\limits_A^\infty \frac{f(bx)}{x}dx = \int\limits_{bA}^\infty \frac{f(t)}{t}dt = F(+\infty)-F(bA)$$

bolar.

$$\int\limits_0^\infty \frac{f(ax)-f(bx)}{x}dx = F(bA)-F(aA) = \int\limits_{aA}^{bA} \frac{f(x)}{x}dx \int\limits_{aA}^{bA} f(x)d(\ln x).$$

Soňky integrala orta baha baradaky teoremany ulanyp, alarys:

$$\int\limits_{Aa}^{bA} f(x)d(\ln x) = f(\xi)\int\limits_{aA}^{bA} d(\ln x) = f(\xi)\ln\frac{b}{a}.$$

Bu ýerde $aA<\xi<bA$ we A nola ymtylsa, onda ξ hem nola ymtylýar. Onda,

$$\int\limits_0^\infty \frac{f(ax)-f(bx)}{x}dx = f(0)\ln\frac{b}{a}.$$

5-nji mysal. Eger $f(x)$ funksiýa $[0,1]$ kesimde üznüksiz bolsa, onda,

$$\int\limits_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int\limits_0^\pi f(\sin x)dx$$

deňligi subut etmeli.

Çözülişi. $\pi - x = t$ bellenişik girizip alarys:

$$\int_0^{\pi} xf(\sin x)dx = \int_{\pi}^{0} (\pi - t) f(\sin(\pi - t)) d(\pi - t) = \pi \int_0^{\pi} f(\sin t)dt - \int_0^{\pi} tf(\sin t)dt;$$

$$\int_0^{\pi} xf(\sin x)dx = \pi \int_0^{\pi} f(\sin x)dx - \int_0^{\pi} xf(\sin x)dx;$$

$$2\int_0^{\pi} xf\left(\sin x\right)dx = \pi \int_0^{\pi} f(\sin x)dx;$$

$$\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x)dx.$$

6-njy mysal. Subut ediň:

$$\int_0^{2\pi} xf(\cos x)\, dx = \pi \int_0^{2\pi} f(\cos x)dx.$$

Çözülişi. $x = 2\pi - t$ bellenişik girizip, alarys:

$$\int_0^{2\pi} xf(\cos x)\, dx = \int_{2\pi}^{0} \left(2\pi - t\right) f\left(\cos\left(2\pi - t\right) d\left(2\pi - t\right)\right) =$$

$$= 2\pi \int_0^{2\pi} f(\cos t)\, dt - \int_0^{2\pi} tf\left(\cos t\right) dt;$$

$$\int_0^{2\pi} xf(\cos x)\, dx = 2\pi \int_0^{2\pi} f\left(\cos x\right) dx - \int_0^{2\pi} xf(\cos x)\, dx;$$

$$2\int_0^{2\pi} xf(\cos x)\, dx = 2\pi \int_0^{2\pi} f\left(\cos x\right) dx;$$

$$\int_0^{2\pi} xf(\cos x)\, dx = \pi \int_0^{2\pi} f\left(\cos x\right) dx.$$

7-nji mysal. Integraly hasaplaň:

$$\int_0^{2\pi} \sin\left(\sin x + nx\right)dx.$$

Çözülişi. Üýtgeýänleri çalşyp, integraly hasaplalyň:

$$\left.\begin{array}{r} x=\pi+t \\ dx=dt \end{array}\right| \begin{array}{ll} x=0 & \quad t=-\pi, \\ x=2\pi & \quad t=\pi. \end{array}$$

$$\int_0^{2\pi} \sin(\sin x + nx)dx = \int_{-\pi}^{\pi} \sin\left[\sin(\pi+t) + n(\pi+t)\right]dt =$$

$$= \int_{-\pi}^{\pi} \sin(-\sin t + nt + n\pi)dt = \int_{-\pi}^{\pi} (-1)^n \sin(nt - \sin t)dt = 0.$$

Jogaby: 0.

8-nji mysal. Aňlatmanyň α bagly däldigini subut ediň:

$$\int_0^{\infty} \frac{dx}{\left(1+x^2\right)\left(1+x^{\alpha}\right)}.$$

Çözülişi. Integralyň häsiýetinden peýdalanyp, berlen integraly aşakdaky görnüşde ýazyp bileris:

$$\int_0^{\infty} \frac{dx}{(1+x^2)(1+x^{\alpha})} = \int_0^{1} \frac{dx}{(1+x^2)(1+x^{\alpha})} + \int_1^{\infty} \frac{dx}{(1+x^2)(1+x^{\alpha})} = I_1 + I_2.$$

I_1 integralda $x = \frac{1}{y}$ üýtgeýäni çalşyryp, alarys:

$$x=1,\ \ y=1,\ \ x=\infty,\ y=0 \text{ we } dx = -\frac{dy}{y^2};$$

$$I_1 = -\int_{\infty}^{1} \frac{\frac{dy}{y^2}}{\left(\frac{1+y^2}{y^2}\right)\left(\frac{1+y^{\alpha}}{y^{\alpha}}\right)} = -\int_{\infty}^{1} \frac{y^{\alpha}dy}{\left(1+y^2\right)\left(1+y^{\alpha}\right)} = \int_1^{\infty} \frac{x^{\alpha}dx}{(1+x^2)(1+x^{\alpha})},$$

$$I_2 = \int_1^{\infty} \frac{dx}{(1+x^2)(1+x^{\alpha})} \text{ we } I_1 = \int_1^{\infty} \frac{x^{\alpha}dx}{(1+x^2)(1+x^{\alpha})},$$

integrallary goşup, alarys:

$$I_1 + I_2 = \int_1^{\infty} \frac{dx}{(1+x^2)(1+x^{\alpha})} + \int_1^{\infty} \frac{x^{\alpha}dx}{(1+x^2)(1+x^{\alpha})} = \int_1^{\infty} \frac{(1+x^{\alpha})dx}{(1+x^2)(1+x^{\alpha})} = \int_1^{\infty} \frac{dx}{1+x^2}.$$

Hakykatdan-da, alnan integral α bagly däldir:

$$\int_1^{\infty}\frac{dx}{1+x^2}=\operatorname{arctg}x\Big|_1^{\infty}=\frac{\pi}{4}.$$

9-njy mysal. Berlen funksiýanyň asyl funksiýasyny tapyň.

$$y=\frac{x^2}{(x\sin x+\cos x)^2}.$$

Çözülişi. Gözlenýän funksiýany Y-bilen belläliň, onda

$$Y=\int y dx=\int\frac{x^2}{(x\sin x+\cos x)^2}dx=\int\frac{x^2\left(\cos^2 x+\sin^2 x\right)}{(x\sin x+\cos x)^2}dx=$$

$$=\int\frac{x^2\cos^2 x-x\sin x\cos x+x\sin x\cos x+x^2\sin^2 x}{(x\sin x+\cos x)^2}dx=$$

$$=\int\frac{x\sin x(x\sin x+\cos x)-x\cos x(\sin x-x\cos x)}{(x\sin x+\cos x)^2}dx=$$

$$=\int\frac{(\cos x+x\sin x-\cos x)(x\sin x+\cos x)-(\sin x+x\cos x-\sin x)(\sin x-x\cos x)}{(x\sin x+\cos x)^2}dx=$$

$$=\int\frac{(\sin x-x\cos x)'(x\sin x+\cos x)-(x\sin x+\cos x)'(\sin x-x\cos x)}{(x\sin x+\cos x)^2}dx=$$

$$=\int d\left[\frac{\sin x-x\cos x}{x\sin x+\cos x}\right]=\frac{\sin x-x\cos x}{x\sin x+\cos x}+c;$$

$$Y=\frac{\sin x-x\cos x}{x\sin x+\cos x}+c.$$

Jogaby: $Y=\dfrac{\sin x-x\cos x}{x\sin x+\cos x}+c.$

10-njy mysal. Hasaplaň

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{x^7-3x^5+7x^3-x+1}{\cos^2 x}dx.$$

Çözülişi. Bilşimiz ýaly $f(-x)=-f(x)$ şert ýerine ýetse,

$$\int_{-a}^{a} f(x)dx = 0$$

bolar. Onda:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^7 - 3x^5 + 7x^3 - x + 1}{\cos^2 x} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^7 - 3x^5 + 7x^3 - x}{\cos^2 x} dx + + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\cos^2 x} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\cos^2 x} dx = 2.$$

Jogaby: 2.

11-nji mysal. $x = 2\cos\varphi$ **,** $y = \sin\varphi$ ellipsiň deňlemesi berlipdir. Bu ýerde $a = 2, b = 1,$ $\rho^2 = x^2 + y^2$. Talyp onuň meýdanyny hasaplap,

$$S = 4\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \rho^2 d\varphi\right) = 2\cdot\int_0^{\frac{\pi}{2}} \left(4\cos^2\varphi + \sin^2\varphi\right)d\varphi = \frac{5\pi}{2}.$$

ýaly netijäni alýar. Emma bu ellipsiň meýdany $S = \pi \cdot a \cdot b = 2\pi$ bolmaly. Ýalňyşlyk nirede?

Çözülişi. $x = \rho\cos\varphi$ we $y = \rho\sin\varphi;$

$$\rho^2 = \frac{4}{\cos^2\varphi + 4\sin^2\varphi}, \quad \rho = \frac{2}{\sqrt{\cos^2\varphi + 4\sin^2\varphi}};$$

$$S = 4\left(\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}} \frac{4}{\cos^2\varphi + 4\sin^2\varphi} d\varphi\right) = 8\int_0^{\frac{\pi}{2}} \frac{d\varphi}{1 + 3\sin^2\varphi}.$$

Soňky integraly integrirlemek üçin t = tgφ bellenişigi geçireliň, onda

$$\sin^2\varphi = \frac{t^2}{1+t^2}, \varphi = \mathrm{arctg}\varphi, d\varphi = \frac{dt}{1+t^2}$$

bolar.

$$8\int_0^{\frac{\pi}{2}} \frac{d\varphi}{1 + 3\sin^2\varphi} \int_0^{\infty} \frac{\frac{dt}{1+t^2}}{1 + 3\frac{t^2}{1+t^2}} = 8\cdot\int_0^{\infty} \frac{dt}{1+4t^2}$$

bellenişik geçirip, alarys:

$$8\int_{0}^{\infty}\frac{dt}{1+4t^2}=4\cdot(\text{arctg}\infty-\text{arctg}0)=4\cdot\frac{\pi}{2}=2\pi.$$

Jogaby: 2π.

12-nji mysal. Integraly hasaplamaly:

$$\int_{-1}^{1}\frac{dx}{(e^x+1)(x^2+1)}.$$

Çözülişi.

$$\int_{-1}^{1}\frac{dx}{(e^x+1)(x^2+1)}=\int_{-1}^{0}\frac{dx}{(e^x+1)(x^2+1)}+\int_{0}^{1}\frac{dx}{(e^x+1)(x^2+1)};$$

$\int_{-1}^{0}\frac{dx}{(e^x+1)(x^2+1)}$ integralda x üýtgeýäni -x bilen çalşyp, alarys:

$$\int_{-1}^{1}\frac{dx}{(e^x+1)(x^2+1)}=\int_{1}^{0}\frac{-dx}{(e^{-x}+1)(x^2+1)}+\int_{0}^{1}\frac{dx}{(e^x+1)(x^2+1)}=$$

$$=\int_{0}^{1}\frac{e^x dx}{(e^x+1)(x^2+1)}+\int_{0}^{1}\frac{dx}{(e^x+1)(x^2+1)}=$$

$$=\int_{0}^{1}\frac{(e^x+1)dx}{(e^x+1)(x^2+1)}=\int_{0}^{1}\frac{dx}{x^2+1}=\text{arctg}x\Big|_{0}^{1}=\frac{\pi}{4}.$$

Jogaby: $\frac{\pi}{4}$.

13-nji mysal. Integraly hasaplamaly

$$\int_{0}^{3}\text{sgn}(x-x^3)dx.$$

Çözülişi. $\text{sgn}\{t\}=t-t^3$ funksiýanyň kesgitlemesine görä

$$\operatorname{sgn}(x-x^3)=\begin{cases}1, \text{eger} \quad 0<x<1 \quad \text{bolsa},\\ 0, \text{eger} \quad x=0, \quad x=1 \quad \text{bolsa},\\ -1, \text{eger} \quad 1<x<3 \quad \text{bolsa}.\end{cases}$$

Onda gözlenýän integraly aşakdaky ýaly ýazyp bolar:

$$\int_0^3 \operatorname{sgn}(x-x^3)dx=\int_0^1 dx-\int_1^3 dx=1-2=-1.$$

Jogaby: –1.

14-nji mysal.

$$y=e^{x^2}\int_0^x e^{-t^2}dt$$

funksiýanyň monoton artýandygyny subut ediň. Bu funksiýa haýsy başlangyç şertlerde nähili differensial deňlemäni kanagatlandyrýar.

Çözülişi. Görnüşi ýaly,

$$y=e^{x^2}\int_0^x e^{-t^2}dt\ge 0.$$

Bu funksiýanyň önümini alalyň we onuň noldan uludygyny görkezeliň.

$$y'=\left(e^{x^2}\int_0^x e^{-t^2}dt\right)'=2xe^{x^2}\int_0^x e^{-t^2}dt+e^{x^2}(-e^{x^2})=2xy+1$$

ýa-da

$$y'=2xy+1>0.$$

Diýmek, berlen funksiýa monoton artýar. Ahyrky deňlikden görnüşi ýaly

$$y=e^{x^2}\int_0^x e^{-t^2}dt$$

funksiýa $y(0)=0$ başlangyç şertde $y'=2xy+1$ deňlemäni kanagatlandyrýar.

15-nji mysal. Eger $f(x)$ we $g(x)$ üznüksiz funksiýalar $[0,1]$ aralykda bilelikde artýan ýa-da kemelýän bolsalar, onda

$$\int_0^x f(x)g(x)dx \geq \int_0^x f(x)dx \int_0^x g(x)dx$$

deňsizligi subut ediň.

Subudy. Ilki bilen aşakdaky kömekçi tassyklamany subut edeliň.

Lemma. *$\{a_n\}$ we $\{b_n\}$ yzygiderlilikler bilelikde artýan ýa-da kemelýän bolsalar, onda aşakdaky deňsizlik ýerine ýeter:*

$$n\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i.$$

Subudy. *Şerte görä $\{a_n\}$ we $\{b_n\}$ yzygiderlilikler bilelikde monoton artýar ýa-da kemelýär. Onda aşakdaky deňsizlik dogrudyr:*

$$(a_i - a_j)(b_i - b_j) \geq 0.$$

Bu deňsizligiň üstünde käbir özgertmeler geçirip, alarys:

$$a_i b_i - a_i b_j - a_j b_i + a_j b_j \geq 0;$$

$$\sum_{i,j=1}^{n} (a_i b_i - a_i b_j - a_j b_i + a_j b_j) \geq 0;$$

$$n \cdot \sum_{i=1}^{n} a_i b_i - \sum_{i,j=1}^{n} a_i b_j - \sum_{i,j=1}^{n} a_j b_i + n\sum_{j=1}^{n} a_j b_j \geq 0;$$

$$2n\sum_{i=1}^{n} a_i b_i \geq \sum_{i,j=1}^{n} a_i b_j + \sum_{i,j=1}^{n} a_j b_i;$$

$$2n\sum_{i=1}^{n} a_i b_i \geq 2\sum_{i,j=1}^{n} a_i b_j;$$

$$n\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i.$$

Indi bu lemmany ulanyp, berlen deňsizligi subut edeliň. Ilki bilen aşakdaky ýaly belleniňişikleri geçireliň:

$$a_i = f(\theta_i),\ b_j = g(\theta_i),\ n = \frac{1}{\Delta x_i}.$$

Bellenişikleri soňky deňsizlikde hasaba alyp,

$$\frac{1}{\Delta x_i}\sum_{i=1}^{n} f(\theta_i)g(\theta_i) \geq \sum_{i=1}^{n} f(\theta_i)\sum_{i=1}^{n} g(\theta_i)$$

alarys. Ahyrky deňsizlikde käbir özgertmeleri geçip, bu deňsizliklerde predele geçeliň:

$$\lim_{n\to+\infty}\sum_{i=1}^{n} f(\theta_i)g(\theta_i)\Delta x_i \geq \lim_{n\to+\infty}\sum_{i=1}^{n} f(\theta_i)\Delta x_i\sum_{i=1}^{n} g(\theta_i)\Delta x_i;$$

$$\lim_{n\to+\infty}\sum_{i=1}^{n} f(\theta_i)g(\theta_i)\Delta x_i \geq \lim_{n\to+\infty}\sum_{i=1}^{n} f(\theta_i)\Delta x_i \cdot \lim_{n\to+\infty}\sum_{i=1}^{n} g(\theta_i)\Delta x_i.$$

Kesgitli integralyň kesgitlemesine laýyklykda ahyrky deňsizligi aşakdaky görnüşde ýazyp bileris:

$$\int_0^x f(x)g(x)dx \geq \int_0^x f(x)dx\int_0^x g(x)dx.$$

16-njy mysal. $f(x)$ funksiýa $[0,1]$ aralykda üznüksiz differensirlenýän we $f(1)-f(0)=1$ şerti kanagatlandyrýan bolsa, aşakdaky deňsizligi subut ediň:

$$\int_0^1 (f'(x))^2dx \geq 1.$$

Subudy. $a_1, a_2, ..., a_n$ sanlar üçin aşakdaky deňsizligiň dogrudygyny matematiki induksiýa usuly arkaly subut etmek bolar:

$$(a_1 + a_2 + ... + a_n)^2 \leq n(a_1^2 + a_2^2 + ... + a_n^2).$$

Bu deňsizlikde $a_i = f'(\theta_i),\ n = \dfrac{1}{\Delta x_i}$ bellenişikleri geçirip, alarys:

$$\left(\sum_{i=1}^{n} f'(\theta_i)\right)^2 \leq \frac{1}{\Delta x_i}\sum_{i=1}^{n}(f'(\theta_i))^2;$$

$$\lim_{n\to+\infty}\left(\sum_{i=1}^{n} f'(\theta_i)\Delta x_i\right)^2 \le \lim_{n\to+\infty}\sum_{i=1}^{n}(f'(\theta_i))^2\Delta x_i;$$

$$\left(\lim_{n\to+\infty}\sum_{i=1}^{n} f'(\theta_i)\Delta x_i\right)^2 \le \lim_{n\to+\infty}\sum_{i=1}^{n}(f'(\theta_i))^2\Delta x_i.$$

Ahyrky deňsizlikde kesgitli integralyň kesgitlemesinden peýdalanyp, ony

$$\int_0^1 (f'(x))^2dx \ge \left(\int_0^1 f'(x)dx\right)^2 = f(x)\Big|_0^1 = f(1)-f(0)=1;$$

ýaly ýazmak bolar. Diýmek,

$$\int_0^1 (f'(x))^2dx \ge 1.$$

17-nji mysal. Eger $f(x)$ funksiýa $[1,+\infty]$ aralykda üznüksiz we

$$\int_1^{\infty} xf(x)dx;$$

integral ýygnanýan bolsa, onda

$$\int_1^{\infty} f(x)dx$$

integralyň hem ýygnanýandygyny subut ediň.

Subudy. Goý, $F(x)$ funksiýa $f(x)$ funksiýanyň asyl funksiýasy bolsun. Onda Teýloryň formulasyna görä, alarys:

$$F(x)=F(1)+F'(\theta(x-1)+1)(x-1),\ \ (0<\theta<1);$$

$$F(x)-F(1)=f\left[\theta(x-1)+1\right](x-1);$$

$$\int_1^x f(x)dx=\int_1^x f\left[\theta(x-1)+1\right](x-1)dx.$$

Ahyrky deňlikde $\theta(x-1)+1=t$ bellenişigi geçirip, alnan deňlikde predele geçeliň,

$$\int_1^{\infty} f(x)dx = \lim_{x\to+\infty}\int_1^{x} f(x)dx = \lim_{x\to+\infty}\int_1^{t} \frac{f(t)(t-1)}{\theta^2}dt;$$

$$(1+\theta^2)\int_1^{\infty} f(x)dx = \int_1^{\infty} f(t)tdt;$$

$$\int_1^{\infty} f(x)dx = \frac{1}{(1+\theta^2)}\int_1^{\infty} f(x)xdx.$$

deňligi alarys. Şerte görä $\int_1^{\infty} f(x)xdx$ integral ýygnanýar, onda ahyrky deňlige görä $\int_1^{\infty} f(x)dx$ integral hem ýygnanýar.

§4. HATARLAR BILEN BAGLANYŞYKLY MESELELER

1-nji mysal. Hasaplaň

$$\sum_{m=0}^{\infty}\left(\prod_{k=0}^{3}\ln\left(xe^{k+m}\right)\right)^{-1}.$$

Çözülişi.

$$\sum_{m=0}^{\infty}\left(\prod_{k=0}^{3}\ln\left(xe^{k+m}\right)\right)^{-1}=\sum_{m=0}^{\infty}\left(\ln\left(xe^{m}\right)\cdot\ln\left(xe^{1+m}\right)\cdot\ln\left(xe^{2+m}\right)\right)^{-1}=$$

$$=\sum_{m=0}^{\infty}\frac{1}{\left(\ln x+\ln e^{m}\right)\cdot\left(\ln x+\ln e^{1+m}\right)\cdot\left(\ln x+\ln e^{2+m}\right)}=$$

$$=\sum_{m=0}^{\infty}\frac{1}{\left(\ln x+m\right)\cdot\left(\ln x+m+1\right)\cdot\left(\ln x+m+2\right)}.$$

$\ln x+m=t$ bellenişik geçirip alarys, $m=0$ bolanda $t=\ln x$ bolar.

$$\sum_{m=0}^{\infty}\left(\prod_{k=0}^{3}\ln\left(xe^{k+m}\right)\right)^{-1}=\sum_{t=\ln x}^{\infty}\frac{1}{t\left(t+1\right)\left(t+2\right)}=$$

$$=\sum_{t=\ln x}^{\infty}\frac{1}{\left(t+1\right)}\left(\frac{1}{t}-\frac{1}{t+2}\right)\frac{1}{2}==\frac{1}{2}\sum_{t=\ln x}^{\infty}\left(\frac{1}{t\left(t+1\right)}-\frac{1}{\left(t+1\right)\left(t+2\right)}\right)=$$

$$=\frac{1}{2}(\frac{1}{\ln x\left(\ln x+1\right)}-\frac{1}{\left(\ln x+1\right)\left(\ln x+2\right)}+\frac{1}{\left(\ln x+1\right)\left(\ln x+2\right)}-$$

$$-\frac{1}{(\ln x+2)(\ln x+3)}+\frac{1}{(\ln x+2)(\ln x+3)}-\cdots)=\frac{1}{2}\frac{1}{\ln x(\ln x+1)}.$$

Jogaby: $\frac{1}{2\ln x(\ln x+1)}.$

2-nji mysal. Toždestwany subut ediň

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(n+1)\dots(n+k)}=\int_0^1\frac{e^x-1}{x}dx.$$

Çözülişi. Subut edilmeli deňligiň çep we sag tarapyny özgerdip, alarys.

$$\left|\sqrt[n]{n}-1\right|<\sqrt{\frac{2}{n-1}}<\varepsilon,$$

$$+\frac{1}{\sqrt{4n-1}}-\frac{1}{\sqrt{2n}}+\dots.$$

$$=\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{1}{1\cdot 2\cdot 3\cdot\ldots\cdot k}-\frac{1}{2\cdot 3\cdot\ldots\cdot(k+1)}+\frac{1}{2\cdot 3\cdot\ldots\cdot(k+1)}-\dots\right)=\sum_{k=1}^{\infty}\frac{1}{k\cdot k!}$$

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\dots+\frac{x^k}{k!}+\dots$$

bolýandygyny göz öňünde tutup, alarys:

$$\int_0^1\frac{e^x-1}{x}dx=\int_0^1\frac{1}{x}\sum_{k=1}^{\infty}\frac{x^k}{k!}dx=\sum_{k=1}^{\infty}\frac{1}{k!}\int_0^1\frac{x^k}{x}dx=\sum_{k=1}^{\infty}\frac{1}{k!}\int_0^1 x^{k-1}dx=$$

$$=\sum_{k=1}^{\infty}\frac{1}{k!}\cdot\frac{x^k}{k}\bigg|_0^1=\sum_{k=1}^{\infty}\frac{1}{k\cdot k!}.$$

3-nji mysal. Hataryň jemini tapyň

$$\sum_{n=1}^{\infty}\frac{2n-1}{2^n}.$$

Çözülişi. Şeýle funksiýa seredeliň

$$f(x)\ln 2=\sum_{n=1}^{\infty}2^{-nx}=\frac{1}{2^x-1};$$

$$f'(x)\ln 2=\ln 2\sum_{n=1}^{\infty}\frac{-n}{2^{nx}}=\frac{-2^x\ln 2}{\left(2^x-1\right)^2}.$$

Alnan deňligiň agzalaryny $-\ln 2$ sana gysgaldyp, alarys:

$$\sum_{n=1}^{\infty}\frac{n}{2^{nx}}=\frac{2^x}{\left(2^x-1\right)^2}.$$

$x=1$ bolanda, alarys

$$\sum_{n=1}^{\infty}\frac{n}{2^n}=\frac{2}{\left(2-1\right)^2}=2;$$

$$\sum_{n=1}^{\infty}\frac{2n-1}{2^n}=\sum_{n=1}^{\infty}\frac{2n}{2^n}+\sum_{n=1}^{\infty}\frac{1}{2^n}=2\cdot 2-\frac{\frac{1}{2}}{1-\frac{1}{2}}=4-1=3.$$

Jogaby: 3.

4-nji mysal. $\sum_{k=1}^{\infty}\frac{\left(2x-x^2\right)^k-2x^k}{k}$ köpagzanyň x^{n+1} bölünýändigini subut ediň.

Çözülişi. Belli bolşy ýaly,

$$-\ln(1-x)=\sum_{k=1}^{n}\frac{x^k}{k}+O\left(x^{n+1}\right);$$

$$-\ln(1-x)^2=\sum_{k=1}^{n}\frac{2x^k}{k}+O\left(x^{n+1}\right).$$

Ýöne,

$$-\ln(1-x)^2=-\ln\left(1-\left(2x-x^2\right)\right)=\sum_{k=1}^{n}\left(\frac{\left(2x-x^2\right)^k}{k}+O\left(\left(2x-x^2\right)^k\right)\right)=$$

$$=\sum_{k=1}^{n}\frac{\left(2x-x^2\right)^k}{k}+O\left(x^{n+1}\right).$$

Onda:

$$\sum_{k=1}^{n}\frac{2x^k}{k}=\sum_{k=1}^{n}\frac{\left(2x-x^2\right)^k}{k}+O\left(x^{n+1}\right);$$

$$\sum_{k=1}^{n}\frac{\left(2x-x^2\right)^k-2x^k}{k}=O\left(x^{n+1}\right).$$

Soňky deňligiň sag tarapy x^{n+1} bölünýär, onda bu deňligiň sag tarapy hem x^{n+1} bölünýär.

5-nji mysal. Hasaplaň

$$\frac{1+\dfrac{\pi^4}{2^4\cdot 4!}+\dfrac{\pi^8}{2^8\cdot 8!}+\dfrac{\pi^{12}}{2^{12}\cdot 12!}+....}{\dfrac{1}{2!}+\dfrac{\pi^4}{2^4\cdot 6!}+\dfrac{\pi^8}{2^8\cdot 10!}+\dfrac{\pi^{12}}{2^{12}\cdot 14!}}.$$

Çözülişi.

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\ldots$$

deňlikde $x=\dfrac{\pi}{2}$ bahany goýup, alarys:

$$\cos\frac{\pi}{2}=1-\frac{\pi^2}{2^2\cdot 2!}+\frac{\pi^4}{2^4\cdot 4!}-\frac{\pi^6}{2^6\cdot 6!}+\frac{\pi^8}{2^8\cdot 8!}-\frac{\pi^{10}}{2^{10}\cdot 10!}+\ldots$$

$$1-\frac{\pi^2}{2^2\cdot 2!}+\frac{\pi^4}{2^4\cdot 4!}-\frac{\pi^6}{2^6\cdot 6!}+\frac{\pi^8}{2^8\cdot 8!}-\frac{\pi x^{10}}{2^{10}\cdot 10!}+....=0;$$

$$1+\frac{\pi^4}{2^4\cdot 4!}+\frac{\pi^8}{2^8\cdot 8!}+\frac{\pi^{12}}{2^{12}\cdot 12!}+....=\frac{\pi^2}{2^2\cdot 2!}+\frac{\pi^6}{2^6\cdot 6!}+\frac{\pi^{10}}{2^{10}\cdot 10!}+...$$

$$1+\frac{\pi^4}{2^4\cdot 4!}+\frac{\pi^8}{2^8\cdot 8!}+\frac{\pi^{12}}{2^{12}\cdot 12!}+....=\frac{\pi^2}{2^2}\left(\frac{1}{2!}+\frac{\pi^4}{2^4\cdot 6!}+\frac{\pi^8}{2^8\cdot 10!}+\frac{\pi^{12}}{2^{12}\cdot 12!}+...\right).$$

Onda bu ýerden,

$$\frac{1+\frac{\pi^4}{2^4\cdot 4!}+\frac{\pi^8}{2^8\cdot 8!}+\frac{\pi^{12}}{2^{12}\cdot 12!}+....}{\frac{1}{2!}+\frac{\pi^4}{2^4\cdot 6!}+\frac{\pi^8}{2^8\cdot 10!}+\frac{\pi^{12}}{2^{12}\cdot 14!}}=\frac{\pi^2}{2^2}=\frac{\pi^2}{4}$$

boljakdygy gelip çykýar.

Jogaby: $\frac{\pi^2}{4}$.

6-njy mysal. Egerde $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ bolýandygy belli bolsa $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ hataryň jemini tapyň.

Çözülişi.

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+...=$$

$$=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...+\frac{1}{(2\cdot 1)^2}+\frac{1}{(2\cdot 2)^2}+\frac{1}{(2\cdot 3)^2}+....=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}+\frac{1}{2^2}\sum_{n=1}^{\infty}\frac{1}{n^2};$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}+\frac{1}{2^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

meseläniň şertine görä $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, onda

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}+\frac{1}{4}\cdot\frac{\pi^2}{6};$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{6}-\frac{1}{4}\cdot\frac{\pi^2}{6}=\frac{\pi^2}{8}.$$

Jogaby: $\frac{\pi^2}{8}$.

7-nji mysal. Hataryň jemini tapyň:

$$1-3x^2+5x^4-7x^6+\ldots+(-1)^n(2n+1)x^{2n}+\ldots,\ (|x|<1).$$

Çözülişi. Hataryň jemini S bilen belläliň we aşakdaky tükeniksiz kemelýän geometrik progressiýanyň jemine seredeliň:

$$x-x^3+x^5-x^7+\ldots+(-1)^n x^{2n+1}+\ldots=\frac{x}{1+x^2}.$$

Bu hataryň iki tarapyny hem differensirläp, alarys:

$$(x-x^3+x^5-x^7+\ldots+(-1)^n x^{2n+1}+\ldots)'=\left(\frac{x}{1+x^2}\right)'$$

$$S=\frac{1-x^2}{(1+x^2)^2}.$$

Jogaby: $\frac{1-x^2}{(1+x^2)^2}$.

8-nji mysal. $\sum_{n=1}^{\infty}\frac{1}{n\sqrt[n]{n}}$ hataryň dargaýandygyny subut etmeli.

Subudy. Bu hatary dargaýan $\sum_{n=1}^{\infty}\frac{1}{n}$ garmoniki hatar bilen deňeşdirýäris. Deňeşdirme nyşanyna göra:

$$\lim_{n\to\infty}\frac{a_n}{b_n}=k,\ (0<k<+\infty)$$

predel bar bolsa, onda $\sum_{n=1}^{\infty}a_n$ we $\sum_{n=1}^{\infty}b_n$ hatarlar şol bir wagtyň özünde ýygnanýar ýa-da dargaýar. Onda

$$\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n\sqrt[n]{n}}}=\lim_{n\to\infty}\sqrt[n]{n}=1$$

bolar. Indi $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$ deňligi subut etmek ýeterlikdir.

Goý, $\varepsilon>0$ islendik san alalyň we $\left|\sqrt[n]{n}-1\right|<\varepsilon$ deňsizlik $n>n_0(\varepsilon)$ belgiden başlap ähli n üçin ýerine ýeter ýaly $n_0(\varepsilon)=n_0\in N$ sanyň bardygyny görkezeliň. Hakykatdan hem

$$n=[1+\sqrt[n]{n}-1]^n=[1+(\sqrt[n]{n}-1)]^n=1+n(\sqrt[n]{n}-1)+\frac{n(n-1)}{2!}(\sqrt[n]{n}-1)^2+$$

$$+\ldots+\frac{n(n-1)(n-2)}{3!}(\sqrt[n]{n}-1)^3+\ldots++(\sqrt[n]{n}-1)^n>\frac{n(n-1)}{2}(\sqrt[n]{n}-1)^2.$$

bolýandygy düşnüklidir. Indi

$$n>\frac{n(n-1)}{2}(\sqrt[n]{n}-1)^2$$

deňsizligi çözüp, $n_0=n_0(\varepsilon)$ belgini kesgitläliň.

$$1>\frac{n-1}{2}(\sqrt[n]{n}-1)^2;$$

$$\left|\sqrt[n]{n}-1\right|<\sqrt{\frac{2}{n-1}}<\varepsilon;$$

$$n>1+\frac{2}{\varepsilon^2},\quad n_0=n_0(\varepsilon)=[1+\frac{2}{\varepsilon^2}].$$

Bu ýerde n_0 belgä derek $1+2\varepsilon^{-2}$ sanyň bitin bölegini almak ýeterlikdir. Diýmek, berlen hatar dargaýar.

9-njy mysal. $\sum\limits_{n=1}^{\infty}\frac{1}{\ln n!}$ hataryň ýygnanmaklygyny derňemeli.

Çözülişi. Islendik $n\geq 2$ üçin $n!<n^n$, oňa görä-de

$$\ln n! < \ln n^n = n\ln n, \quad \ln n! < n\ln n, \quad \frac{1}{n\ln n} < \frac{1}{\ln n!}.$$

Emma, $\sum_{n=1}^{\infty}\frac{1}{n\ln n}$ hatar dargaýar. Munuň şeýledigini görkezmek üçin Koşiniň integral nyşanyny ulanmak ýeterlikdir, ýagny,

$$\int_2^{+\infty}\frac{dx}{x\ln x} = \int_2^{\infty}\frac{d(\ln x)}{\ln x} = \ln(\ln x)\Big|_2^{+\infty} = +\infty.$$

Dargaýan hataryň degişli agzalaryndan uly bolup durýanlygy zerarly, berlen hatar hem dargaýar.

Jogaby: hatar dargaýar.

10-njy mysal. Umumy agzasy $a_n = \frac{(-1)^{n+1}}{\sqrt{n}}$ ýygnanýan hataryň agzalarynyň ornuny çalyşmak bilen dargaýan hatar almaly**.**

Çözülişi. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ hatar Leýbnisiň nyşanyna görä ýygnanýar. Bu hataryň iki položitel agzasyndan soň bir otrisatel alamatly agzasyny, soňra ýene-de iki položitel alamatly agzalaryndan soň bir otrisatel alamatly agzasyny ýerleşdireliň we şu prosesi dowam edip, aşakdaky ýaly hatary alarys:

$$1+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{7}}-\frac{1}{\sqrt{4}}+...+\frac{1}{\sqrt{4n-3}}+\frac{1}{\sqrt{4n-1}}-\frac{1}{\sqrt{2n}}+...$$

Bu hataryň dargaýandygyny subut edeliň. Onuň umumy agzasyny

$$b_n = \frac{1}{\sqrt{4n-3}}+\frac{1}{\sqrt{4n-1}}-\frac{1}{\sqrt{2n}}$$

diýsek, onda

$$b_n = \frac{1}{\sqrt{4n-3}}+\frac{1}{\sqrt{4n-1}}-\frac{1}{\sqrt{2n}} > \frac{1}{\sqrt{4n-1}}+\frac{1}{\sqrt{4n-1}}-\frac{1}{\sqrt{2n}} >$$

$$> \frac{2}{\sqrt{4n-1}} - \frac{1}{\sqrt{2n}} > \frac{2}{\sqrt{4n}} - \frac{1}{\sqrt{2n}} = (1 - \frac{1}{\sqrt{2}}) \frac{1}{\sqrt{n}} = a_n$$

boljakdygy düşnüklidir. Emma $\sum_{n=1}^{\infty} (1 - \frac{1}{\sqrt{2}}) \frac{1}{\sqrt{n}}$ hatar dargaýar, onda degişli san hataryň degişli agzalaryndan uly agzalary bolan hatar hem dargaýar.

Jogaby: $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (\frac{1}{\sqrt{4n-3}} + \frac{1}{\sqrt{4n-1}} - \frac{1}{\sqrt{2n}}).$

11-nji mysal. Umumy agzasy nola ymtylýan, emma özi dargaýan bolan, alamaty gezekleşýän hatara mysal getiriň.

Çözülişi. Meselem,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \ln \frac{n}{n+1} = \ln \frac{1}{2} - \ln \frac{2}{3} + \ln \frac{3}{4} - \ln \frac{4}{5} + \ldots$$

hataryň umumy agzasy $a_n = \ln \frac{n}{n+1}$ nola ymtylýar, emma muňa garamazdan, bu hatar dargaýar. Çünki, ol Leýbnisiň nyşanyny kanagatlandyrmaýar, ýagny,

$$\ln \frac{1}{2} < \ln \frac{2}{3} < \ln \frac{3}{4} < \ln \frac{4}{5} < \ldots$$

Jogaby: $\sum_{n=1}^{\infty} (-1)^{n-1} \ln \frac{n}{n+1}.$

12-nji mysal.

$\sum_{n=1}^{\infty} a_n \ln n$ hatar dargar ýaly, ýygnanýan $\sum_{n=1}^{\infty} a_n$ hatara mysal getiriň.

Çözülişi. Koşiniň integral nyşanyna görä,

$$\int_1^{\infty} \frac{1}{(x+1)\ln^2(x+1)} dx = -\frac{1}{\ln(x+1)} \Bigg|_1^{\infty} = \frac{1}{\ln 2};$$

$\sum\limits_{n=1}^{\infty} \dfrac{1}{(n+1)\ln^2(n+1)}$ hatar ýygnanýar. Emma $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{(n+1)\ln^2(n+1)}$ hatar welin dargaýar. Sebäbi,

$$\sum_{n=1}^{\infty}\frac{\ln n}{(n+1)\ln^2(n+1)} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\ln n^2}{(n+1)\ln^2(n+1)} >$$

$$> \frac{1}{2}\sum_{n=1}^{\infty} \frac{\ln(n+1)}{(n+1)\ln^2(n+1)} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(n+1)\ln(n+1)};$$

soňky alnan hatar Koşiniň integral nyşanyna görä dargaýar. Diýmek, mysalyň şertini

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln^2(n+1)}$$

hatar kanagatlandyrýar.

Jogaby: $\sum\limits_{n=1}^{\infty} \dfrac{1}{(n+1)\ln^2(n+1)}$.

13-nji mysal.

$\sum\limits_{n=1}^{\infty} a_n^{2k+1}, k = 1, 2, 3, \ldots$ hatar dargar ýaly, ýygnanýan $\sum\limits_{n=1}^{\infty} a_n$ hatar barmy?

Çözülişi. Şeýle bir hatar düzeliň:

$$1 = 1 + \left(\frac{1}{\sqrt[2k+1]{2}} - \frac{1}{2\cdot\sqrt[2k+1]{2}} - \frac{1}{2\cdot\sqrt[2k+1]{2}} \right) +$$

$$+ \left(\frac{1}{\sqrt[2k+1]{3}} - \frac{1}{3\cdot\sqrt[2k+1]{3}} - \frac{1}{3\cdot\sqrt[2k+1]{3}} - \frac{1}{3\cdot\sqrt[2k+1]{3}} \right) + \ldots$$

Görnüşi ýaly, bu hatar ýygnanýar we onuň jemi 1-e deň. Geliň, indi onuň her bir agzasynyň $(2k+1)$ derejä göterip, aşakdaky hatary düzeliň:

$$1 + \frac{1}{2} - \frac{1}{2^{2k+2}} - \frac{1}{2^{2k+2}} + \frac{1}{3} - \frac{1}{3^{2k+2}} - \frac{1}{3^{2k+2}} - \frac{1}{3^{2k+2}} + \ldots$$

Bu hatary iki hataryň tapawudy görnüşinde ýazalyň.

$$\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots\right)-\left(\frac{1}{2^{2k+1}}+\frac{1}{3^{2k+1}}+\frac{1}{4^{2k+1}}+\ldots\right).$$

Emma bu jemdäki birinji hatar dargaýan, ikinji hatar bolsa ýygnanýan hatar. Bu hatarlaryň tapawudy hem ýene-de dargaýan hatar bolýar. Diýmek, düzülen hatar meseläniň şertini kanagatlandyrýar.

Jogaby: bar.

14-nji mysal. Hataryň ýygnanmaklygyny derňemeli:

$$\sum_{n=1}^{\infty}\left(1-\cos\frac{\pi}{n}\right).$$

Çözülişi. Belli bolşy ýaly, $\forall x \in \left[0,\frac{\pi}{2}\right]: \sin x \le x.$ Bu deňligi göz öňünde tutup,

$$\sum_{n=1}^{\infty}\left(1-\cos\frac{\pi}{n}\right)=\sum_{n=1}^{\infty}2\sin^2\frac{\pi}{2n}\le\sum_{n=1}^{\infty}\frac{\pi^2}{2n^2}$$

alarys. Ahyrky deňsizligiň sag tarapyndaky hatar, mälim bolşy ýaly, ýygnanýan umumylaşdyrylan garmon iki hatar. Diýmek, deňeşdirme nyşanyna görä berlen hatar hem ýygnanýar.

15-nji mysal. $x = tg\sqrt{x}$ deňlemäniň ähli položitel köklerini artýan tertipde ýerleşdirip $\{x_n\}$ yzygiderlilik alnypdyr, $\sum_{n=1}^{\infty}\frac{1}{x_n}$ hataryň ýygnanmaklygyny derňemeli.

Çözülişi. $y_1 = x$ we $y_2 = tg\sqrt{x}$ funksiýalaryň položitel kesişme nokatlary y_1 funksiýanyň periodikligine görä aşakdaky şerti kanagatlandyrar:

$$x_n \in \left((\pi n)^2;\left(\frac{\pi}{2}+\pi n\right)^2\right).$$

Ahyrky şerti ulanyp,

$$\frac{1}{x_n}\le\frac{1}{(\pi n)^2},\sum_{n=1}^{\infty}\frac{1}{x_n}\le\sum_{n=1}^{\infty}\frac{1}{(\pi n)^2}$$

deňsizligi alarys. Ahyrky deňsizligiň sag tarapyndaky hatar ýygnanýan umumylaşdyrylan garmoniki hatar bolanlygy üçin, deňeşdirme nyşanyna görä berlen hatar hem ýygnanýar.

16-njy mysal. Goý, $\sum_{n=1}^{\infty} a_n$ hatar dargaýan bolsun, $a_n > 0$, $S_n = a_1 + a_2 + ... + a_n$. $\sum_{n=1}^{\infty} \frac{a_n}{S_n}$ hataryň hem dargaýandygyny subut ediň.

Çözülişi. Eger $\lim_{n\to\infty} \frac{a_n}{S_n} \neq 0$ bolsa, onda $\sum_{n=1}^{\infty} \frac{a_n}{S_n}$ hatar dargaýan hatar bolar. Goý, $\lim_{n\to\infty} \frac{a_n}{S_n} \neq 0$ bolsun, ýagny,

$$\frac{a_n}{S_n} \approx O(n), \text{ haçanda } n \to \infty .$$

Şerte görä $\sum_{n=1}^{\infty} a_n$ hatar dargaýar, onda Dalamberiň nyşanyna görä $\frac{a_n + 1}{a_n} \sim 1 + \alpha(n)$, haçanda $n \to \infty$, bu ýerde $\alpha(n) \geq 0$.

Bu şertleri ulanyp, $\sum_{n=1}^{\infty} \frac{a_n}{S_n}$ hataryň Dalamberiň nyşanyna görä dargaýandygyny görkezmek bolar:

$$\lim_{n\to\infty} \frac{\frac{a_{n+1}}{S_{n+1}}}{\frac{a_n}{S_n}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \frac{S_n}{S_{n+1}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_{n+1}} \frac{S_n}{S_n + a_{n+1}} =$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \cdot \frac{1}{1 + \frac{a_{n+1}}{S_n}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \cdot \frac{1}{1 + \frac{a_n}{S_n} \cdot \frac{a_{n+1}}{a_n}} =$$

$$\lim_{n\to\infty} \left(1 + \alpha(n)\right)) \frac{1}{1 + O(n)(1 + \alpha(n))} = \lim_{n\to\infty} \frac{1}{(1 + \alpha(n))^{-1} + O(n)} =$$

$$\lim_{n\to\infty}\frac{1}{(1+\alpha(n))^{-1}}=1+\lim_{n\to\infty}\alpha(n)>1.$$

Diýmek, $\sum\limits_{n=1}^{\infty}\frac{a_n}{S_n}$ hatar dargaýar.

17-nji mysal. Hataryň ýygnanmaklygyny derňemeli:

$$\sqrt{2}+\sqrt{2-\sqrt{2}}+\sqrt{2-\sqrt{2+\sqrt{2}}}+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}+\ldots$$

Çözülişi. Görnüşi ýaly,

$$x_1=\sqrt{2}=2\sin\frac{\pi}{2^2};\;\; x_2=\sqrt{2-\sqrt{2}}=2\sin\frac{\pi}{2^3};$$

$$x_n=\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}=2\sin\frac{\pi}{2^{n+1}}\cdot$$

Dalamberiň nyşanyna görä,

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{2\sin\frac{\pi}{2^{n+2}}}{2\sin\frac{\pi}{2^{n+1}}}=\lim_{n\to\infty}\frac{\frac{\pi}{2^{n+2}}}{\frac{\pi}{2^{n+1}}}=\frac{1}{2}<1$$

hatar ýygnanýar.

18-nji mysal.

$$f(x)=\sum_{n=1}^{\infty}\frac{e^{-nx}}{1+n^2}$$

funksiýanyň $x\geq 0$ şöhlede üznüksizdigini, $x>0$ interwalda differensirlenýändigini görkeziň.

Çözülişi. Ilki bilen $\sum\limits_{n=1}^{\infty}\frac{e^{-nx}}{1+n^2}$ hataryň ýygnanmaklyk oblastyny derňäliň. $x=0$ bolanda, bu hataryň ýygnanýandygyny görmek kyn däldir:

$$f(0)=\sum_{n=1}^{\infty}\frac{e^{-n\cdot 0}}{1+n^2}=\sum_{n=1}^{\infty}\frac{1}{1+n^2}\leq\sum_{n=1}^{\infty}\frac{1}{n^2};$$

$$\lim_{n\to\infty}\frac{g_{n+1}}{g_n}=\lim_{n\to\infty}\frac{\dfrac{e^{-(n-1)x}}{1+(n+1)^2}}{\dfrac{e^{-nx}}{1+n^2}}=\lim_{n\to\infty}\frac{1+n^2}{1+(n+1)^2}\cdot\frac{1}{e^x}=\frac{1}{e^x}.$$

Bu ýerden görnüşi ýaly, ýokarky hatar $x\ge 0$ şöhlede ýygnanýar, $x<0$ interwalda bolsa dargaýar.

Alnan netijeleri peýdalanyp, meselede talap edilýän şertleri görkezeliň:

$$\lim_{\Delta x\to 0}\Delta f=\lim_{\Delta x\to 0}[f\left(x+\Delta x\right)-f(x)];$$

$$\lim_{\Delta x\to 0}\Delta f=\lim_{\Delta x\to 0}\left[\sum_{n=1}^{\infty}\frac{e^{-n(x+\Delta x)}}{1+n^2}-\sum_{n=1}^{\infty}\frac{e^{-nx}}{1+n^2}\right]=$$

$$=\lim_{\Delta x\to 0}\sum_{n=1}^{\infty}\frac{e^{-nx}}{1+n^2}\left[\frac{1}{e^{n\Delta x}}-1\right]=\sum_{n=1}^{\infty}\frac{e^{-nx}}{1+n^2}\lim_{\Delta x\to 0}\left[\frac{1}{e^{n\Delta x}}-1\right]=0.$$

Diýmek, $\lim\limits_{\Delta x\to 0}\Delta f=0$. Ýagny, $f(x)$ funksiýa $x\ge 0$ şöhlede üznüksiz. Bu funksiýanyň $x>0$ interwalda differensirlenýändigini görkezmek üçin bolsa, $x>0$ interwalda $\lim\limits_{\Delta x\to 0}\dfrac{\Delta f}{\Delta x}=0$ predeliň bardygyny subut etmek ýeterlikdir:

$$\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}=\sum_{n=1}^{\infty}\frac{e^{-nx}}{1+n^2}\lim_{\Delta x\to 0}\frac{\left[\dfrac{1}{e^{n\Delta x}}-1\right]}{\Delta x}=\sum_{n=1}^{\infty}\frac{-n}{1+n^2}e^{-nx}<\infty.$$

§5. ÝOKARY ALGEBRA BILEN BAGLANYŞYKLY MESELELER

1-nji mysal. $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ matrisanyň

$$X^2-(a+d)X+(ad-bc)E=0.$$

deňlemäni kanagatlandyrýandygyny subut ediň, bu ýerde $E=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Çözülişi. Berlen deňlemede $X=A$ bahany goýup, onuň deňlemäni kanagatlandyrýandygyny görmek bolýar:

$$A^2-(a+d)A+(ad-bc)E=$$

$$=\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2-(a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix}+(ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=$$

$$=\begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix}-\begin{pmatrix} a^2+ad & ab+db \\ ac+dc & ad+d^2 \end{pmatrix}+$$

$$+\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=0.$$

2-nji mysal. Kwadraty nol matrisa deň bolan, ikinji tertipli matrisalaryň ählisini tapyň.

Çözülişi. Gözlenilýän matrisany $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ görnüşde saýlap alalyň, onda:

$$A^2=0,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2=0,\ \begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

bu ýerden, alarys:

$$\begin{cases} a^2+bc=0, \\ d^2+bc=0, \\ (a+d)b=0, \\ (a+d)c=0. \end{cases}$$

Bu ulgamy çözüp, mysalyň şertini kanagatlandyrýan matrisany taparys: $A=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ ýa-da $\forall a,b\in R, A=\begin{pmatrix} a & b \\ \dfrac{-a^2}{b} & -a \end{pmatrix}$.

3-nji mysal. Hasaplaň:

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{100}.$$

Çözülişi. Derejäniň esasy $A=\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Onda:

$$A^2=\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}^2=\begin{pmatrix} 2^2 & 1+2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^2 \end{pmatrix};$$

$$A^3=\begin{pmatrix} 2^2 & 1+2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^2 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}=\begin{pmatrix} 2^3 & 1+2+2^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^3 \end{pmatrix}.$$

Bu prosesi dowam edip, alarys:

$$A^{100} = \begin{pmatrix} 2^{100} & 1+2+2^2+...+2^{99} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{100} \end{pmatrix} = \begin{pmatrix} 2^{100} & 2^{100}-1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{100} \end{pmatrix}.$$

Jogaby: $\begin{pmatrix} 2^{100} & 2^{100}-1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{100} \end{pmatrix}.$

4-nji mysal. Goý,

$$f(x) = \begin{vmatrix} 1 & 1 & 1 \\ 3-x & 5-3x^2 & 3x^3-1 \\ 2x^2-1 & 3x^5-1 & 7x^8-1 \end{vmatrix}$$

bolsun. $f^{'}(c)=0$ bolar ýaly, şeýle bir $c(0<c<1)$ sanyň tapyljakdygyny subut ediň.

Çözülişi. Berlen $f(x)$ funksiýanyň (0, 1) interwalyň uçky nokatlaryndaky bahalaryny hasaplalyň:

$$f(0) = \begin{vmatrix} 1 & 1 & 1 \\ 3-0 & 5-0 & 0-1 \\ 0-1 & 0-1 & 0-1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 5 & -1 \\ -1 & -1 & -1 \end{vmatrix} = 0;$$

$$f(1) = \begin{vmatrix} 1 & 1 & 1 \\ 3-1 & 5-1 & 1-1 \\ 1-1 & 1-1 & 1-1 \end{vmatrix} = f \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 5 \end{vmatrix} = 0.$$

Görnüşi ýaly, $f(0)=f(1)=0$, onda Rollyň teoremasyna görä bu interwaldan şeýle bir nokat tapylyp, $f(0)=f(1)=0$, deňlik ýerine ýeter.

5-nji mysal. Predeli tapyň:

$$\lim_{n\to\infty}\begin{pmatrix} \frac{1}{2} & 1 & 1 \\ 0 & \frac{1}{3} & 1 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}^n .$$

Çözülişi. A we B matrisalara seredeliň:

$$A=\begin{pmatrix} \frac{1}{2} & 1 & 1 \\ 0 & \frac{1}{3} & 1 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}, \quad B=\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

B matrisanyň 3-nji derejesiniň

$$B^3=\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}=$$

$$=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}=0$$

nol matrisa deňligini göz öňünde tutup, alarys:

$$(A+B)^n=\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}^n;$$

$$A^n + nA^{n-1}B + \frac{n(n-1)}{2}A^{n-2}B^2 + 0 = \begin{pmatrix} \frac{1}{2^n} & 0 & 0 \\ 0 & \frac{1}{3^n} & 0 \\ 0 & 0 & \frac{1}{5^n} \end{pmatrix}.$$

Ahyrky deňlikde predele geçip, alarys:

$$\lim_{n\to\infty}\left(A^n + nA^{n-1}B + \frac{n(n-1)}{2}A^{n-2}B^2 + 0 \right) = \lim_{n\to\infty}\begin{pmatrix} \frac{1}{2^n} & 0 & 0 \\ 0 & \frac{1}{3^n} & 0 \\ 0 & 0 & \frac{1}{5^n} \end{pmatrix};$$

$$\lim_{n\to\infty} A^n \left(E + nB + \frac{n(n-1)}{2}B^2 \right) = \lim_{n\to\infty}\begin{pmatrix} \frac{1}{2^n} & 0 & 0 \\ 0 & \frac{1}{3^n} & 0 \\ 0 & 0 & \frac{1}{5^n} \end{pmatrix};$$

$$\lim_{n\to\infty} A^n \begin{pmatrix} 1 & -n & \frac{n(n-3)}{2} \\ 0 & 1 & -n \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\begin{pmatrix} 1 & -\infty & \infty \\ 0 & 1 & -\infty \\ 0 & 0 & 1 \end{pmatrix} \lim_{n\to\infty} A^n = 0.$$

Bu deňlikden görnüşi ýaly, $\lim\limits_{n\to\infty} A^n$ matrisa noluň bölüjisi bolmaly, emma $\det A \neq 0$ bolýanlygyny göz öňünde tutsak, $\lim\limits_{n\to\infty} A^n$ matrisa diňe

$$\lim_{n\to\infty} A^n = 0.$$

şertde noluň bölüjisi bolup biler. Diýmek,

$$\lim_{n\to\infty}\begin{pmatrix} \frac{1}{2} & 1 & 1 \\ 0 & \frac{1}{3} & 1 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}^n = 0.$$

Jogaby: 0.

6-njy mysal. $a_1 = 1, a_2 = 2, n \geq 1$ üçin $a_{n+2} = a_{n+1} + a_n$ şertler bilen kesgitlenýän yzygiderlilige Fibonnaçiniň yzygiderliligi diýilýär. Subut ediň:

$$a_n = \begin{vmatrix} 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & \ldots & 0 & 0 & 0 \\ . & . & . & . & \ldots & . & . & . \\ 0 & 0 & 0 & 0 & \ldots & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 1 \end{vmatrix}_n .$$

Çözülişi. Mysalda berlen deňligi matematiki induksiýa usulynyň kömegi bilen subut edeliň:

$n = 1$ bolanda, $a_1 = 1$ deňlik dogry.

$n = 2$ bolanda, $a_2 = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$ deňlik dogry.

Goý, $n \le k+1$ üçin deňlik dogry bolsun. Onda $n = k+2$ bolanda,

$$a_{k+2} = a_{k+1} + a_k =$$

$$= \begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & \dots & 0 & 0 & 0 \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{vmatrix}_{k+1} + \begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & \dots & 0 & 0 & 0 \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{vmatrix}_{k} =$$

$$= (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & \dots & 0 & 0 & 0 \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{vmatrix}_{k+1} +$$

$$+(-1)^{1+2} \cdot (-1) \cdot \begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & \dots & 0 & 0 & 0 \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{vmatrix}_{k+1}$$

deňlikden peýdalanyp,

$$a_{k+2} = \begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & \dots & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{vmatrix}_{k+2}$$

bolýandygyny alarys.

8-nji mysal. Her bir setirinde we her bir sütüninde diňe bir elementi 1, galan elementleri nollar bolan n-nji tertipli kesgitleýjileriň ählisiniň jemini tapyň. Şeýle kesgitleýjileriň sany näçe?

Çözülişi. $\det E$ kesgitleýji mysalyň şertini kanagatlandyrýar. Bu ýerde,

$$\det E = \begin{vmatrix} 1 & 0 & & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1.$$

Mysalyň şertini kanagatlandyrýan beýleki kesgitleýjileriň ählisi bu kesgitleýjiniň setirleriniň (ýa-da sütünleriniň) çalşyrmalary netijesinde alynýar. $E_{ij}(i \le j)$ bilen $\det E$ kesgitleýjiniň i we j setirleriniň orunlarynyň çalşyrylmagy netijesinde alynýan kesgitleýjini belläliň. Kesgitleýjiniň esasy häsiýetlerinden peýdalanyp,

$$E_{ij} = (-1)^{j-i} \det E = (-1)^{j-i}$$

deňligi alarys..

n sany setiriň dürli çalşyrmalarynyň sanynyň $n!$ bolýanlygy üçin, E_{ij} kesgitleýjileriň sany hem $n!$ bolar. $n!$ sanyň jübütligini göz öňünde tutsak, ähli E_{ij} kesgitleýjileriň jemi:

$$\sum_{j=1}^{n}\sum_{i=1}^{j} E_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{j} (-1)^{j-i} = \sum_{k=1}^{n!} (-1)^k = 0$$

bolar.

Jogaby: kesgitleýjileriň sany $n!$, jemi 0 deň.

9-njy mysal. Goý, α, β, γ – sanlar $x^3 + px + q = 0$ deňlemäniň kökleri bolsun. Hasaplaň:

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{vmatrix}.$$

Çözülişi. Wiýetiň teoremasyna görä:

$$\alpha+\beta+\gamma=0.$$

Bu deňligi peýdalanyp, berlen kesgitleýjini hasaplalyň:

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{vmatrix} = \begin{vmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \alpha+\gamma+\beta & \beta+\alpha+\gamma & \gamma+\beta+\alpha \end{vmatrix} = \begin{vmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

Jogaby: 0.

10-njy mysal. Goý, A n-nji tertipli kwadrat matrisa

$$\begin{pmatrix} a & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & a & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & a \end{pmatrix}$$

görnüşe eýe bolsun. A^m matrisanyň birinji setirindäki elementleriň jemini tapyň, bu ýerde $m \le n$.

Çözülişi. Matematiki induksiýa usulynyň kömegi bilen A^m matrisanyň

$$A^m = \begin{pmatrix} a^m & C_m^1 a^{m-1} & C_m^2 a^{m-2} & \cdots & 1 & \cdots & 0 \\ & \vdots & & & \vdots & & \vdots \end{pmatrix}$$

görnüşe eýedigini subut edeliň.

$m=1$ bolanda,

$$A^1 = \begin{pmatrix} a^1 & C_1^1 a^0 & 0 & \cdots & & \cdots & 0 \\ & \vdots & & \vdots & & \vdots & \end{pmatrix}$$ ýerine ýetýär.

Goý, A^k matrisa şol görnüşe eýe bolsun, onda:

$$A^{k+1} = A^k \cdot A =$$

$$= \begin{pmatrix} a^k & C_k^1 a^{k-1} & C_k^2 a^{k-2} & \cdots & 1 & \cdots & 0 \\ & \vdots & & & \vdots & & \vdots \end{pmatrix} \begin{pmatrix} a & 1 & 0 & & 0 & 0 & 0 \\ 0 & a & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & a & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & a & 1 \\ 0 & 0 & 0 & & 0 & 0 & a \end{pmatrix}$$

bolar. Bu köpeltmek hasylda $C_k^i + C_k^{i+1} = C_{k+1}^{i+1}$ deňligi peýdalanyp,

$$A^{k+1} = \begin{pmatrix} a^{k+1} & C_{k+1}^1 a^k & C_{k+1}^2 a^{k-1} & \cdots & 1 & \cdots & 0 \\ & \vdots & & & \vdots & & \vdots \end{pmatrix}$$

deňligi alarys. Diýmek, A^m matrisanyň birinji setirindäki elementleriň jemi:

$$a^m + C_m^1 a^{m-1} + C_m^2 a^{m-2} + \ldots + 1 + 0 + \ldots + 0 = (a+1)^m$$

bolar.

Jogaby: $(a+1)^m$.

11-nji mysal. $AB - BA = E$ (E birlik matrisa) deňligi kanagatlandyrýan A we B matrisalaryň ýoklugyny subut ediň.

Çözülişi. Tersine güman edeliň, ýagny, $AB - BA = E$ deňligi kanagatlandyrýan

$$A = (a_{ij})_{n \times n} \, we \, B = (b_{ij})_{n \times n}$$

matrisalar bar bolsun. Berlen deňlikde aşakdaky amallary geçirip,

$$Tr(AB - BA) = TrE;$$

$$Tr(AB) - Tr(BA) = \sum_{i=1}^{n} 1;$$

$$Tr(AB) - Tr(BA) = \sum_{i=1}^{n} 1;$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}b_{ji}-\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}a_{ji}=n \quad \Rightarrow \quad n=0$$

nädogry deňligi alarys, alnan garşylyk biziň eden gümanymyzyň nädogrudygyny görkezýär. Diýmek, $AB-BA=E$ deňligi kanagatlandyrýan A we B matrisalar ýok.

12-nji mysal. Goý, A-n-nji tertipli kwadrat matrisa bolsun. Eger, $A^2=E$ bolsa, onda $A+E$ we $A-E$ matrisalaryň ranglarynyň jeminiň n deňdigini subut ediň.

Çözülişi. Mysalyň şertindäki $A^2=E$ deňlikden alarys:

$$\det A^2=\det E,\ (\det A)^2=1 \text{ ýa-da } A=\pm 1\neq 0.$$

Bu alnan netije, A matrisanyň rangynyň n deňdigini görkezýär. Ýagny,

$$r_A=n.$$

Belli bolşy ýaly, A we B matrisalar üçin aşakdaky goşa deňsizlik dogrudyr:

$$r_{A+B}\leq r_A+r_B\leq r_{AB}+n.$$

Bu deňsizlikde $A=A+E, B=A-E$ bahalary goýup, alarys:

$$r_{A+E+A-E}\leq r_{A+E}+r_{A-E}\leq r_{(A+E)(A-E)}+n,$$

$$r_{2A}\leq r_{A+E}+r_{A-E}\leq r_{A^2-E}+n,$$

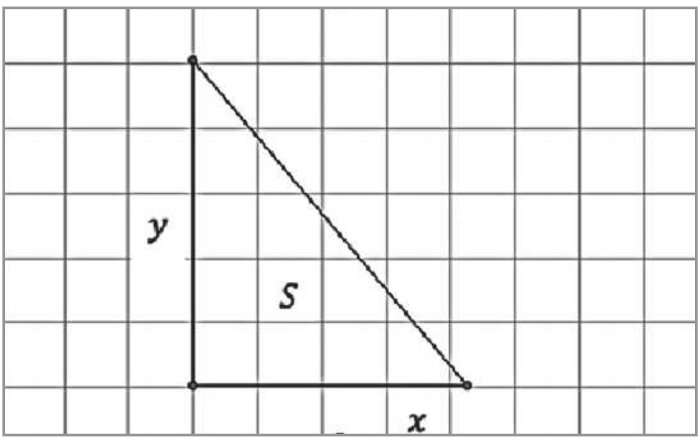
$$r_A=r_{2A}\leq r_{A+E}+r_{A-E}\leq r_0+n,$$

$$n\leq r_{A+E}+r_{A-E}\leq n \text{ ýa-da } r_{A+E}+r_{A-E}=n.$$

§6. GEOMETRIÝA BILEN BAGLANYŞYKLY MESELELER

1-nji mesele. Meýdany S deň bolan gönüburçly üçburçluklaryň iň kiçi perimetrlisiniň taraplaryny tapyň.

Çözülişi. 1-nji suratda meýdany S, katetleri x, y bolan gönüburçly üçburçluk şekillendirilen.



1-nji surat

Şerte görä,

$$S = \frac{1}{2}xy \text{ bu ýerden } y = \frac{2S}{x} \text{ alarys.}$$

Onda, bu üçburçlugyň perimetri

$$P = x + y + \sqrt{x^2 + y^2} = x + \frac{2S}{x} + \sqrt{x^2 + \left(\frac{2S}{x}\right)^2}$$

bolar. Perimetriň iň kiçi bahasy

$$P = x + \frac{2S}{x} + \sqrt{x^2 + \left(\frac{2S}{x}\right)^2} \geq 2\sqrt{x \cdot \frac{2S}{x}} + \sqrt{2\sqrt{x^2\left(\frac{2S}{x}\right)^2}}$$

ýa-da

$$P \geq 2(\sqrt{2}+1)S$$

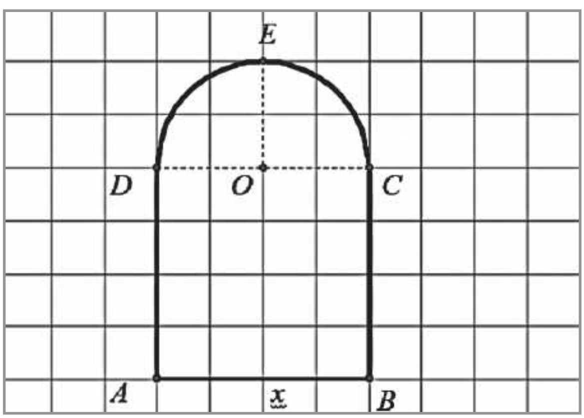
deňsizlige görä $2(\sqrt{2}+1)$ deň. Bu ýagdaý bolsa, diňe

$$x = \frac{2S}{x} \text{ ýa-da } x = \sqrt{2S}$$

bolanda ýerine ýetip biler. Oňa görä-de gözlenilýän üçburçlugyň taraplary, $x = \sqrt{2S}$, $y = \frac{2S}{x} = \sqrt{2S}$, $\sqrt{x^2 + y^2} = 2\sqrt{S}$ bolar.

2-nji mesele. Aýnanyň formasy gönüburçlukdan we onuň ýokarsyna daýanýan ýarym tegelekden ybarat. Onuň formasynyň perimetri P deň. Aýnanyň ininiň haýsy bahasynda onuň meýdany iň uly baha eýe bolar?

Çözülişi. Meseläniň şertini peýdalanyp, aýnany 2-nji suratdaky ýaly şekillendirmek bolar.



2-nji surat

$AB = DC = 2OD = 2OE = x, AD = BC = y$ bellenişikleri z şerte görä, alarys:

$$P = 2y + x + \pi\frac{x}{2} \text{ ýa-da } y = \frac{1}{2}\left(P - \frac{\pi + 2}{2}x\right).$$

Onda, aýnanyň formasynyň meýdany,

$$S = S_{ABCD} + S_{sektor} = xy + \frac{\pi x^2}{8} = \frac{x}{2}\left(P - \frac{\pi + 2}{2}x\right) + \frac{\pi x^2}{8}.$$

ýa-da

$$S = S(x) = \frac{P}{2}x - \frac{\pi + 4}{8}x^2.$$

Aýnanyň meýdanyna x görä üýtgeýän funksiýa hökmünde seredip, onuň maksimumyny tapalyň:

$$S'(x) = \left(\frac{P}{2}x - \frac{\pi + 4}{8}x^2\right)' = \frac{p}{2} - \frac{\pi + 4}{4}x;$$

$$S'(x) = 0, \frac{p}{2} - \frac{\pi + 4}{4}x = 0, \text{ ýa-da } x = \frac{2P}{\pi + 4}.$$

Görnüşi ýaly, $\forall\delta > 0$ san üçin:

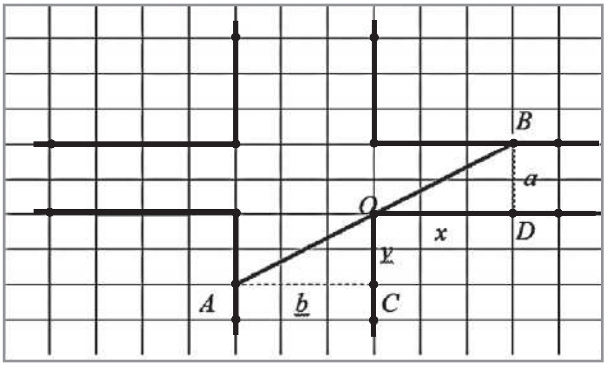
$$S'(x + \delta) > 0 \text{ we } S'(x - \delta) < 0.$$

Diýmek, $x = \dfrac{2P}{\pi + 4}$ nokatda $S(x) = \dfrac{P}{2}x - \dfrac{\pi + 4}{8}x^2$ funksiýanyň maksimumy bar.

Jogaby: $x = \dfrac{2P}{\pi + 4}.$

3-nji mesele. Inleri degişlilikde a we b deň bolan koridorlar gönüburç boýunça kesişýär. Bir koridordan beýleki koridora gorizontal ýagdaýda geçiriljek merdiwanyň iň uly uzynlygyny kesgitlemeli.

Çözülişi. 3-nji suratda meseläniň şertinde berlen koridorlar şekillendirilen.



3-nji surat

Görnüşi ýaly, meseläni çözmek üçin AB kesimiň uzynlygynyň iň uly bahasyny tapmaly. Şerte görä, $BD = a, AC = b.$ Eger, $OD = x$, $OC = y$ diýip bellesek, $\Delta AOC \sim \Delta BOD$ bolýanlygyndan, alarys:

$$\frac{a}{x} = \frac{y}{b} \text{ ýa-da } y = \frac{ab}{x}.$$

Onda AB kesimiň uzynlygy

$$AB = AO + OB = \sqrt{a^2 + x^2} + \sqrt{y^2 + b^2} =$$

$$AB = \sqrt{a^2 + x^2} + \sqrt{\left(\frac{ab}{x}\right)^2 + b^2} = \left(\frac{b}{x} + 1\right)\sqrt{a^2 + x^2}$$

bolar. Diýmek, AB kesimiň uzynlygynyň iň uly bahasyny tapmak üçin

$$f(x) = \left(\frac{b}{x} + 1\right)\sqrt{a^2 + x^2}$$

funksiýanyň maksimumyny tapmak gerek bolýar.

$$f'(x)=\left(\frac{b}{x}+1\right)'\sqrt{a^2+x^2}+\left(\frac{b}{x}+1\right)\left(\sqrt{a^2+x^2}\right)';$$

$$f'(x)=\frac{x^3-a^2b}{x^2\sqrt{a^2+x^2}};$$

$$f'(x)=0, \frac{x^3-a^2b}{x^2\sqrt{a^2+x^2}}=0, x=\sqrt[3]{a^2b}.$$

$x=\sqrt[3]{a^2b}$ nokadyň $f(x)$ funksiýanyň maksimum nokady bolýandygyny barlamak kyn däldir. Oňa görä-de AB kesimiň iň uly bahasy

$$AB=\left(\frac{b}{\sqrt[3]{a^2b}}+1\right)\sqrt{a^2+(\sqrt[3]{a^2b})^2}=\sqrt{\left(\frac{b}{\sqrt[3]{a^2b}}+1\right)^2\left(a^2+\sqrt[3]{a^4b^2}\right)};$$

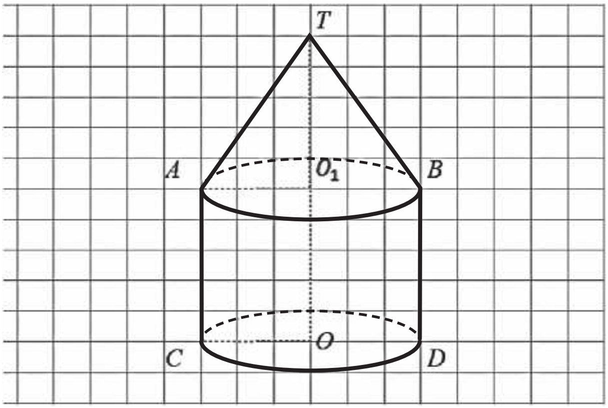
ýa-da $AB=\left(\sqrt[3]{a^2}+\sqrt[3]{b^2}\right)^{\frac{3}{2}}.$

Jogaby: $\left(\sqrt[3]{a^2}+\sqrt[3]{b^2}\right)^{\frac{3}{2}}.$

4-nji mesele. Ýangyç guýulýan sisternanyň 4-nji suratdaky ýaly görnüşi bar we onuň üstüniň meýdany S deň. Eger $\angle ATB=90^0$ bolsa, onda sisternanyň göwrüminiň iň uly baha eýe bolmagy üçin silindriň beýikligi näçä deň bolmaly?

Çözülişi. Meseläni şertine görä $\angle ATB=90°$. Eger $AO = OB = R$, $AC=BD=OO_1=H$ bellenişikleri geçirsek, $\angle ATO=\angle BTO=45°$ bolýandygyndan, alarys:

$$AO_1=TO_1=R \text{ we } AT=R\sqrt{2}.$$



4-nji surat

Sisternanyň doly üstüniň meýdanynyň konusyň we silindriň gapdal üstleriniň meýdanlaryndan, hem-de silindriň esasynyň meýdanyndan ybaratdygyny göz öňünde tutup, alarys:

$$S = S_{k.g.m} + S_{s.g.m} + \pi R^2 = \frac{\pi (AT)^2}{4} + 2\pi R \cdot (AC) + \pi R^2;$$

$$S = \frac{\pi (R\sqrt{2})^2}{4} + 2\pi R \cdot (H) + \pi R^2 \text{ ýa-da } S = \frac{3}{2}\pi R^2 + 2\pi R\ H.$$

Bu ýerden hem, $H = \dfrac{S}{2\pi R} - \dfrac{3}{4}R$ deňligi alarys. Bu deňligi sisternanyň göwrümini tapmak üçin ulanalyň:

$$V = V_S + V_k = \frac{1}{6}\pi R^2 R + \pi R^2 H = \pi R^2 \left(\frac{R}{6} + \frac{S}{2\pi R} - \frac{3R}{4} \right)$$

ýa-da

$$V = \frac{S}{2} R - \frac{7\pi}{12} R^3.$$

Indi, sisternanyň göwrümine, R görä üýtgeýän funksiýa hökmünde seredip, onuň iň uly bahasyny tapalyň:

$$V = V(R) = \frac{S}{2}R - \frac{7\pi}{12}R^3;$$

$$V'(R) = 0, \frac{S}{2} - \frac{7\pi}{4}R^2 = 0, \text{ýa-da } R = \sqrt{\frac{2S}{7\pi}};$$

$$V_{\max} = V\left(\sqrt{\frac{2S}{7\pi}}\right) = \frac{S}{2}\cdot\sqrt{\frac{2S}{7\pi}} - \frac{7\pi}{12}\left(\sqrt{\frac{2S}{7\pi}}\right)^3 = \sqrt{\frac{2S}{7\pi}}\left(\frac{S}{2} - \frac{S}{6}\right) = \frac{S}{3}\sqrt{\frac{2S}{7\pi}}.$$

Diýmek, sisternanyň göwrümi iň uly baha,

$$H = \frac{S}{2\pi R} - \frac{3}{4}R = \frac{S}{2\pi}\sqrt{\frac{7\pi}{2S}} - \frac{3}{4}\cdot\sqrt{\frac{2S}{7\pi}} = \sqrt{\frac{2S}{7\pi}};$$

ýa-da

$$H = \sqrt{\frac{2S}{7\pi}}$$

bolanda eýe bolar.

Jogaby: $\sqrt{\frac{2S}{7\pi}}.$

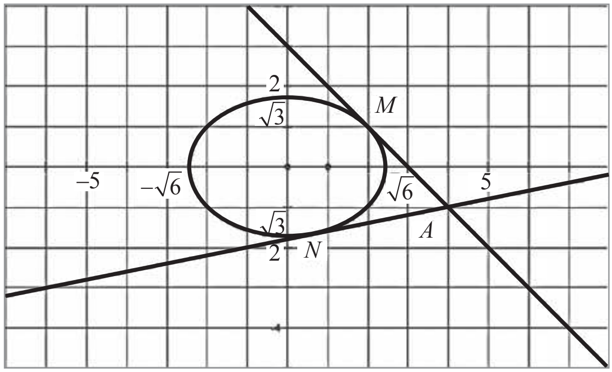
5-nji mesele. (4;-1) nokatdan $\frac{x^2}{6} + \frac{y^2}{3} = 1$ ellipse geçirilen galtaşýan göni çyzyklaryň deňlemelerini tapyň.

Çözülişi. Goý, $A(4;–1)$ nokatdan berlen ellipse geçirilen galtaşýanlar oňa M, N nokatlarda galtaşýan bolsunlar (*5-nji surat*).

Belli bolşy ýaly, berlen ellipsiň (u,υ) nokadynda oňa geçirilen galtaşýan göni çyzygyň deňlemesi

$$\frac{u}{6}x + \frac{\upsilon}{3}y = 1$$

görnüşde bolar.



5-nji surat

Meseläniň şertini kanagatlandyrýan göni çyzyklar berlen A nokatdan geçýär. Oňa görä-de,

$$\frac{u}{6}\cdot 4+\frac{\upsilon}{3}(-1)=1 \text{ ýa-da } \upsilon=2u-3.$$

Diýmek, M, N nokatlaryň koordinatlary $(u, 2u-3)$ görnüşde bolmaly. Başga bir tarapdan, M, N nokatlar ellipse degişli, ýagny

$$\frac{u^2}{6}+\frac{(2u-3)^2}{3}=1,$$

$$3u^2-8u+4=0,$$

$$u_1=2,\ u_2=\frac{2}{3}, \text{bu ýerden } \upsilon_1=1,\ \upsilon_2=-\frac{5}{3}.$$

Bu tapylanlary ulanyp, (MA) we (NA) galtaşýan gönüleriň deňlemelerini taparys:

$$(MA):\ \frac{2}{6}x+\frac{1}{3}y=1 \text{ ýa-da } x+y-3=0;$$

$$(NA):\ \frac{18}{2}x+\frac{5}{9}y=1 \text{ ýa-da } x-5y-9=0.$$

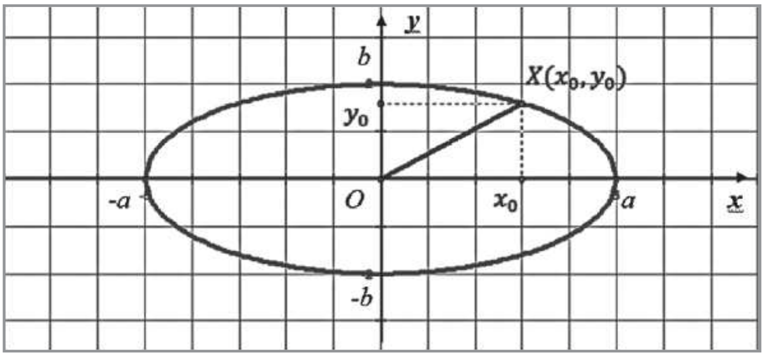
Jogaby: $x+y-3=0,\ \ x-5y-9=0.$

6-njy mesele. Ellipsiň merkezini onuň erkin nokady bilen birleşdirýän kesimiň uzynlygynyň uly we kiçi ýarym oklaryň arasynda ýerleşýändigini subut ediň.

Subudy. Goý,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,(a\geq b)$$

ellips berlen bolsun (*6-njy surat*).



6-njy surat

Görnüşi ýaly,

$$\left|OX\right|=\sqrt{x_0^2+y_0^2}=a\sqrt{\frac{x_0^2+y_0^2}{a^2}}\leq a\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}=a \text{ ýa-da } \left|OX\right|\leq a;$$

$$\left|OX\right|=\sqrt{x_0^2+y_0^2}=b\sqrt{\frac{x_0^2+y_0^2}{b^2}}\leq b\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}=b \text{ ýa-da } \left|OX\right|\geq b.$$

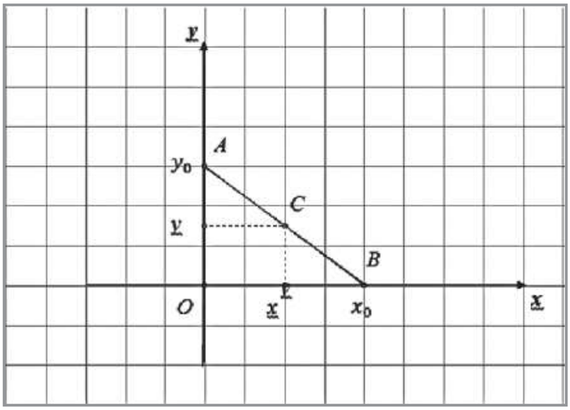
Diýmek,

$$b\leq\left|OX\right|\leq a.$$

7-nji mesele. Uzynlygy 3-e deň bolan AB kesimiň uçky nokatlary koordinat oklarynda (A – nokat Oy okda, B – nokat Ox okda) ýerleşen we degişli oklar boýunça hereket edýärler. Bu ýagdaýda A nokatdan 1 birlik uzaklykda ýerleşen C nokat nähili egrini çyzýar.

Çözülişi. 7-nji suratdan görnüşi ýaly,

$$A(0,y_0),B(x_0,0),C(x,y).$$



7-nji surat

Şerte görä, $|AB| = 3, \overrightarrow{AB} = 3\overrightarrow{AC}$. Bu deňliklerde A, B, we C nokatlaryň koordinatlaryny goýup, alarys:

$$\begin{cases} x_0^2 + y_0^2 = 9; \\ x_0 = 3x; \\ -y_0 = 3(y - y_0); \end{cases}$$

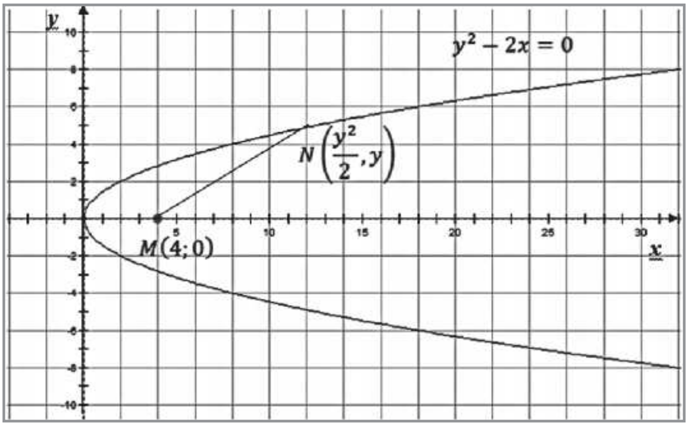
$$\begin{cases} x_0^2 + y_0^2 = 9; \\ x_0 = 3x; \\ y_0 = \dfrac{3}{2}y \end{cases} \quad \text{ýa-da} \quad x^2 + \frac{y^2}{4} = 1.$$

Diýmek, berlen C nokat, merkezi $O(0;0)$ nokatda, uly ýarym oky 2 we kiçi ýarym oky 1-e deň bolan ellipsi çyzýar.

Jogaby: Ellips. $x^2 + \dfrac{y^2}{4} = 1.$

8-nji mesele. $M(4;0)$ nokatdan $y^2 - 2x = 0$ egri çyzyga çenli uzaklygy tapmaly.

Çözülişi. Goý, N nokat berlen egrä degişli bolsun, onda egriniň deňlemesinden $N\left(\dfrac{y^2}{2}, y\right)$ bolýandyny görmek kyn däldir.



8-nji surat

Bu ýagdaýda MN kesimiň uzynlygy

$$|MN| = \sqrt{\left(\frac{y^2}{2} - 4\right)^2 + y^2} = f(y)$$

bolar. M nokatdan berlen egrä çenli uzaklyk diýip, f (y) funksiýanyň minimum bahasyna aýdylýandygy bize ozaldan mälimdir. Oňa görä-de,

$$f'(y) = \frac{\left(\frac{y^2}{2} - 4\right)y + y}{\sqrt{\left(\frac{y^2}{2} - 4\right)^2 + y^2}} = \frac{\left(\frac{y^2}{2} - 3\right)y}{f(y)};$$

$$f'(y) = 0, \ \frac{\left(\frac{y^2}{2} - 3\right)y}{f(y)} = 0, \text{ ýa-da } y_1 = 0; \ y_{2,3} = \pm\sqrt{6}.$$

Bu ýerden $y_{2,3} = \pm\sqrt{6}$ nokatlaryň f (y) funksiýanyň minimum nokatlary bolýandygyny görmek kyn däldir. Oňa görä-de, berlen M nokatdan berlen egrä çenli uzaklyk,

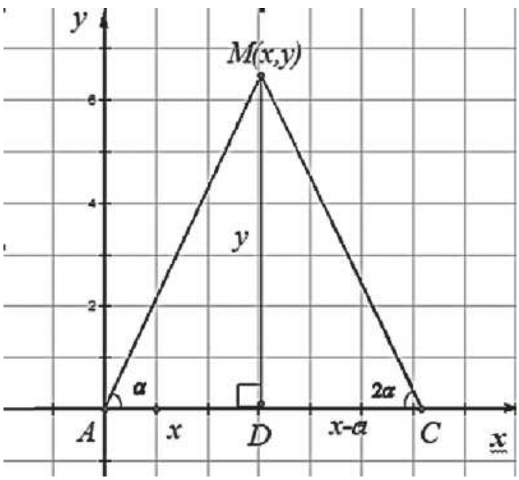
$$|MN| = f_{\min} = f\left(\pm\sqrt{6}\right) = \sqrt{\left(\frac{\left(\pm\sqrt{6}\right)^2}{2} - 4\right)^2 + \left(\pm\sqrt{6}\right)^2} = \sqrt{7}.$$

Jogaby: $\sqrt{7}$.

9-njy mesele. Üçburçlugyň iki depesi üýtgewsiz, üçünji depesi garşysyndaky burçlaryň gatnaşygy 2 deň bolar ýaly hereket edýän bolsa, bu depäniň geometriki ornuny kesgitlemeli.

Çözülişi. 9-njy suratdan görnüşi ýaly,

$$\text{tg}\alpha = \frac{y}{x} \text{ we } \text{tg}2\alpha = \frac{y}{x-a}.$$



9-njy surat

Emma, ikinji bir tarapdan

$$\text{tg}2\alpha = \frac{2\text{tg}\alpha}{1-\text{tg}^2\alpha}$$

bolýanlygyny peýdalansak, *C* nokadyň hereketi üçin, aşakdaky deňlemäni alarys:

$$\frac{y}{x-a}=\frac{2\dfrac{y}{x}}{1-\left(\dfrac{y}{x}\right)^2} \text{ ýa-da } 3x^2-y^2-2ax=0.$$

Jogaby: $3x^2-y^2-2ax=0.$

10-njy mesele. Üçburçlugyň taraplary özüniň

$$a_ix+b_iy+c_i=0$$

deňlemeleri bilen berlen. Onuň meýdany üçin aşakdaky deňligi subut ediň:

$$S=\frac{\Delta^2}{2|\Delta_1\Delta_2\Delta_3|};$$

bu ýerde $\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ we her bir Δ_i degişlilikde c_i elementiň algebraik doldurgyjy.

Subudy. Goý, berlen üçburçlugyň depeleri A_1, A_2, A_3 we

$$A_1:\begin{cases} a_2x+b_2y+c_2=0, \\ a_3x+b_3y+c_3=0; \end{cases}$$

$$A_2:\begin{cases} a_1x+b_1y+c_1=0, \\ a_3x+b_3y+c_3=0; \end{cases}$$

$$A_3:\begin{cases} a_1x+b_1y+c_1=0, \\ a_2x+b_2y+c_2=0 \end{cases}$$

bolsun. Bu ulgamlary çözüp, alarys:

$$A_1\left(\frac{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}; -\frac{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}\right);$$

$$A_2\left(\frac{\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}}; -\frac{\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}}\right);$$

$$A_3\left(\frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}; -\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}\right).$$

A_1, A_2, A_3 depeleriň tapylan koordinatlary boýunça A_1, A_2, A_3 – üçburçlugyň meýdanyny tapalyň:

$$S = \frac{1}{2}\mathrm{mod}\begin{Vmatrix} \dfrac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} - \dfrac{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} & \dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} - \dfrac{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} \\ \dfrac{\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}} - \dfrac{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} & \dfrac{\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}} - \dfrac{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} \end{Vmatrix};$$

Bu deňlikdäki kesgitleýjini ykjam görnüşe getirip, alarys:

$$S=\frac{1}{2}\operatorname{mod}\frac{\begin{vmatrix} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} & -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} & -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \\ \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} & -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}};$$

Bu deňligi $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ kesgitleýjä köpeldip böleliň, netije-de, alarys:

$$S=\frac{1}{2}\operatorname{mod}\frac{\begin{vmatrix} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} & -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} & -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \\ \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} & -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{vmatrix}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}};$$

$$S=\frac{1}{2}\operatorname{mod}\frac{\begin{vmatrix}\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} & 0 & 0\\ 0 & \begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} & 0\\ 0 & 0 & \begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}\end{vmatrix}}{\begin{vmatrix}a_2 & b_2\\ a_3 & b_3\end{vmatrix}\begin{vmatrix}a_1 & b_1\\ a_3 & b_3\end{vmatrix}\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}};$$

$$S=\frac{1}{2}\operatorname{mod}\frac{\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}^3}{\begin{vmatrix}a_2 & b_2\\ a_3 & b_3\end{vmatrix}\begin{vmatrix}a_1 & b_1\\ a_3 & b_3\end{vmatrix}\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}}$$

ýa-da

$$S=\frac{1}{2}\operatorname{mod}\frac{\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}^2}{\begin{vmatrix}a_2 & b_2\\ a_3 & b_3\end{vmatrix}\begin{vmatrix}a_1 & b_1\\ a_3 & b_3\end{vmatrix}\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}.$$

Ahyrky deňlikde, meseläniň şertindäki bellenişikleri göz öňünde tutsak,

$$S = \frac{\Delta^2}{2|\Delta_1\Delta_2\Delta_3|}$$

deňligi alarys.

Bellik. n- ölçegli simpleksiň gipergranlary

$$a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n + a_{in+1} = 0, (i = \overline{1, n+1})$$

deňlemeleri bilen berlen. Onuň göwrümi üçin aşakdaky deňligi subut ediň:

$$S = \frac{|\Delta|^n}{n!|\Delta_1\Delta_2\Delta_3...\Delta_{n+1}|},$$

bu ýerde $\Delta = |a_{ij}|_{(n+1)\times(n+1)}$ we her bir Δ_i degişlilikde $a_{i,n+1}$ elementiň algebraik doldurgyjy.

Çözülişi.

Lemma. $n+1$ tertipli öwrülişikli kwadrat (a_{ij}) matrisa üçin

$$|\Delta_{ij}| = |a_{ij}|^n$$

deňlik dogrudyr. Bu ýerde Δ_{ij} elementler a_{ij} elementleriň algebraik doldurgyçlary.

Subudy. (a_{ij}) matrisanyň öwrülişiklidigini peýdalanyp, alarys:

$$1 = |(a_{ij})(a_{ij})^{-1}| = |a_{ij}|\left|\frac{\Delta_{ji}}{a_{ij}}\right| = |a_{ij}|\frac{|\Delta_{ji}|}{|a_{ij}|^{n+1}} = \frac{|\Delta_{ji}|^T}{|a_{ij}|^n} = \frac{|\Delta_{ij}|}{|a_{ij}|^n}$$

ýa-da

$$|\Delta_{ij}| = |a_{ij}|^n.$$

Simpleksiň $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n + a_{in+1} = 0$ gipertekizliklerde ýatan granlaryny degişlilikde G_i-ler bilen, onuň depelerini bolsa A_i-ler bilen belgiläliň. Her bir A_i depe üçin

$$A_i = G_1 \cap G_2 \cap ... \cap G_{i-1} \cap G_{i+1} \cap ... \cap G_{n+1}$$

diýip şertleşeliň. Onda A_i depe üçin

$$A_1 : \begin{cases} a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n + a_{2,n+1} = 0; \\ a_{31}x_1 + a_{32}x_2 + ... + a_{3n}x_n + a_{3,n+1} = 0; \\ ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \\ a_{n+1,1}x_1 + a_{n+2,2}x_2 + ... + a_{n+1,n}x_n + a_{n+1,n+1} = 0 \end{cases}$$

deňlemeler ulgamyny alarys. Bu ulgamy Krameriň usuly bilen çözüp we lemmadaky bellenişikleri ulanyp, A_1 depäniň koordinatlaryny aşakdaky ýaly aňlatmak bolar:

$$A_1\left(\frac{\Delta_{11}}{\Delta_{1,n+1}}, \frac{\Delta_{12}}{\Delta_{1,n+1}}, ..., \frac{\Delta_{1n}}{\Delta_{1,n+1}}\right).$$

Edil şuňa meňzeşlikde beýleki A_i $(i = \overline{2, n+1})$ depeleriň hem koordinatlaryny taparys:

$$A_i\left(\frac{\Delta_{i1}}{\Delta_{i,n+1}}, \frac{\Delta_{i2}}{\Delta_{i,n+1}}, ..., \frac{\Delta_{in}}{\Delta_{i,n+1}}\right).$$

Alnan netijeleri ulanyp, n-ölçegli simpleksiň göwrümini tapalyň:

$$V_n = \frac{1}{n!}\operatorname{mod}(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}, ..., \overrightarrow{A_1A_n});$$

$$V_n = \frac{1}{n!}\mathrm{mod}\begin{vmatrix} \frac{\Delta_{21}}{\Delta_{2,n+1}} - \frac{\Delta_{11}}{\Delta_{1,n+1}} & \cdots & \frac{\Delta_{2n}}{\Delta_{2,n+1}} - \frac{\Delta_{1n}}{\Delta_{1,n+1}} \\ \vdots & \ddots & \vdots \\ \frac{\Delta_{n+1,1}}{\Delta_{n+1,n+1}} - \frac{\Delta_{11}}{\Delta_{1,n+1}} & \cdots & \frac{\Delta_{n+1,n}}{\Delta_{n+1,n+1}} - \frac{\Delta_{1n}}{\Delta_{1,n+1}} \end{vmatrix}.$$

Alnan deňlikdäki kesgitleýjini oňaýly görnüşe getirip, alarys:

$$V_n = \frac{1}{n!\left|\Delta_{1,n+1}\Delta_{2,n+1}\ldots\Delta_{n+1,n+1}\right|}\begin{vmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1,n+1} \\ \Delta_{21} & \Delta_{22} & & \Delta_{2,n+1} \\ & \vdots & \ddots & \vdots \\ \Delta_{n+1,1} & \Delta_{n+1,2} & \vdots & \Delta_{n+1,n+1} \end{vmatrix}.$$

Bu deňlikde lemmany ulansak,

$$V_n = \frac{1}{n!\left|\Delta_{1,n+1}\Delta_{2,n+1}\ldots\Delta_{n+1,n+1}\right|}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n+1} \\ a_{21} & a_{22} & & a_{2,n+1} \\ & \vdots & \ddots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \vdots & a_{3,n+1} \end{vmatrix}^n$$

bolar. Meseläniň şertindäki bellenişikleri peýdalanyp, soňky deňligi

$$V_n = \frac{\left|\Delta\right|^n}{n!\left|\Delta_1\Delta_2\Delta_3\ldots\Delta_{n+1}\right|}$$

görnüşde ýazmak bolar.

11-nji mesele. Ellipsoide degişli bolmadyk nokatdan, oňa mümkin bolan galtaşýan gönüçyzyklaryň ählisi geçirilen. Ähli galtaşma nokatlaryň bir tekizlikde ýatýandygyny subut ediň.

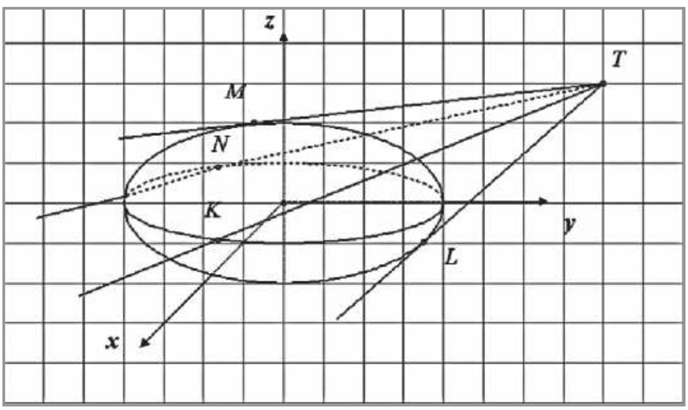
Çözülişi. Berlen ellipsoidiň deňlemesini

$$\frac{x^2}{a^2} + \frac{x^2}{b^2} + \frac{z^2}{c^2} = 1$$

görnüşde saýlap alalyň we berlen nokady $T(t_1,t_2,t_3)$ bilen belläliň. Ähli galtaşma nokatlaryň bir tekizlikde ýatýandygyny subut etmek üçin, ol galtaşma nokatlaryň islendik dördüsiniň bir tekizlikde ýatýandygyny subut etmek ýeterlikdir. Geliň, şol galtaşma nokatlardan islendik dördüsini alalyň we olary $M(m_1,m_2,m_3)$, $N(n_1,n_2,n_3)$, $K(k_1,k_2,k_3)$, $L(l_1,l_2,l_3)$ diýip belläliň (10-njy surat). *M*,*N*,*K*,*L* nokatlaryň bir tekizlikde ýatmagy üçin

$$\left(\overline{MN},\overline{MK},\overline{ML}\right)=\begin{vmatrix} n_1-m_1 & n_2-m_2 & n_3-m_3 \\ k_1-m_1 & k_2-m_2 & k_3-m_3 \\ l_1-m_1 & l_2-m_2 & l_3-m_3 \end{vmatrix}=0$$

bolmaly.



10-njy surat

Belli bolşy, ýaly ellipsoidiň $X(x_0,y_0,z_0)$ nokadynda oňa geçirilen galtaşýan gönüçyzygyň deňlemesi

$$\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z=1$$

bolýar. Oňa görä-de,

$$(TM): \frac{m_1}{a^2}x + \frac{m_2}{b^2}y + \frac{m_3}{c^2}z = 1;$$

$$(TN): \frac{n_1}{a^2}x + \frac{n_2}{b^2}y + \frac{n_3}{c^2}z = 1;$$

$$(TK): \frac{k_1}{a^2}x + \frac{k_2}{b^2}y + \frac{k_3}{c^2}z = 1;$$

$$(TL): \frac{l_1}{a^2}x + \frac{l_2}{b^2}y + \frac{l_3}{c^2}z = 1$$

bolar. Bu gönüleriň umumy nokadynyň barlygy üçin aşakdaky ulgam noldan tapawutly ýeke-täk çözüwe eýedir:

$$\begin{cases} \frac{m_1}{a^2}x + \frac{m_2}{b^2}y + \frac{m_3}{c^2}z = 1; \\ \frac{n_1}{a^2}x + \frac{n_2}{b^2}y + \frac{n_3}{c^2}z = 1; \\ \frac{k_1}{a^2}x + \frac{k_2}{b^2}y + \frac{k_3}{c^2}z = 1; \\ \frac{l_1}{a^2}x + \frac{l_2}{b^2}y + \frac{l_3}{c^2}z = 1 \end{cases}$$

ýa-da

$$\begin{cases} \left(\frac{n_1}{a^2} - \frac{m_1}{a^2}\right)x + \left(\frac{n_2}{b^2} - \frac{m_2}{b^2}\right)y + \left(\frac{n_3}{c^2} - \frac{m_3}{c^2}\right)z = 0; \\ \left(\frac{k_1}{a^2} - \frac{m_1}{a^2}\right)x + \left(\frac{k_2}{b^2} - \frac{m_2}{b^2}\right)y + \left(\frac{k_3}{c^2} - \frac{m_3}{c^2}\right)z = 0; \\ \left(\frac{l_1}{a^2} - \frac{m_1}{a^2}\right)x + \left(\frac{l_2}{b^2} - \frac{m_2}{b^2}\right)y + \left(\frac{l_3}{c^2} - \frac{m_3}{c^2}\right)z = 0. \end{cases}$$

Ahyrky birjynsly ulgamyň noldan tapawutly çözüwiniň bolmagy üçin onuň esasy kesgitleýjisi nola deň bolmaly, ýagny,

$$\begin{vmatrix} \dfrac{n_1}{a^2}-\dfrac{m_1}{a^2} & \dfrac{n_2}{b^2}-\dfrac{m_2}{b^2} & \dfrac{n_3}{c^2}-\dfrac{m_3}{c^2} \\ \dfrac{k_1}{a^2}-\dfrac{m_1}{a^2} & \dfrac{k_2}{b^2}-\dfrac{m_2}{b^2} & \dfrac{k_3}{c^2}-\dfrac{m_3}{c^2} \\ \dfrac{l_1}{a^2}-\dfrac{m_1}{a^2} & \dfrac{l_2}{b^2}-\dfrac{m_2}{b^2} & \dfrac{l_3}{c^2}-\dfrac{m_3}{c^2} \end{vmatrix} = 0.$$

Bu ýerden,

$$\frac{1}{a^2b^2c^2}\begin{vmatrix} n_1-m_1 & n_2-m_2 & n_3-m_3 \\ k_1-m_1 & k_2-m_2 & k_3-m_3 \\ l_1-m_1 & l_2-m_2 & l_3-m_3 \end{vmatrix} = 0$$

ýa-da

$$\begin{vmatrix} n_1-m_1 & n_2-m_2 & n_3-m_3 \\ k_1-m_1 & k_2-m_2 & k_3-m_3 \\ l_1-m_1 & l_2-m_2 & l_3-m_3 \end{vmatrix} = 0$$

bolýandygyny alarys. Diýmek, nokatlar bir tekizlikde ýatýar.

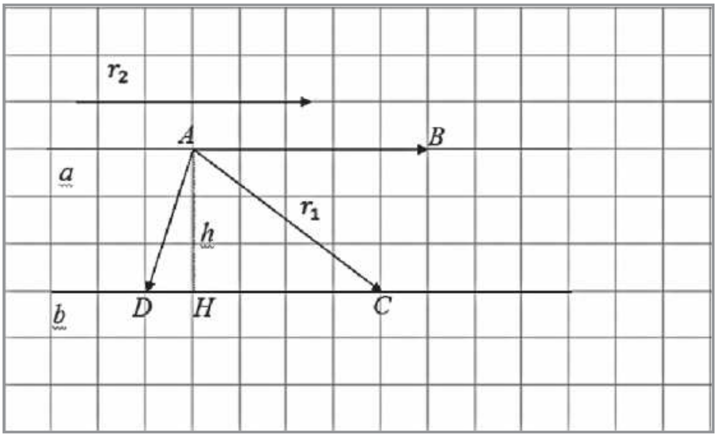
12-nji mesele. Iki parallel gönüçyzygyň arasyndaky h uzaklygy

$$h = \frac{|r_1 \times r_2|}{|r_2|}$$

formula bilen aňladyp bolýandygyny subut ediň. Bu ýerde r_1- başlangyjy berlen gönüleriň birinde, ahyry beýlekisinde bolan wektor, r_2-berlen gönülere parallel wektor.

Çözülişi. 11-nji suratdan görnüşi ýaly,

$$\overrightarrow{AB} = r_2 \quad \overrightarrow{AC} = r_1 \quad |AH| = h, \quad \overrightarrow{AD} = r_1 - r_2 .$$



11-nji surat

Belli bolşy ýaly, parallellogramyň meýdany üçin

$$S_{ABCD} = |AH||DC| \quad we \quad S_{ABCD} = |\overrightarrow{AB} \times \overrightarrow{AD}|$$

formulalar dogrudyr. Bu iki formulany deňläp, alarys:

$$|AH||DC| = |\overrightarrow{AB} \times \overrightarrow{AD}|;$$

$$h|r_2| = |r_2 \times (r_1 - r_2)|;$$

$$h = \frac{|r_2 \times r_1 - r_2 \times r_2|}{r_2} = \frac{|r_2 \times r_1 - 0|}{r_2} = \frac{|r_2 \times r_1|}{r_2} = \frac{|r_1 \times r_2|}{r_2}.$$

Diýmek,

$$h = \frac{|r_1 \times r_2|}{r_2}.$$

13-nji mesele. Giňişlikde *a, b, c, x, y, z* – wektorlar berlipdir. Aşakdaky toždestwony subut ediň:

$$(a,b,c)(x,y,z)=\begin{vmatrix}(a,x) & (a,y) & (a,z)\\ (b,x) & (b,y) & (b,z)\\ (c,x) & (c,y) & (c,z)\end{vmatrix}.$$

Çözülişi. Belli bolşy ýaly, $a(a_1,a_2,a_3)$, $b(b_1,b_2,b_3)$, $c(c_1,c_2,c_3)$ wektorlar üçin, a,b skalýar we a,b,c gatyşyk köpeltmek hasyllar aşakdaky ýaly kesgitlenýärler:

$$(a,b)=a_1b_1+a_2b_2+a_3b_3;$$

$$(a,b,c)=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}.$$

Berlen mysalda $a(a_1,a_2,a_3)$, $b(b_1,b_2,b_3)$, $c(c_1,c_2,c_3)$, $x(x_1,x_2,x_3)$, $y(y_1,y_2,y_3)$, $z(z_1,z_2,z_3)$ bellenişikleri geçirip we ýokardaky deňlikleri peýdalanyp, alarys:

$$(a,b,c)(x,y,z)=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}\cdot\begin{vmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{vmatrix}=$$

$$=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}\cdot\begin{vmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{vmatrix}^T=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}\cdot\begin{vmatrix}x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{vmatrix}=$$

$$=\left|\begin{pmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix}\begin{pmatrix}x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{pmatrix}\right|$$

ýa-da

$$(a,b,c)(x,y,z)=\begin{vmatrix}(a,x) & (a,y) & (a,z)\\ (b,x) & (b,y) & (b,z)\\ (c,x) & (c,y) & (c,y)\end{vmatrix}.$$

14-nji mesele. $\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC}$ wektorlar

$$\overrightarrow{OA}\times\overrightarrow{OB}+\overrightarrow{OB}\times\overrightarrow{OC}+\overrightarrow{OC}\times\overrightarrow{OA}=0$$

şerti kanagatlandyrýarlar.

a) $\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC}$ wektorlaryň komplanardygyny;

A, B, C nokatlaryň bir gönüde ýatýandygyny subut ediň.

Çözülişi. Ozaldan mälim bolşuna görä $\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC}$ wektorlaryň komplanar bolmagy üçin

$$(\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC})=0$$

deňlik ýerine ýetmegi zerur we ýeterlidir. Meseläniň şertinde berlen deňligi peýdalanyp, bu deňligiň ýerine ýetýändigini görkezeliň:

$$(\overrightarrow{OA}\times\overrightarrow{OB}+\overrightarrow{OB}\times\overrightarrow{OC}+\overrightarrow{OC}\times\overrightarrow{OA})\cdot\overrightarrow{OC}=0\cdot\overrightarrow{OC};$$

$$(\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC})+(\overrightarrow{OB},\overrightarrow{OC},\overrightarrow{OC})+(\overrightarrow{OC},\overrightarrow{OA},\overrightarrow{OC})=0;$$

$$(\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC})+0+0=0 \text{ ýa-da } (\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC})=0.$$

Diýmek, wektorlaryň komplanar..

b) A, B, C nokatlaryň bir gönüde ýatmagy üçin

$$\overrightarrow{AB}\times\overrightarrow{AC}=0$$

bolmagy ýeterlidir. Berlen deňligi peýdalanyp, bu deňligi ýerine ýetýändigini görkezmek bolar:

$$\overrightarrow{OA}\times\overrightarrow{OB}+\overrightarrow{OB}\times\overrightarrow{OC}+\overrightarrow{OC}\times\overrightarrow{OA}=0;$$

$$-\overrightarrow{OB}\times\overrightarrow{OA}+\overrightarrow{OB}\times\overrightarrow{OC}-\overrightarrow{OA}\times\overrightarrow{OC}=-\overrightarrow{OA}\times\overrightarrow{OA};$$

$$(-\overrightarrow{OB}\times\overrightarrow{OA}+\overrightarrow{OB}\times\overrightarrow{OC})-(\overrightarrow{OA}\times\overrightarrow{OC}-\overrightarrow{OA}\times\overrightarrow{OA})=0;$$

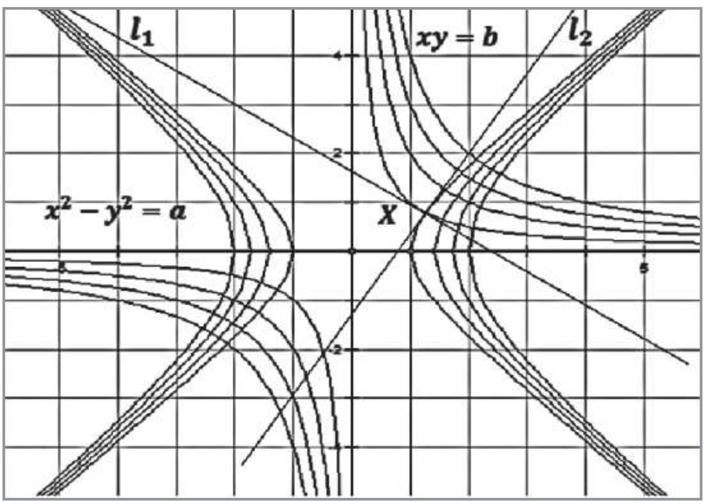
$$\overrightarrow{OB}\times(-\overrightarrow{OA}+\overrightarrow{OC})-\overrightarrow{OA}\times(\overrightarrow{OC}-\overrightarrow{OA})=0;$$

$$(\overrightarrow{AO}+\overrightarrow{OC})\times(\overrightarrow{AO}+\overrightarrow{OB})=0 \text{ ýa-da } \overrightarrow{AB}\times\overrightarrow{AC}=0.$$

Diýmek, A, B, C nokatlar bir gönüde ýatýar.

15-nji mesele. $x^2 - y^2 = a$ we $xy = b$ giperbolalaryň maşgalasynyň ortoganal tory emele getirýändigini, ýagny, egrileriň gönüburç boýunça kesişýändigini subut etmeli.

Çözülişi. $x^2 - y^2 = a$ we $xy = b$ giperbolalaryň maşgalasynyň ortoganal tory emele getirýändigini subut etmek üçin, bu giperbolalaryň kesişme nokatlarynda olara geçirilen galtaşýan gönüleriň özara perpendikulýardygyny subut etmek ýeterlidir (*12-nji surat*).



12-nji surat

Geliň, indi we egrileriň maşgalasyna degişli bolan islendik iki we egrini alyp, olaryň

$$X(x_0, y_0,): \begin{cases} x_0^2 - y_0^2 = a_0; \\ x_0 y_0 = b_0 \end{cases}$$

kesişme nokadynda, egrilere geçirilen degişlilikde l_1 we l_2 galtaşýan gönüçyzyklaryň burç koeffisiýentlerini tapalyň:

$$l_1 : y = k_1 x + p_{1,} \qquad k_1 = y^{'}(x_0) = \frac{x_0}{y_0} = \frac{x_0^2}{b_0};$$

$$l_2 : y = k_2 x + p_{2,} \qquad k_2 = y^{'}(x_0) = -\frac{b_0}{x_0^2}.$$

Bu ýerden görnüşi ýaly,. Bu bolsa we l_2 gönüleriň özara perpendikulýardygyny aňladýar. Diýmek, we giperbolalaryň maşgalasy gönü-burç boýunça kesişýärler, ýagny, olar ortoganal tory emele getirýärler.

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